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EXISTENCE OF EXPLICIT ASYMPTOTICALLY NORMAL ESTIMATORS IN A MULTIPLE LOGARITHMIC REGRESSION PROBLEM

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Abstract. We construct and investigate a class of explicit estimators for an unknown multidimensional parameter in a logarithmic regression problem. We present general conditions for these estimators to be asymptotical-ly normal. It is the fourth class of non-linear regression problems for which such explicit estimators are found.

Keywords: multiple logarithmic regression, difficulties in the least squares method, explicit estimators of the parameters, asymptotically normal estimators

1. Introduction and main results

1.1. **Statement of the problem.** Consider a sequence of random variables $\{Y_i\}$ for which the following representations hold:

$$Y_i = \log(x_{oi} + \sum_{j=1}^{k} \alpha_j x_{ji}) + \varepsilon_i,\ i = 1, 2 \ldots, \tag{1}$$

where $\{x_{ij}\}$ are known constants, $\{\alpha_j\}$ are unknown parameters and $\{\varepsilon_i\}$ are unobservable independent and identically distributed random variables. Suppose now that we observe only first n elements $Y_1, \ldots, Y_n$ of the sequence. The research objective is to get explicit estimator $\boldsymbol{\alpha}_n^* = (\alpha_1^*, \ldots, \alpha_k^*)^\top$ for the multidimensional parameter $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)^\top$ and investigate its properties.

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The case $k = 1$ of the one-dimensional unknown parameter was considered in [1] (see also [2]). Here we are going to extend this results to the general case of arbitrary $k > 1$.

The ordinary least squares method is a standard approach of estimator constructing. In this case implicit estimator $\hat{\boldsymbol{\alpha}}_n = (\hat{\alpha}_1, \dots, \hat{\alpha}_k)^\top$ has the following form:

$$\hat{\boldsymbol{\alpha}}_n := arg \min_{\alpha_1,\dots,\alpha_k} \sum_{i=1}^n \Big(Y_i - \log(x_{oi} + \sum_{j=1}^k \alpha_j x_{ji})\Big)^2, \tag{2}$$

i.e. $\hat{\boldsymbol{\alpha}}_n$ is the point where the function on the right hand-side of (2) delivers its minimum.

However, we may face with serious computational difficulties (see [3]), because the function on the right-hand side of (2) may possesses unpredictable growing number of local minima. To avoid this problem we construct below in (11) a class of explicit estimators for parameter $\boldsymbol{\alpha}$, which has several remarkable properties.

1.2. **Transformation of observations.** Before constructing estimators we rewrite (1) in a more convenient form. Introduce the following notations:

$$\theta = \mathbb{E}e^{\varepsilon_i}, \quad \eta_i = e^{\varepsilon_i} - \theta, \quad \sigma_i = x_{oi} + \sum_{j=1}^k \alpha_j x_{ji}, \quad Z_i = e^{Y_i}. \tag{3}$$

So, for all $i = 1, 2 \dots$

$$Z_i = x_{oi}\theta + \theta \sum_{j=1}^k \alpha_j x_{ji} + \sigma_i \eta_i. \tag{4}$$

Rewrite (4) in a more compact form using matrices and vectors notations:

$$\mathbf{z}_n = \theta \mathbf{c}_n + \theta \mathbf{X}_n^\top \boldsymbol{\alpha} + \mathbf{D}_n \boldsymbol{\eta}_n, \quad \text{where} \tag{5}$$

$$\mathbf{z}_n = (Z_1, \dots Z_n)^\top, \ \mathbf{c}_n = (x_{01}, \dots, x_{on})^\top, \ \ \boldsymbol{\eta}_n = (\eta_1, \dots, \eta_n)^\top,$$

$$\mathbf{X}_n = \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{k1} & \dots & x_{kn} \end{pmatrix}, \quad \mathbf{D}_n = \begin{pmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{pmatrix}.$$

Later on we suppose that the following assumption is true.

ASSUMPTION (A). Random variables $\{\varepsilon_i\}$ are independent and identically distributed with

$$0 < \sigma^2 := \mathbb{D}e^{\varepsilon_1} < \infty. \tag{6}$$

1.3. **Basic assumptions.** Denote by $\dot{\mathbf{X}}_n$ the enlarged $(k+1) \times n$ matrix which we obtain if we add the new string $\mathbf{c}_n$ to k strings of the $k \times n$ matrix $\mathbf{X}_n$.

ASSUMPTION (B). Rank $\dot{\mathbf{X}}_{n_0}$ is $k+1$ for some $n_0 < \infty$, we assume that $n \geq n_0$ and that $\sigma_j \neq 0$ for all $j \geq 1$.

Later on the main problem for statistician is to choose a constant $k \times n$ matrix $\mathbf{A}_n$ and a constant vector $\mathbf{b}_n = (b_1, \dots, b_n)^\top$ such that the following four assumptions hold:

$$\mathbf{A}_n \mathbf{X}_n^\top = \mathbf{I}_k, \quad \mathbf{A}_n \mathbf{c}_n = \mathbf{0}_k, \tag{7}$$

$$\mathbf{b}_n^\top \mathbf{c}_n = 1, \quad \mathbf{b}_n^\top \mathbf{X}_n^\top = \mathbf{0}_k, \tag{8}$$

where $\mathbf{0}_k$ is zero vector with k components and $\mathbf{I}_k$ is $k \times k$ unit matrix.

The existence of such $\mathbf{A}_n$ and $\mathbf{b}_n$ for all $n \geq n_0$ follows from the next

Example 1. Since $\mathbf{c}_n^\top \mathbf{c}_n = ||\mathbf{c}_n||^2 > 0$ we may introduce the following matrix

$$\widetilde{\mathbf{X}}_n = \mathbf{X}_n \Big[\mathbf{I_n} - \frac{\mathbf{c}_n \mathbf{c}_n^\top}{\mathbf{c}_n^\top \mathbf{c}_n}\Big] = \mathbf{X}_n - \frac{\mathbf{X}_n \mathbf{c}_n}{\mathbf{c}_n^\top \mathbf{c}_n} \mathbf{c}_n^\top.$$

It will be shown in Lemma 1 that matrix $\widetilde{\mathbf{X}}_n \mathbf{X}_n^\top$ is not degenerate. So, we may introduce

$$\mathbf{A}_n = \Big(\widetilde{\mathbf{X}}_n \mathbf{X}_n^\top\Big)^{-1} \widetilde{\mathbf{X}}_n, \quad \mathbf{b}_n^\top = \frac{\mathbf{c}_n^\top}{\mathbf{c}_n^\top \mathbf{c}_n} \Big[\mathbf{I}_n - \mathbf{X}_n^\top \mathbf{A}_n\Big].$$

It is straightforward to see that the matrix $\mathbf{A}_n$ and the vector $\mathbf{b}_n$ satisfy conditions (7) and (8) respectively.

1.4. **Construction of the estimator.** Let us describe the main ideas which allow us to get estimator for parameter $\boldsymbol{\alpha}$. Choose arbitrary matrix $\mathbf{A}_n$ and vector $\mathbf{b}_n^\top$ satisfying (7) and (8) respectively. Multiply (5) by $\mathbf{A}_n$ and $\mathbf{b}_n^\top$ from the left hand-side. We obtain

$$\mathbf{A}_n \mathbf{z}_n = \boldsymbol{\beta} + \mathbf{A}_n \mathbf{D}_n \boldsymbol{\eta}_n, \quad \mathbf{b}_n^\top \mathbf{z}_n = \theta + \mathbf{b}_n^\top \mathbf{D}_n \boldsymbol{\eta}_n, \quad \text{with} \quad \boldsymbol{\beta} = \theta \boldsymbol{\alpha}. \tag{9}$$

Under wide assumptions by the law of large numbers the last summands in both expressions from (9) tend to 0 as $n \to \infty$. Thus, it is natural to get estimators for parameters in the following forms

$$\boldsymbol{\beta}_n^* = \mathbf{A}_n \mathbf{z}_n, \quad \theta_n^* = \mathbf{b}_n^\top \mathbf{z}_n. \tag{10}$$

Thereby, we obtained estimators for parameters θ and $\boldsymbol{\beta} = \theta\boldsymbol{\alpha}$, so the desired estimator of the parameter $\boldsymbol{\alpha} = \boldsymbol{\beta}/\theta$ we can take in the following form

$$\boldsymbol{\alpha}_n^* = \frac{\boldsymbol{\beta}_n^*}{\theta_n^*} = \frac{\mathbf{A}_n \mathbf{z}_n}{\mathbf{b}_n^\top \mathbf{z}_n}. \tag{11}$$

Remark 1. *The main idea of the construction is to introduce an artificial parameter θ and estimate it primarily.*

1.5. **Main ideas of proofs.** We have from (9) and (10) that

$$\boldsymbol{\beta}_n^* - \boldsymbol{\beta} = \mathbf{A}_n \mathbf{D}_n \boldsymbol{\eta}_n, \quad \theta_n^* - \theta = \mathbf{b}_n^\top \mathbf{D}_n \boldsymbol{\eta}_n. \tag{12}$$

Using (12), consider expression

$$\boldsymbol{\alpha}_n^* - \boldsymbol{\alpha} = \frac{\boldsymbol{\beta}_n^*}{\theta^*} - \boldsymbol{\alpha} = \frac{\boldsymbol{\beta}_n^* - \boldsymbol{\alpha}\theta^*}{\theta^*} = \frac{(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}) - (\boldsymbol{\alpha}\theta^* - \boldsymbol{\alpha}\theta)}{\theta^*} =$$

$$= \frac{\mathbf{A}_n \mathbf{D}_n \boldsymbol{\eta}_n - \boldsymbol{\alpha} \mathbf{b}_n^\top \mathbf{D}_n \boldsymbol{\eta}_n}{\theta^*} = \frac{(\mathbf{A}_n - \boldsymbol{\alpha} \mathbf{b}_n^\top) \mathbf{D}_n \boldsymbol{\eta}_n}{\theta^*}.$$

Thus

$$\boldsymbol{\alpha}_n^* - \boldsymbol{\alpha} = \frac{\mathbf{Q}_n \mathbf{D}_n \boldsymbol{\eta}_n}{\theta^*} = \frac{\mathbf{Q}_n \mathbf{D}_n \boldsymbol{\eta}_n}{\theta + \mathbf{b}_n^\top \mathbf{D}_n \boldsymbol{\eta}_n}, \quad \text{where} \quad \mathbf{Q}_n = \mathbf{A}_n - \boldsymbol{\alpha} \mathbf{b}_n^\top. \tag{13}$$

Underline that

$$\mathbb{E} \mathbf{Q}_n \mathbf{D}_n \boldsymbol{\eta}_n = \mathbf{0}_k, \quad \mathrm{Cov}(\mathbf{Q}_n \mathbf{D}_n \boldsymbol{\eta}_n) = \mathbf{C}_n(\mathbf{Q}_n) := \sigma^2 \mathbf{Q}_n \mathbf{D}_n^2 \mathbf{Q}_n^\top,$$

$$\mathbb{E}\theta_n^* = \theta, \quad \mathbb{D}\theta_n^* = \mathbb{D}(\mathbf{b}_n^\top \mathbf{z}_n) = \sigma^2 \mathbf{b}_n^\top \mathbf{D}_n^2 \mathbf{b}_n = d_n(\mathbf{b}_n) =: \sigma^2 \sum_{i=1}^n b_i^2 \sigma_i^2. \tag{14}$$

Remark 2. *We do not suppose that errors $\{\varepsilon_i\}$ have zero expectations.*

1.6. **Asymptotic normality of estimators.** It will be shown in Theorem 2 below that for $n \geq n_0$ covariance matrix $\mathrm{Cov}(\mathbf{Q}_n\mathbf{D}_n\boldsymbol{\eta}_n)$ is not degenerate. Let all limits below be taken as $n \to \infty$. So, the main result of the research is

Theorem 1. *Let assumptions (A) and (B) hold and random variables* $\mathbf{Q}_n\mathbf{D}_n\boldsymbol{\eta}_n$ *satisfy CLT, i.e.*

$$\mathbf{w}_n := \Big(\mathbf{C}_n(\mathbf{Q}_n)\Big)^{-1/2}\mathbf{Q}_n\mathbf{D}_n\boldsymbol{\eta}_n \Rightarrow N_{(0,\mathbf{I}_k)}. \tag{15}$$

If, in addition

$$d_n(\mathbf{b}_n) \to 0, \tag{16}$$

then $\boldsymbol{\alpha}_n^*$ *is an asymptotically normal estimator of the unknown parameter* $\boldsymbol{\alpha}$ *i.e.*

$$\theta\,\Big(\mathbf{C}_n(\mathbf{Q}_n)\Big)^{-1/2}(\boldsymbol{\alpha}_n^* - \boldsymbol{\alpha}) \Rightarrow N_{(0,\mathbf{I}_k)}, \tag{17}$$

where $d_n(\mathbf{b}_n)$ *and* $\mathbf{C}_n(\mathbf{Q}_n)$ *were defined in (14).*

Remark 3. *Denote by* $\mathbf{q}_i^{(n)}$ *the* i*-th column of matrix* $\Big(\mathbf{C}_n(\mathbf{Q}_n)\Big)^{-1/2}\mathbf{Q}_n\boldsymbol{D}_n$*. By this notation*

$$\Big(\mathbf{C}_n(\mathbf{Q}_n)\Big)^{-1/2}\mathbf{Q}_n\mathbf{D}_n\boldsymbol{\eta}_n = \sum_{i=1}^{n}\eta_i\mathbf{q}_i^{(n)},$$

where $\{\eta_i\}$ *are independent and identically distributed random variables with zero mean. So, we may use CLT for weighted random variables. Under the assumptions (A) and (B) the condition*

$$\max_{i\leq n}\sigma_i^2||\mathbf{q}_i^{(n)}||^2 \to 0, \tag{18}$$

is sufficient for $\{\eta_i\mathbf{q}_i^{(n)}\}$ *to satisfy the Lindeberg condition and (15) holds.*

1.7. **Optimization of the estimators.** Theorem 1 implies that the less covariance matrix $\mathbf{C}_n(\mathbf{Q}_n)$ is the more precise estimator $\boldsymbol{\alpha}^*$ approximates the unknown parameter. So the minimization problem of covariance matrix from (14) with respect to matrix $\mathbf{A}_n$ and vector $\mathbf{b}_n$ naturally arises. Introduce notations

$$\mathbf{M}_n = \mathbf{D}_n^{-2} - \frac{\mathbf{D}_n^{-2}\mathbf{c}_n\mathbf{c}_n^\top\mathbf{D}_n^{-2}}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}, \quad \mathbf{A}_n^{(opt)} = \Big(\mathbf{X}_n\mathbf{M}_n\mathbf{X}_n^\top\Big)^{-1}\mathbf{X}_n\mathbf{M}_n,$$

$$\mathbf{b}_n^{(opt)\top} = \frac{\mathbf{c}_n^\top}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}\mathbf{D}_n^{-2}\Big[\mathbf{I}_n - \mathbf{X}_n^\top\mathbf{A}_n^{(opt)}\Big], \quad \mathbf{Q}_n^{(opt)} = \mathbf{A}_n^{(opt)} - \boldsymbol{\alpha}\mathbf{b}_n^{(opt)\top}. \tag{19}$$

It will be shown in Lemma 1 that matrix $\mathbf{X}_n\mathbf{M}_n\mathbf{X}_n^\top$ is not degenerate.

Also introduce arbitrary $k \times n$ matrix $\mathbf{A}_n$ and arbitrary vector $\mathbf{b}_n$ with n components satisfying (7) and (8) respectively. Using notations (19) denote

$$\boldsymbol{\delta}_n = \mathbf{b}_n - \mathbf{b}_n^{(opt)}, \quad \boldsymbol{\Delta}_n = \mathbf{A}_n - \mathbf{A}_n^{(opt)}, \quad \boldsymbol{\Delta}_{\mathbf{Q},n} = \boldsymbol{\Delta}_n - \boldsymbol{\alpha}\boldsymbol{\delta}_n^\top. \tag{20}$$

Later on notation $B_n \geq C_n$ in case of matrices B_n and C_n means that $B_n - C_n$ is positive semi-definite matrix.

Theorem 2. *Under assumption (B) matrix* $\mathbf{C}_n(\mathbf{Q}_n^{(opt)})$ *is positive defined and for each matrix* $\mathbf{A}_n$ *and vector* $\mathbf{b}_n$ *satisfying (7) and (8)*

$$\mathbf{C}_n(\mathbf{Q}_n) = \mathbf{C}_n(\mathbf{Q}_n^{(opt)}) + \mathbf{C}_n(\boldsymbol{\Delta}_{\mathbf{Q},n}) \geq \mathbf{C}_n(\mathbf{Q}_n^{(opt)}). \tag{21}$$

In addition,

$$\mathbb{D}\theta_n^* = d_n(\mathbf{b}_n) = d_n(\mathbf{b}_n^{(opt)}) + d_n(\boldsymbol{\delta}_n) \geq d_n(\mathbf{b}_n^{(opt)}) > 0. \tag{22}$$

Remark 4. *First, for each fixed vector $\mathbf{b}_n$ we may take*

$$\mathbf{A}_n = \mathbf{A}_n^{(\mathbf{b}_n)} = \mathbf{A}_n^{(opt)} + \boldsymbol{\alpha}(\mathbf{b}_n - \mathbf{b}_n^{(opt)})^\top. \tag{23}$$

In this case

$$\mathbf{A}_n - \boldsymbol{\alpha}\mathbf{b}_n^\top = \mathbf{A}_n^{(\mathbf{b}_n)} - \boldsymbol{\alpha}\mathbf{b}_n^\top = \mathbf{Q}_n^{opt}.$$

And, hence, with $\mathbf{A}_n$ from (23) we obtain minimum of $\mathbf{C}_n(\mathbf{Q}_n)$.

Second, we may always take $\mathbf{b}_n = \mathbf{b}_n^{opt}$ to minimize $d_n(\mathbf{b}_n) = \mathbb{D}\theta_n^*$. Thus, taking $\mathbf{A}_n = \mathbf{A}_n^{opt}$ and $\mathbf{b}_n = \mathbf{b}_n^{opt}$ we minimize the covariance matrix and the variance simultaneously.

Remark 5. *Note that $\mathbf{A}_n^{opt} = \mathbf{A}_n^{opt}(\boldsymbol{\alpha})$ and $\mathbf{b}_n^{opt} = \mathbf{b}_n^{opt}(\boldsymbol{\alpha})$ depend on unknown parameter $\boldsymbol{\alpha}$. This fact makes difficulties to apply such optimal $\mathbf{A}_n$ and $\mathbf{b}_n$ in practice. Nevertheless, we may recommend use the estimator $\boldsymbol{\alpha}_n^*$ with $\mathbf{A}_n^{opt} = \mathbf{A}_n^{opt}(\boldsymbol{\alpha}_0)$ and $\mathbf{b}_n^{opt} = \mathbf{b}_n^{opt}(\boldsymbol{\alpha}_0)$, where the value $\boldsymbol{\alpha}_0$ should be chosen as close to the unknown $\boldsymbol{\alpha}$ as possible.*

1.8. **Concluding remarks.** We constructed the class of estimators $\boldsymbol{\alpha}_n^*$ which depend on some constant matrix $\mathbf{A}_n$ and constant vector $\mathbf{b}_n$. Also we found conditions under which the estimators are asymptotically normal. In search of the best estimator in this class we found optimal matrix $\mathbf{A}_n^{opt}$ and optimal vector $\mathbf{b}_n^{opt}$ which deliver minimum of covariance matrix and minimum of variance from (14) simultaneously.

Earlier one-dimensional explicit asymptotically normal estimators of unknown parameters were obtained only for four classes of non-linear regressions: fractionally linear [4], partially linear [5], power [6] and logarithmic [1]. With regard to explicit estimators for multidimensional unknown parameter in non-linear regressions, earlier only estimators of fractionally linear regressions were obtained and investigated (see [7] and [8]). The survey of the results in this direction may be found in [2].

Note that using Fisher's idea from [9] we may improve our estimator $\boldsymbol{\alpha}_n^*$. Hypothesis is that after one step we may obtain an asymptotically optimal estimator in a wider class of estimators. This estimator will have the same covariance matrix as classical least squares estimator (2) whose calculation often delivers computational difficulties. Especially in the case when the sum of squares has unpredictable growing number of local minima (see [3]).

2. Proofs of theorems

2.1. **Auxiliary lemmas.** First of all, we will prove the next

Lemma 1. *Under assumption (B) for all $n \geq n_0$ matrices $\widetilde{\mathbf{X}}_n\mathbf{X}_n^\top$ and $\mathbf{X}_n\mathbf{M}_n\mathbf{X}_n^\top$ are not degenerate.*

Proof. Note that $\mathbf{X}_n\mathbf{M}_n\mathbf{D}_n^2 = \mathbf{X}_n - \mathbf{v}_k\mathbf{c}_n^\top$ for some vector $\mathbf{v}_k$ with k components. But such matrices have rank k by assumption (B). Hence, matrix

$$\mathbf{X}_n\mathbf{M}_n\mathbf{X}_n^\top = (\mathbf{X}_n\mathbf{M}_n\mathbf{D}_n^2)\mathbf{D}_n^{-2}\mathbf{X}_n^\top$$

has rank k as a product of matrices with ranks k and $n \geq n_0 \geq k$.

The same arguments with $\mathbf{D}_n^2 = \mathbf{I}_n$ imply that $\widetilde{\mathbf{X}}_n\mathbf{X}_n^\top$ is also not degenerate. □

Lemma 2. *Under assumption (B) for all $n \geq n_0$ conditions (7) and (8) hold for $\mathbf{A}_n = \mathbf{A}_n^{(opt)}$ and $\mathbf{b}_n = \mathbf{b}_n^{(opt)}$ from (19).*

Proof. Let us check that matrix $\mathbf{A}_n^{(opt)}$ satisfies conditions from (7). Firstly,

$$\mathbf{A}_n^{(opt)}\mathbf{X}_n^\top = \left(\mathbf{X}_n\mathbf{M}_n\mathbf{X}_n^\top\right)^{-1}\mathbf{X}_n\mathbf{M}_n\mathbf{X}_n^\top = \mathbf{I}_k. \tag{24}$$

For the second condition note that

$$\mathbf{M}_n\mathbf{c}_n = \mathbf{D}_n^{-2}\mathbf{c}_n - \frac{\mathbf{D}_n^{-2}\mathbf{c}_n\mathbf{c}_n^\top\mathbf{D}_n^{-2}}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}\mathbf{c}_n = \mathbf{D}_n^{-2}\mathbf{c}_n - \mathbf{D}_n^{-2}\mathbf{c}_n = \mathbf{0}_k.$$

So,

$$\mathbf{A}_n^{(opt)}\mathbf{c}_n = \left(\mathbf{X}_n\mathbf{M}_n\mathbf{X}_n^\top\right)^{-1}\mathbf{X}_n\mathbf{M}_n\mathbf{c}_n = \mathbf{0}_k. \tag{25}$$

Now consider (8) for $\mathbf{b}_n^{(opt)\top}$ from (19). Note that by (25)

$$\mathbf{b}_n^{(opt)\top}\mathbf{c}_n = \frac{\mathbf{c}_n^\top\mathbf{D}_n^{-2}}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}\left[\mathbf{I}_n - \mathbf{X}_n^\top\mathbf{A}_n^{(opt)}\right]\mathbf{c}_n = \frac{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n} = 1. \tag{26}$$

And by (24)

$$\mathbf{b}_n^{(opt)\top}\mathbf{X}_n^\top = \frac{\mathbf{c}_n^\top\mathbf{D}_n^{-2}}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}\left[\mathbf{I}_n - \mathbf{X}_n^\top\mathbf{A}_n^{(opt)}\right]\mathbf{X}_n^\top = \frac{\mathbf{c}_n^\top\mathbf{D}_n^{-2}}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}\left[\mathbf{X}_n^\top - \mathbf{X}_n^\top\right] = \mathbf{0}_k. \tag{27}$$

Thus, (7) and (8) are true for $\mathbf{A}_n = \mathbf{A}_n^{(opt)}$ and $\mathbf{b}_n = \mathbf{b}_n^{(opt)}$ from (19). □

Lemma 3. *Under assumption (B) for $\boldsymbol{\delta}_n$ and $\boldsymbol{\Delta}_n$ from (20) next conditions hold:*

$$\boldsymbol{\delta}_n^\top\mathbf{c}_n = 0, \quad \boldsymbol{\delta}_n^\top\mathbf{X}_n^\top = \mathbf{0}_k, \tag{28}$$

$$\boldsymbol{\Delta}_n\mathbf{c}_n = \mathbf{0}_k, \quad \boldsymbol{\Delta}_n\mathbf{X}_n^\top = \mathbf{0}_{k\times k}. \tag{29}$$

In addition

$$\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{c}_n = \mathbf{0}_k, \quad \boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{X}_n^\top = \mathbf{0}_{k\times k}, \tag{30}$$

where $\mathbf{0}_{k\times k}$ is zero $k \times k$ matrix.

Proof. Using (8) and Lemma 2 we obtain

$$\boldsymbol{\delta}_n^\top\mathbf{c}_n = \mathbf{b}_n^\top\mathbf{c}_n - \mathbf{b}_n^{(opt)\top}\mathbf{c}_n = 1 - 1 = 0,$$

$$\boldsymbol{\delta}_n^\top\mathbf{X}_n^\top = \mathbf{b}_n^\top\mathbf{X}_n^\top - \mathbf{b}_n^{(opt)\top}\mathbf{X}_n^\top = \mathbf{0}_k.$$

From (7) and Lemma 2 we have

$$\boldsymbol{\Delta}_n\mathbf{X}_n^\top = \mathbf{A}_n\mathbf{X}_n^\top - \mathbf{A}_n^{(opt)}\mathbf{X}_n^\top = \mathbf{I}_k - \mathbf{I}_k = \mathbf{0}_{k\times k},$$

$$\boldsymbol{\Delta}_n\mathbf{c}_n = \mathbf{A}_n\mathbf{c}_n - \mathbf{A}_n^{(opt)}\mathbf{c}_n = \mathbf{0}_k.$$

Equalities in (30) can be easily obtained from (28) and (29).

$$\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{c}_n = \boldsymbol{\Delta}_n\mathbf{c}_n - \boldsymbol{\alpha}\boldsymbol{\delta}_n^\top\mathbf{c}_n = \mathbf{0}_k,$$

$$\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{X}_n^\top = \boldsymbol{\Delta}_n\mathbf{X}_n^\top - \boldsymbol{\alpha}\boldsymbol{\delta}_n^\top\mathbf{X}_n^\top = \mathbf{0}_{k\times k}.$$

□

Lemma 4. *Under assumption (B) the following conditions hold*

$$\boldsymbol{\delta}_n^\top\mathbf{D}_n^2\mathbf{b}_n^{(opt)} = 0, \tag{31}$$

$$\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\mathbf{Q}_n^{(opt)\top} = \mathbf{0}_{k\times k}. \tag{32}$$

Proof. Note that by notations (19)

$$\mathbf{D}_n^2\mathbf{M}_n = \mathbf{I}_n - \mathbf{c}_n\frac{\mathbf{c}_n^\top\mathbf{D}_n^{-2}}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}. \tag{33}$$

By (28), (29) and (30) we obtain the following equalities from (33)

$$\boldsymbol{\delta}_n^\top\mathbf{D}_n^2\mathbf{M}_n\mathbf{X}_n^\top = \boldsymbol{\delta}_n^\top\mathbf{X}_n^\top = \mathbf{0}_k, \quad \boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\mathbf{M}_n\mathbf{X}_n^\top = \boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{X}_n^\top = \mathbf{0}_{k\times k}. \tag{34}$$

Consider the following expression

$$\mathbf{D}_n^2\mathbf{A}_n^{(opt)\top} = \mathbf{D}_n^2\mathbf{M}_n\mathbf{X}_n^\top\Big(\mathbf{X}_n\mathbf{M}_n\mathbf{X}_n^\top\Big)^{-1}. \tag{35}$$

Now (34) and (35) imply

$$\boldsymbol{\delta}_n^\top\mathbf{D}_n^2\mathbf{A}_n^{(opt)\top} = \mathbf{0}_k, \quad \boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\mathbf{A}_n^{(opt)\top} = \mathbf{0}_{k\times k}. \tag{36}$$

Let us consider

$$\mathbf{D}_n^2\mathbf{b}_n^{(opt)} = \frac{\mathbf{c}_n}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n} - \frac{\mathbf{D}_n^2\mathbf{A}_n^{(opt)\top}\mathbf{X}_n\mathbf{D}_n^{-2}\mathbf{c}_n}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n}. \tag{37}$$

Multiply (37) by $\boldsymbol{\delta}_n^\top$ on the left hand-side. From (28), (30) and (36) we obtain

$$\boldsymbol{\delta}_n^\top\mathbf{D}_n^2\mathbf{b}_n^{(opt)} = \frac{\boldsymbol{\delta}_n^\top\mathbf{c}_n}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n} - 0 = 0.$$

So, (31) is proved.

For (32) note that from (28), (30), (36) and (37) we also have the following

$$\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\mathbf{b}_n^{(opt)} = \frac{\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{c}_n}{\mathbf{c}_n^\top\mathbf{D}_n^{-2}\mathbf{c}_n} - \mathbf{0}_k = \mathbf{0}_k. \tag{38}$$

By definition of $\mathbf{Q}_n^{(opt)}$ from (36) and (38) we obtain

$$\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\mathbf{Q}_n^{(opt)\top} = \boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2(\mathbf{A}_n^{(opt)\top} - \mathbf{b}_n^{(opt)}\boldsymbol{\alpha}^\top) = \mathbf{0}_{k\times k}.$$

Thus, Lemma 4 is completely proved. □

2.2. **Proof of Theorem 2.** Show equality in (22). By notation (20) we have the following expression

$$d_n(\mathbf{b}_n) = d_n(\mathbf{b}_n^{(opt)} + \boldsymbol{\delta}_n) = \sigma^2(\mathbf{b}_n^{(opt)} + \boldsymbol{\delta}_n)^\top\mathbf{D}_n^2(\mathbf{b}_n^{(opt)} + \boldsymbol{\delta}_n) =$$

$$= \sigma^2\Big(\mathbf{b}_n^{(opt)\top}\mathbf{D}_n^2\mathbf{b}_n^{(opt)} + \mathbf{b}_n^{(opt)\top}\mathbf{D}_n^2\boldsymbol{\delta}_n + \boldsymbol{\delta}_n^\top\mathbf{D}_n^2\mathbf{b}_n^{(opt)} + \boldsymbol{\delta}_n^\top\mathbf{D}_n^2\boldsymbol{\delta}_n\Big) =$$

$$= d_n(\mathbf{b}_n^{(opt)}) + d_n(\boldsymbol{\delta}_n) + \sigma^2(\boldsymbol{\delta}_n^\top\mathbf{D}_n^2\mathbf{b}_n^{(opt)})^\top + \sigma^2\boldsymbol{\delta}_n^\top\mathbf{D}_n^2\mathbf{b}_n^{(opt)}.$$

Note that equality in (22) holds by (31).

Show equality in (21). By notation (20) we obtain

$$\mathbf{C}_n(\mathbf{Q}_n) = \mathbf{C}_n(\mathbf{Q}_n^{(opt)} + \boldsymbol{\Delta}_{\mathbf{Q},n}) = \sigma^2(\mathbf{Q}_n^{(opt)} + \boldsymbol{\Delta}_{\mathbf{Q},n})\mathbf{D}_n^2(\mathbf{Q}_n^{(opt)} + \boldsymbol{\Delta}_{\mathbf{Q},n})^\top =$$

$$= \sigma^2\mathbf{Q}_n^{(opt)}\mathbf{D}_n^2\mathbf{Q}_n^{(opt)\top} + \sigma^2\mathbf{Q}_n^{(opt)}\mathbf{D}_n^2\boldsymbol{\Delta}^\top{}_{\mathbf{Q},n} + \sigma^2\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\mathbf{Q}_n^{(opt)\top} + \sigma^2\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\boldsymbol{\Delta}_{\mathbf{Q},n}^\top =$$

$$= \mathbf{C}_n(\mathbf{Q}_n^{(opt)}) + \mathbf{C}_n(\boldsymbol{\Delta}_{\mathbf{Q},n}) + \sigma^2(\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\mathbf{Q}_n^{(opt)\top})^\top + \sigma^2\boldsymbol{\Delta}_{\mathbf{Q},n}\mathbf{D}_n^2\mathbf{Q}_n^{(opt)\top}.$$

Equality in (21) follows from (32).

2.3. **Proof of Theorem 1.** Multiply (13) by matrix $\mathbf{C}_n(\mathbf{Q}_n)^{-1/2}$ from the left hand-side we obtain the following representation

$$\mathbf{C}_n(\mathbf{Q}_n)^{-1/2}(\boldsymbol{\alpha}_n^* - \boldsymbol{\alpha}) = \frac{\mathbf{w}_n}{\boldsymbol{\theta}_n^*}. \tag{39}$$

Since $\mathbb{E}\theta_n^* = \theta$ and $\mathbb{D}\theta_n^* = d_n(\mathbf{b}_n)$, we have from (16) and Chebyshev inequality that $\theta_n^* \xrightarrow{\mathbb{P}} \theta$. This fact, (15) and (39) imply (17). So, Theorem 1 is proved.

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