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EXPLICIT EXPRESSION FOR A FIRST INTEGRAL FOR SOME CLASSES OF TWO-DIMENSIONAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we are interested in studying the existence of first integrals and then the trajectories for classes of two-dimensional differential systems of the forms

$$\begin{cases} x' = \frac{P(x,y)^\alpha}{T(x,y)^\beta} + x\frac{R(x,y)^\gamma}{S(x,y)^\delta}, \\ y' = \frac{Q(x,y)^\alpha}{K(x,y)^\beta} + y\frac{R(x,y)^\gamma}{S(x,y)^\delta}, \end{cases}$$

and

$$\begin{cases} x' = x\left(\frac{P(x,y)^\alpha}{T(x,y)^\beta} + \frac{R(x,y)^\gamma}{S(x,y)^\delta}\right), \\ y' = y\left(\frac{Q(x,y)^\alpha}{K(x,y)^\beta} + \frac{R(x,y)^\gamma}{S(x,y)^\delta}\right), \end{cases}$$

where a, b, n, m are positive integers, α, β, γ, $\delta \in \mathbb{Q}$ and $P(x,y)$, $Q(x,y)$, $R(x,y)$, $T(x,y)$, $K(x,y)$, $S(x,y)$ are homogeneous polynomials of degree n, n, m, a, a, b respectively. Concrete examples exhibiting the applicability of our result are introduced.

Keywords: Autonomous differential system, Kolmogorov system, first integral, trajectories, Hilbert 16th problem.

1. Introduction

By definition a planar autonomous differential system is a differential system of the form

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$$(1) \qquad \begin{cases} x' = \dfrac{dx}{dt} = F(x,y), \\ y' = \dfrac{dy}{dt} = G(x,y), \end{cases}$$

where $F(x,y)$ and $G(x,y)$ are functions from an open subset Ω of $\mathbb{R}^2$ to $\mathbb{R}$. The set Ω is said to be the domain of definition of the planar differential systems (1), and $X = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y}$ is the vector field defined on Ω and associated to differential systems (1).

There exist three main open problems in the qualitative theory of real planar differential systems [1, 7, 8, 11, 18], the distinction between a center and a focus, the determination of the number of limit cycles and their distribution, and the determination of their integrability. The importance for searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain polynomial or rational first integrals [19]. One of the classical tools in the classification of all trajectories of a dynamical system is to find first integrals, for or more details about first integrals, see for instance [3, 4, 9, 12, 14, 15, 20, 21]. In the qualitative theory of planar dynamical systems one of the most important topics is related to the second part of the unsolved Hilbert 16th problem [10], there is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly [2].

System (1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω, which is constant on the trajectories of the system (1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x}F(x,y) + \frac{\partial H(x,y)}{\partial y}G(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait [6].

In this paper we introduce an explicit expression of first integral and then the trajectories for classes of two-dimensional differential systems of the forms

$$(2) \qquad \begin{cases} x' = \frac{P(x,y)^\alpha}{T(x,y)^\beta} + x\frac{R(x,y)^\gamma}{S(x,y)^\delta}, \\ y' = \frac{Q(x,y)^\alpha}{K(x,y)^\beta} + y\frac{R(x,y)^\gamma}{S(x,y)^\delta}, \end{cases}$$

and

$$(3) \qquad \begin{cases} x' = x\left(\frac{P(x,y)^\alpha}{T(x,y)^\beta} + \frac{R(x,y)^\gamma}{S(x,y)^\delta}\right), \\ y' = y\left(\frac{Q(x,y)^\alpha}{K(x,y)^\beta} + \frac{R(x,y)^\gamma}{S(x,y)^\delta}\right), \end{cases}$$

where a, b, n, m are positive integers, $\alpha, \beta, \gamma, \delta \in \mathbb{Q}$ and $P(x,y), Q(x,y), R(x,y)$, $T(x,y)$, $K(x,y)$, $S(x,y)$ homogeneous polynomials of degree n, n, m, a, a, b respectively.

The autonomous differential system (3) on the plane known as Kolmogorov system, is frequently used to model the interaction of two species occupying the same ecological niche [16]. There are many natural phenomena which can be modeled by Kolmogorov systems such as mathematical ecology and population dynamics [17] chemical reactions, plasma physics [13], hydrodynamics [5], economics, etc..

In the classical Lotka- Volterra-Gause model, F and G are linear and it is well known that there are no limit cycles. There can, of course, only be one critical point in the interior of the realistic quadrant $\{(x,y) : x > 0, y > 0\}$ in this case, but this can be a center, however, there are no isolated periodic solutions.

Let us define the trigonometric functions

$f_1(\theta) = \frac{P(\cos\theta,\sin\theta)^\alpha}{T(\cos\theta,\sin\theta)^\beta}\cos\theta + \frac{Q(\cos\theta,\sin\theta)^\alpha}{K(\cos\theta,\sin\theta)^\beta}\sin\theta,\ f_2(\theta) = \frac{R(\cos\theta,\sin\theta)^\gamma}{S(\cos\theta,\sin\theta)^\delta},$

$f_3(\theta) = \frac{Q(\cos\theta,\sin\theta)^\alpha}{K(\cos\theta,\sin\theta)^\beta}\cos\theta - \frac{P(\cos\theta,\sin\theta)^\alpha}{T(\cos\theta,\sin\theta)^\beta}\sin\theta,$

$g_1(\theta) = \frac{P(\cos\theta,\sin\theta)^\alpha}{T(\cos\theta,\sin\theta)^\beta}\cos^2\theta + \frac{Q(\cos\theta,\sin\theta)^\alpha}{K(\cos\theta,\sin\theta)^\beta}\sin^2\theta,$

$g_2(\theta) = \frac{Q(\cos\theta,\sin\theta)^\alpha}{K(\cos\theta,\sin\theta)^\beta}\cos\theta\sin\theta - \frac{P(\cos\theta,\sin\theta)^\alpha}{T(\cos\theta,\sin\theta)^\beta}\cos\theta\sin\theta.$

2. Main result

Our main result on the existence of a first integral and then the trajectories of the two-dimensional differential systems (2) and (3) is the following.

Theorem 1. *Consider the two-dimensional differential systems* (2) *and* (3)*, then the following statements hold.*

(*a*) *If* $f_3(\theta) \neq 0$ *for* $\theta \in \left(0, \frac{\pi}{2}\right)$, $\lambda \neq 0$, $P(\cos\theta,\sin\theta) \geq 0$, $Q(\cos\theta,\sin\theta) \geq 0$, $R(\cos\theta,\sin\theta) \geq 0$, $K(\cos\theta,\sin\theta) > 0, S(\cos\theta,\sin\theta) > 0$ *and* $T(\cos\theta,\sin\theta) > 0$, *then system* (2) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{\lambda}{2}} \exp\left(-\lambda \int_{\theta_0}^{\arctan\frac{y}{x}} A(s)\,ds\right) - \lambda \int_{\theta_0}^{\arctan\frac{y}{x}} \exp\left(-\lambda\int_{\theta_0}^{w} A(s)\,ds\right) B(w)\,dw,$$

where $A(s) = \frac{f_1(s)}{f_3(s)}$, $B(\theta) = \frac{f_2(s)}{f_3(s)}$, $\lambda = b\delta - a\beta + n\alpha - m\gamma - 1$, *and* θ_0 *is a number from the interval* $\left(0, \frac{\pi}{2}\right)$.

Moreover, the system (2) *has no limit cycle contained in the realistic quadrant* $\{(x,y) : x > 0, y > 0\}$.

(*b*) *If* $f_3(\theta) \neq 0$ *for* $\theta \in \left(0, \frac{\pi}{2}\right)$, $P(\cos\theta,\sin\theta) \geq 0$, $Q(\cos\theta,\sin\theta) \geq 0$, $R(\cos\theta,\sin\theta) \geq 0$, $K(\cos\theta,\sin\theta) > 0$, $S(\cos\theta,\sin\theta) > 0$,$T(\cos\theta,\sin\theta) > 0$ *and* $\lambda = 0$, *then system* (2) *has the first integral*

$$H(x,y) = \sqrt{x^2+y^2}\exp\left(-\int_{\theta_0}^{\arctan\frac{y}{x}} (A(s)+B(s))\,ds\right).$$

Moreover, the system (2) *has no limit cycle contained in the realistic quadrant* $\{(x,y) : x > 0, y > 0\}$.

(*c*) *If* $f_3(\theta) = 0$ *for all* $\theta \in \left(0, \frac{\pi}{2}\right)$, *then system* (2) *has the first integral* $H = \frac{y}{x}$. *Moreover, the system* (2) *has no limit cycle.*

(*d*) *If* $g_2(\theta) \neq 0$ *for* $\theta \in \left(0, \frac{\pi}{2}\right)$, $P(\cos\theta,\sin\theta) \geq 0$, $Q(\cos\theta,\sin\theta) \geq 0$, $R(\cos\theta,\sin\theta) \geq 0$, $K(\cos\theta,\sin\theta) > 0$, $S(\cos\theta,\sin\theta) > 0$, $T(\cos\theta,\sin\theta) > 0$ *and* $1+\lambda \neq 0$, *then* *system* (3) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{1+\lambda}{2}} \exp\left(-(1+\lambda)\int_{\theta_0}^{\arctan\frac{y}{x}} C(s)\,ds\right) - (1+\lambda)\int_{\theta_0}^{\arctan\frac{y}{x}} \exp\left(-(1+\lambda)\int_{\theta_0}^{w} C(s)\,ds\right) D(w)\,dw,$$

where $C(s)=\frac{g_1(s)}{g_2(s)}$, $D(s)=\frac{f_2(s)}{g_2(s)}$, $\lambda = b\delta - a\beta + n\alpha - m\gamma - 1$, *and* θ_0 *is a number from the interval* $\left(0,\frac{\pi}{2}\right)$.

Moreover, the system (3) *has no limit cycle contained in the realistic quadrant* $\{(x,y): x>0, y>0\}$.

(*e*) *If* $g_2(\theta)\neq 0$ *for* $\theta\in\left(0,\frac{\pi}{2}\right)$, $P(\cos\theta,\sin\theta)\geq 0$, $Q(\cos\theta,\sin\theta)\geq 0$, $R(\cos\theta,\sin\theta)\geq 0$, $K(\cos\theta,\sin\theta)>0$, $S(\cos\theta,\sin\theta)>0$, $T(\cos\theta,\sin\theta)>0$ *and* $1+\lambda=0$, *then system* (3) *has the first integral*

$$H(x,y)=\sqrt{x^2+y^2}\exp\left(-\int_{\theta_0}^{\arctan\frac{y}{x}}(C(s)+D(s))\,ds\right).$$

Moreover, the system (3) *has no limit cycle contained in the realistic quadrant* $\{(x,y): x>0, y>0\}$.

(*f*) *If* $g_2(\theta)=0$ *for all* $\theta\in\left(0,\frac{\pi}{2}\right)$, *then system* (3) *has the first integral* $H=\frac{y}{x}$. *Moreover, the system* (3) *has no limit cycle.*

Доказательство. In order to prove our results (*a*), (*b*) and (*c*) we write the differential system (2) in polar coordinates (r,θ), defined by $x=r\cos\theta$ and $y=r\sin\theta$, then system (2) becomes

$$\left\{\begin{array}{l} r' = f_1(\theta)\,r^{n\alpha-a\beta}+f_2(\theta)\,r^{m\gamma-b\delta+1},\\ \theta' = f_3(\theta)\,r^{n\alpha-a\beta-1}, \end{array}\right. \tag{4}$$

where the trigonometric functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ are given in introduction, $r'=\frac{dr}{dt}$ and $\theta'=\frac{d\theta}{dt}$.

Assume that $f_3(\theta)\neq 0$ for $\theta\in\left(0,\frac{\pi}{2}\right)$, $P(\cos\theta,\sin\theta)\geq 0$, $Q(\cos\theta,\sin\theta)\geq 0$, $R(\cos\theta,\sin\theta)\geq 0$, $K(\cos\theta,\sin\theta)>0$, $S(\cos\theta,\sin\theta)>0$, $T(\cos\theta,\sin\theta)>0$ and $\lambda\neq 0$.

Taking as independent variable the coordinate θ, then the differential system (4) writes

$$\frac{dr}{d\theta}=A(\theta)\,r+B(\theta)\,r^{1-\lambda},$$

where $A(\theta)=\frac{f_1(\theta)}{f_3(\theta)}$, $B(\theta)=\frac{f_2(\theta)}{f_3(\theta)}$ and $\lambda=b\delta-a\beta+n\alpha-m\gamma-1$, which is a Bernoulli equation. By introducing the standard change of variables $\rho=r^{\lambda}$ we obtain the linear equation

$$\frac{d\rho}{d\theta}=\lambda\left(A(\theta)\,\rho+B(\theta)\right). \tag{5}$$

The general solution of linear equation (5) is

$$\rho(\theta)=\exp\left(\lambda\int_{\theta_0}^{\theta}A(s)\,ds\right)\left(\mu+\lambda\int_{\theta_0}^{\theta}\exp\left(-\lambda\int_{\theta_0}^{w}A(s)\,ds\right)B(w)\,dw\right),$$

where $\mu \in \mathbb{R}$, which has the first integral

$$\begin{aligned} H(x,y) &= \left(x^2+y^2\right)^{\frac{\lambda}{2}} \exp\left(-\lambda \int_{\theta_0}^{\arctan \frac{y}{x}} A(s)\, ds\right) - \\ & \lambda \int_{\theta_0}^{\arctan \frac{y}{x}} \exp\left(-\lambda \int_{\theta_0}^{w} A(s)\, ds\right) B(w)\, dw. \end{aligned}$$

The curves $H = h$ with $h \in \mathbb{R}$, are formed by trajectories of the differential system (2). These trajectories can be written in cartesian coordinates as

$$x^2+y^2 = \left(\begin{array}{c} h \exp\left(\lambda \int_{\theta_0}^{\arctan \frac{y}{x}} A(s)\, ds\right) + \\ \lambda \exp\left(\lambda \int_{\theta_0}^{\arctan \frac{y}{x}} A(s)\, ds\right) \\ \int_{\theta_0}^{\arctan \frac{y}{x}} \exp\left(-\lambda \int_{\theta_0}^{w} A(s)\, ds\right) B(w)\, dw \end{array}\right)^{\frac{2}{\lambda}}.$$

Therefore the periodic orbit Γ is contained in the curve

$$x^2+y^2 = \left(\begin{array}{c} h_\Gamma \exp\left(\lambda \int_{\theta_0}^{\arctan \frac{y}{x}} A(s)\, ds\right) + \\ \lambda \exp\left(\lambda \int_{\theta_0}^{\arctan \frac{y}{x}} A(s)\, ds\right) \\ \int_{\theta_0}^{\arctan \frac{y}{x}} \exp\left(-\lambda \int_{\theta_0}^{w} A(s)\, ds\right) B(w)\, dw \end{array}\right)^{\frac{2}{\lambda}}.$$

But this curve cannot contain the periodic orbit Γ and consequently no limit cycle contained in the realistic quadrant $\{(x,y) : x > 0, y > 0\}$, because this curve in realistic quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in \left]0, +\infty\right[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ for all $\eta \in \left]0, +\infty\right[$. The abscissa is given by

$$x = \frac{1}{\sqrt{1+\eta^2}} \left(\begin{array}{c} h_\Gamma \exp\left(\lambda \int_{\theta_0}^{\arctan \eta} A(s)\, ds\right) + \\ \lambda \exp\left(\lambda \int_{\theta_0}^{\arctan \eta} A(s)\, ds\right) \\ \int_{\theta_0}^{\arctan \eta} \exp\left(-\lambda \int_{\theta_0}^{w} A(s)\, ds\right) B(w)\, dw \end{array}\right)^{\frac{2}{\lambda}}.$$

There is at most a unique value of x on every half straight OX^+. Consequently, there is at most a unique point in the realistic quadrant $\{(x,y) : x > 0, y > 0\}$. So this curve cannot contain the periodic orbit and consequently there is no limit cycle.

Hence statement (a) of Theorem 1 is proved.

Suppose now that $f_3(\theta) \neq 0$ for $\theta \in \left(0, \frac{\pi}{2}\right)$, $P(\cos\theta, \sin\theta) \geq 0$, $Q(\cos\theta, \sin\theta) \geq 0$, $R(\cos\theta, \sin\theta) \geq 0$, $K(\cos\theta, \sin\theta) > 0$, $S(\cos\theta, \sin\theta) > 0$, $T(\cos\theta, \sin\theta) > 0$ and $\lambda = 0$.

Taking as independent variable the coordinate θ, then the differential system (3) writes

$$\frac{dr}{d\theta} = \left(A(\theta) + B(\theta)\right) r, \tag{6}$$

where $A(\theta) = \frac{f_1(\theta)}{f_3(\theta)}$, $B(\theta) = \frac{f_2(\theta)}{f_3(\theta)}$.

The general solution of equation (6) is

$$r(\theta) = \mu \exp\left(\int_{\theta_0}^{\theta} \left(A(s) + B(s)\right) ds\right),$$

where $\mu \in \mathbb{R}$, which has the first integral

$$H(x,y) = \sqrt{x^2+y^2}\exp\left(-\int_{\theta_0}^{\arctan\frac{y}{x}} (A(s)+B(s))\,ds\right).$$

The curves $H = h$ with $h \in \mathbb{R}$, are formed by trajectories of the differential system (2). These trajectories can be written in cartesian coordinates as

$$\sqrt{x^2+y^2} = h\exp\left(\int_{\theta_0}^{\arctan\frac{y}{x}} (A(s)+B(s))\,ds\right).$$

Therefore the periodic orbit Γ is contained in the curve

$$\sqrt{x^2+y^2} = h_\Gamma\exp\left(\int_{\theta_0}^{\arctan\frac{y}{x}} (C(s)+D(s))\,ds\right).$$

But this curve cannot contain the periodic orbit Γ, and consequently no limit cycle contained in the realistic quadrant $\{(x,y) : x > 0, y > 0\}$, because this curve in realistic quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in \left]0,+\infty\right[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ for all $\eta \in \left]0,+\infty\right[$. The abscissa is given by

$$x = \frac{h_\Gamma}{\sqrt{1+\eta^2}}\exp\left(\int_{\theta_0}^{\arctan\eta} (C(s)+D(s))\,ds\right).$$

There is at most a unique value of x on every half straight OX^+. Consequently, there is at most a unique point in realistic quadrant $\{(x,y) : x > 0, y > 0\}$. So this curve cannot contain the periodic orbit and consequently there is no limit cycle.

Hence statement (b) of Theorem 1 is proved.

Assume now that $f_3(\theta) = 0$ for all $\theta \in \left(0,\frac{\pi}{2}\right)$, then from (4) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system. Then since all straight lines through the origin are formed by trajectories, which can be written in cartesian coordinates as $y = hx$ where $h \in \mathbb{R}$. Consequently, there is no limit cycle.

This completes the proof of statement (c) of Theorem 1.

In order to prove our results (d), (e) and (f), we write the polynomial differential system (3) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then system (3) becomes

$$\begin{cases} r' = g_1(\theta)\,r^{n\alpha-a\beta+1} + f_2(\theta)\,r^{m\gamma-b\delta+1}, \\ \theta' = g_2(\theta)\,r^{n\alpha-a\beta}, \end{cases} \tag{7}$$

where the trigonometric functions $f_2(\theta)$, $g_1(\theta)$, $g_2(\theta)$ are given in introduction, $r' = \frac{dr}{dt}$ and $\theta' = \frac{d\theta}{dt}$.

Assume that $g_2(\theta) \neq 0$ for $\theta \in \left(0,\frac{\pi}{2}\right)$, $P(\cos\theta,\sin\theta) \geq 0$, $Q(\cos\theta,\sin\theta) \geq 0$, $R(\cos\theta,\sin\theta) \geq 0$, $K(\cos\theta,\sin\theta) > 0$, $S(\cos\theta,\sin\theta) > 0$, $T(\cos\theta,\sin\theta) > 0$ and $1+\lambda \neq 0$.

Taking as new independent variable the coordinate θ, then the differential system (7) writes

$$\frac{dr}{d\theta} = C(\theta)\,r + D(\theta)\,r^{-\lambda},$$

where $C(\theta) = \frac{g_1(\theta)}{g_2(\theta)}$, $D(\theta) = \frac{f_2(\theta)}{g_2(\theta)}$ and $\lambda = b\delta - a\beta + n\alpha - m\gamma - 1$ which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^{1+\lambda}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = (1+\lambda)\left(C(\theta)\rho + D(\theta)\right). \tag{8}$$

The general solution of linear equation (8) is

$$\begin{aligned}\rho(\theta) &= \exp\left((1+\lambda)\int_{\theta_0}^{\theta} C(s)\,ds\right)\\ &\quad \left(\mu + (1+\lambda)\int_{\theta_0}^{\theta} \exp\left(-(1+\lambda)\int_{\theta_0}^{w} C(s)\,ds\right) D(w)\,dw\right),\end{aligned}$$

where $\mu \in \mathbb{R}$, which has the first integral

$$\begin{aligned}H(x,y) &= \left(x^2+y^2\right)^{\frac{1+\lambda}{2}} \exp\left(-(1+\lambda)\int_{\theta_0}^{\arctan\frac{y}{x}} C(s)\,ds\right) - \\ &\quad (1+\lambda)\int_{\theta_0}^{\arctan\frac{y}{x}} \exp\left(-(1+\lambda)\int_{\theta_0}^{w} C(s)\,ds\right) D(w)\,dw.\end{aligned}$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, are formed by trajectories of the differential system (3), These trajectories can be written in cartesian coordinates as

$$x^2+y^2 = \begin{pmatrix} h\exp\left((1+\lambda)\int_{\theta_0}^{\arctan\frac{y}{x}} C(s)\,ds\right) + \\ (1+\lambda)\exp\left((1+\lambda)\int_{\theta_0}^{\arctan\frac{y}{x}} C(s)\,ds\right) \\ \int_{\theta_0}^{\arctan\frac{y}{x}} \exp\left(-(1+\lambda)\int_{\theta_0}^{w} C(s)\,ds\right) D(w)\,dw \end{pmatrix}^{\frac{2}{1+\lambda}},$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$x^2+y^2 = \begin{pmatrix} h_\Gamma\exp\left((1+\lambda)\int_{\theta_0}^{\arctan\frac{y}{x}} C(s)\,ds\right) + \\ (1+\lambda)\exp\left((1+\lambda)\int_{\theta_0}^{\arctan\frac{y}{x}} C(s)\,ds\right) \\ \int_{\theta_0}^{\arctan\frac{y}{x}} \exp\left(-(1+\lambda)\int_{\theta_0}^{w} C(s)\,ds\right) D(w)\,dw \end{pmatrix}^{\frac{2}{1+\lambda}}.$$

But this curve cannot contain the periodic orbit Γ and consequently no limit cycle contained in the realistic quadrant $\{(x,y) : x > 0, y > 0\}$, because this curve in realistic quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in]0,+\infty[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ for all $\eta \in]0,+\infty[$. The abscissa is given by

$$x = \frac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} h_\Gamma\exp\left((1+\lambda)\int_{\theta_0}^{\arctan\eta} C(s)\,ds\right) + \\ (1+\lambda)\exp\left((1+\lambda)\int_{\theta_0}^{\arctan\eta} C(s)\,ds\right) \\ \int_{\theta_0}^{\arctan\eta} \exp\left(-(1+\lambda)\int_{\theta_0}^{w} C(s)\,ds\right) D(w)\,dw \end{pmatrix}^{\frac{2}{1+\lambda}}.$$

There is at most a unique value of x on every half straight OX^+. Consequently, there is at most a unique point in realistic quadrant $\{(x,y) : x > 0, y > 0\}$. So this curve cannot contain the periodic orbit and consequently there is no limit cycle.

Hence statement (d) of Theorem 1 is proved.

Suppose now that $g_2(\theta) \neq 0$ for $\theta \in \left(0, \frac{\pi}{2}\right)$, $P(\cos\theta, \sin\theta) \geq 0$, $Q(\cos\theta, \sin\theta) \geq 0$, $R(\cos\theta, \sin\theta) \geq 0$, $K(\cos\theta, \sin\theta) > 0$, $S(\cos\theta, \sin\theta) > 0$, $T(\cos\theta, \sin\theta) > 0$ and $1 + \lambda = 0$

Taking as new independent variable the coordinate θ, then the differential system (7) writes

$$\frac{dr}{d\theta} = (C(\theta) + D(\theta))\, r. \tag{9}$$

The general solution of equation (9) is

$$r(\theta) = \mu \exp\left(\int_{\theta_0}^{\theta} (C(s) + D(s))\, ds\right),$$

where $\mu \in \mathbb{R}$, which has the first integral

$$H(x,y) = \sqrt{x^2+y^2} \exp\left(-\int_{\theta_0}^{\arctan \frac{y}{x}} (C(s) + D(s))\, ds\right).$$

Let Γ be a periodic orbit surrounding an equilibrium located in one of the realistic quadrant $\{(x,y) : x > 0, y > 0\}$ and let $h_\Gamma = H(\Gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, are formed by trajectories of the differential system (3). These trajectories can be written in cartesian coordinates as

$$\sqrt{x^2+y^2} = h \exp\left(\int_{\theta_0}^{\arctan \frac{y}{x}} (C(s) + D(s))\, d\omega\right),$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$\sqrt{x^2+y^2} = h_\Gamma \exp\left(\int_{\theta_0}^{\arctan \frac{y}{x}} (C(s) + D(s))\, ds\right).$$

But this curve cannot contain the periodic orbit Γ, and consequently no limit cycle contained in the realistic quadrant $\{(x,y) : x > 0, y > 0\}$, because this curve in realistic quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in \left]0, +\infty\right[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ for all $\eta \in \left]0, +\infty\right[$. The abscissa is given by

$$x = \frac{h_\Gamma}{\sqrt{1+\eta^2}} \exp\left(\int_{\theta_0}^{\arctan \eta} (C(s) + D(s))\, ds\right).$$

There is at most a unique value of x on every half straight OX^+. Consequently, there is at most a unique point in realistic quadrant $\{(x,y) : x > 0, y > 0\}$. So this curve cannot contain the periodic orbit and consequently there is no limit cycle.

Hence statement (e) of Theorem 1 is proved.

Assume now that $g_2(\theta) = 0$ for all $\theta \in \left(0, \frac{\pi}{2}\right)$. Then, from differential system (7) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (3) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system. Then since all straight lines through the origin are formed by trajectories, which can be written in cartesian coordinates as $y = hx$ where $h \in \mathbb{R}$. Consequently, there is no limit cycle.

This completes the proof of statement (f) of Theorem 1. $\square$

3. Examples

The following examples are given to illustrate our results.

Example 1 If we take $P(x,y) = 3x^5 + 3x^3y^2$, $T(x,y) = x^4 + 2x^2y^2 + y^4$, $R(x,y) = 4x^4 + 8x^2y^2 + 4y^4$, $S(x,y) = x^2 + y^2$, $Q(x,y) = 6x^4y + 9x^2y^3 + 3y^5$, $K(x,y) = x^4 + 2x^2y^2 + y^4$ and $\alpha = 1$, $\beta = \gamma = \frac{1}{2}$, $\delta = \frac{3}{2}$, then system (2) reads

$$\begin{cases} x' = \frac{3x^5+3x^3y^2}{(x^4+2x^2y^2+y^4)^{\frac{1}{2}}} + x\frac{(4x^4+8x^2y^2+4y^4)^{\frac{1}{2}}}{(x^2+y^2)^{\frac{3}{2}}}, \\ y' = \frac{6x^4y+9x^2y^3+3y^5}{(x^4+2x^2y^2+y^4)^{\frac{1}{2}}} + y\frac{(4x^4+8x^2y^2+4y^4)^{\frac{1}{2}}}{(x^2+y^2)^{\frac{3}{2}}}. \end{cases} \tag{10}$$

The Kolmogorov system (10) in polar coordinates (r,θ) becomes

$$\begin{cases} r' = 3r^3 + 2, \\ \theta' = \left(\frac{3}{2}\sin 2\theta\right) r^2, \end{cases}$$

here $f_1(\theta) = 3$, $f_2(\theta) = 2$ and $f_3(\theta) = \frac{3}{2}\sin 2\theta$. This corresponds to the case (a) of the Theorem 1. Then the Kolmogorov system (10) has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{3}{2}} \exp\left(\int_{\theta_0}^{\arctan\frac{y}{x}} \frac{-6}{\sin 2s} ds\right) - 4\int_{\theta_0}^{\arctan\frac{y}{x}} \frac{\exp\left(\int_0^w \frac{-6}{\sin 2s} ds\right)}{\sin 2w} dw.$$

where θ_0 is a number from the interval $\left(0, \frac{\pi}{2}\right)$.

The curves $H = h$ with $h \in \mathbb{R}$, are formed by trajectories of the differential system (10). These trajectories can be written in cartesian coordinates as

$$\begin{aligned} \left(x^2+y^2\right)^{\frac{3}{2}} &= h\exp\left(\int_{\theta_0}^{\arctan\frac{y}{x}} \frac{6}{\sin 2s} ds\right) + \\ & 4\exp\left(\int_{\theta_0}^{\arctan\frac{y}{x}} \frac{6}{\sin 2s} ds\right) \int_{\theta_0}^{\arctan\frac{y}{x}} \frac{\exp\left(\int_0^w \frac{-6}{\sin 2s} ds\right)}{\sin 2w} dw, \end{aligned}$$

where $h \in \mathbb{R}$. The system (10) has no periodic orbits, and consequently no limit cycle contained in the realistic quadrant $\{(x,y) : x > 0, y > 0\}$.

Example 2 If we take $P(x,y) = 10x^4 + 9y^4 + 18x^2y^2$, $T(x,y) = x^3y + xy^3$, $R(x,y) = 4x^4 + 8x^2y^2 + 4y^4$, $S(x,y) = x^2 + y^2$, $Q(x,y) = x^4 + y^4 + 2x^2y^2$, $K(x,y) = x^3y + xy^3$ and $\alpha = 1$, $\beta = 1, \gamma = \frac{3}{2}$, $\delta = 2$, then system (3) reads

$$\begin{cases} x' = x\left(\frac{10x^4+9y^4+18x^2y^2}{x^3y+xy^3} + \frac{(4x^4+8x^2y^2+4y^4)^{\frac{3}{2}}}{(x^2+y^2)^2}\right), \\ y' = y\left(\frac{x^4+y^4+2x^2y^2}{x^3y+xy^3} + \frac{(4x^4+8x^2y^2+4y^4)^{\frac{3}{2}}}{(x^2+y^2)^2}\right). \end{cases} \tag{11}$$

The Kolmogorov system (11) in polar coordinates (r,θ) becomes

$$\begin{cases} r' = \left(\left(9+\cos^4\theta\right)\cot\theta + \tan\theta\right) r + 8r^3, \\ \theta' = -\left(8+\cos^4\theta\right), \end{cases}$$

here $g_1(\theta) = \left(9 + \cos^4\theta\right)\cot\theta + \tan\theta$, $f_2(\theta) = 8$ and $g_2(\theta) = -\left(8 + \cos^4\theta\right)$. This corresponds to the case (d) of the Theorem 1. Then the Kolmogorov system (11) has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{\frac{-1}{2}} \exp\left(\int_{\theta_0}^{\arctan\frac{y}{x}} \frac{\left(9+\cos^4 s\right)\cot s + \tan s}{-8-\cos^4 s} ds\right) + \int_{\theta_0}^{\arctan\frac{y}{x}} \frac{8\exp\left(\int_{\theta_0}^{w} \frac{\left(9+\cos^4 s\right)\cot s+\tan s}{-8-\cos^4 s} ds\right)}{-8-\cos^4 w} dw,$$

where θ_0 is a number from the interval $\left(0, \frac{\pi}{2}\right)$.

The curves $H = h$ with $h \in \mathbb{R}$, are formed by trajectories of the differential system (11). These trajectories can be written in cartesian coordinates as

$$\left(x^2+y^2\right)^{\frac{-1}{2}} = \exp\left(\int_{\theta_0}^{\arctan\frac{y}{x}} \frac{\left(9+\cos^4 s\right)\cot s + \tan s}{8+\cos^4 s} ds\right) \left(h + \int_{\theta_0}^{\arctan\frac{y}{x}} \frac{8\exp\left(\int_{\theta_0}^{w} \frac{\left(9+\cos^4 s\right)\cot s+\tan s}{-8-\cos^4 s} ds\right)}{8+\cos^4 w} dw\right),$$

where $h \in \mathbb{R}$. The system (11) has no periodic orbits, and consequently no limit cycle contained in the realistic quadrant $\{(x,y) : x > 0, y > 0\}$.

Example 3 If we take $P(x,y) = x^2 - 2xy + 3y^2$, $T(x,y) = x^2 - 2xy + 3y^2$, $R(x,y) = x+y$, $S(x,y) = x^2+y^2$, $Q(x,y) = x^2-2xy+3y^2$, $K(x,y) = x^2-2xy+3y^2$ and $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\gamma = 3$, $\delta = 2$, then system (3) reads

$$\text{(12)} \quad \begin{cases} x' = x\left(\frac{\left(x^2-2xy+3y^2\right)^{\frac{5}{2}}}{\left(x^2-2xy+3y^2\right)^{\frac{1}{2}}} + \frac{(x+y)^3}{(x^2+y^2)^2}\right), \\ y' = y\left(\frac{\left(x^2-2xy+3y^2\right)^{\frac{5}{2}}}{\left(x^2-2xy+3y^2\right)^{\frac{1}{2}}} + \frac{(x+y)^3}{(x^2+y^2)^2}\right). \end{cases}$$

The Kolmogorov system (12) in polar coordinates (r, θ) becomes

$$\begin{cases} r' = (5 + \sin 4\theta - 4\sin 2\theta - 4\cos 2\theta)\, r^5 + (1 + 2\cos\theta\sin\theta)(\sin\theta + \cos\theta), \\ \theta' = 0, \end{cases}$$

here $g_1(\theta) = 5 + \sin 4\theta - 4\sin 2\theta - 4\cos 2\theta$, $f_2(\theta) = (1 + 2\cos\theta\sin\theta)(\sin\theta + \cos\theta)$ and $g_2(\theta) = 0$.This corresponds to the case (f) of the Theorem 1. Then the Kolmogorov system (12) has the first integral $H(x,y) = \frac{y}{x}$.

The curves $H = h$ with $h \in \mathbb{R}$, are formed by trajectories of the differential system (12), which can be written in cartesian coordinates as $y = hx$ where $h \in \mathbb{R}$. Consequently, there is no limit cycle.

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