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ON SEMILATTICES AND LATTICES FOR FAMILIES OF THEORIES

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ABSTRACT. We define and study semilattices and lattices for E-closed families of theories. Properties of these semilattices and lattices are investigated. It is shown that lattices for families of theories with the least generating sets are distributive.

Keywords: E-operator, combination of theories, family of theories, least generating set, semilattice, lattice.

We continue to study structural properties of combinations of structures and their theories [3, 4, 5, 6] defining semilattices and lattices for families of theories. Properties of these semilattices and lattices are investigated. It is shown that lattices for families of theories with the least generating sets are distributive.

1. Preliminaries

Throughout the paper we use the following terminology in [3, 4, 6].

Let $P = (P_i)_{i\in I}$, be a family of nonempty unary predicates, $(\mathcal{A}_i)_{i\in I}$ be a family of structures such that P_i is the universe of $\mathcal{A}_i$, $i \in I$, and the symbols P_i are disjoint with languages for the structures $\mathcal{A}_j$, $j \in I$. The structure $\mathcal{A}_P \rightleftharpoons \bigcup\limits_{i\in I} \mathcal{A}_i$ expanded by the predicates P_i is the *P-union* of the structures $\mathcal{A}_i$, and the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_P$ is the *P-operator*. The structure $\mathcal{A}_P$ is called the *P-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ if $\mathcal{A}_i = (\mathcal{A}_P \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$,

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$i \in I$. Structures $\mathcal{A}'$, which are elementary equivalent to $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$, will be also considered as P-combinations.

Clearly, all structures $\mathcal{A}' \equiv \mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ are represented as unions of their restrictions $\mathcal{A}'_i = (\mathcal{A}' \upharpoonright P_i) \upharpoonright \Sigma(\mathcal{A}_i)$ if and only if the set $p_\infty(x) = \{\neg P_i(x) \mid i \in I\}$ is inconsistent. If $\mathcal{A}' \neq \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I}$, we write $\mathcal{A}' = \mathrm{Comb}_P(\mathcal{A}'_i)_{i\in I\cup\{\infty\}}$, where $\mathcal{A}'_\infty = \mathcal{A}' \upharpoonright \bigcap\limits_{i\in I} \overline{P_i}$, maybe applying Morleyzation.

Moreover, we write $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I\cup\{\infty\}}$ for $\mathrm{Comb}_P(\mathcal{A}_i)_{i\in I}$ with the empty structure $\mathcal{A}_\infty$.

Note that if all predicates P_i are disjoint, a structure $\mathcal{A}_P$ is a P-combination and a disjoint union of structures $\mathcal{A}_i$. In this case the P-combination $\mathcal{A}_P$ is called *disjoint*. Clearly, for any disjoint P-combination $\mathcal{A}_P$, $\mathrm{Th}(\mathcal{A}_P) = \mathrm{Th}(\mathcal{A}'_P)$, where $\mathcal{A}'_P$ is obtained from $\mathcal{A}_P$ replacing $\mathcal{A}_i$ by pairwise disjoint $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. Thus, in this case, similar to structures the P-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_P = \mathrm{Th}(\mathcal{A}_P)$, being *$P$-combination* of T_i, which is denoted by $\mathrm{Comb}_P(T_i)_{i\in I}$.

For an equivalence relation E replacing disjoint predicates P_i by E-classes we get the structure $\mathcal{A}_E$ being the *E-union* of the structures $\mathcal{A}_i$. In this case the operator mapping $(\mathcal{A}_i)_{i\in I}$ to $\mathcal{A}_E$ is the *E-operator*. The structure $\mathcal{A}_E$ is also called the *E-combination* of the structures $\mathcal{A}_i$ and denoted by $\mathrm{Comb}_E(\mathcal{A}_i)_{i\in I}$; here $\mathcal{A}_i = (\mathcal{A}_E \upharpoonright A_i) \upharpoonright \Sigma(\mathcal{A}_i)$, $i \in I$. Similar above, structures $\mathcal{A}'$, which are elementary equivalent to $\mathcal{A}_E$, are denoted by $\mathrm{Comb}_E(\mathcal{A}'_j)_{j\in J}$, where $\mathcal{A}'_j$ are restrictions of $\mathcal{A}'$ to its E-classes. The E-operator works for the theories $T_i = \mathrm{Th}(\mathcal{A}_i)$ producing the theory $T_E = \mathrm{Th}(\mathcal{A}_E)$, being *$E$-combination* of T_i, which is denoted by $\mathrm{Comb}_E(T_i)_{i\in I}$ or by $\mathrm{Comb}_E(\mathcal{T})$, where $\mathcal{T} = \{T_i \mid i \in I\}$.

Clearly, $\mathcal{A}' \equiv \mathcal{A}_P$ realizing $p_\infty(x)$ is not elementary embeddable into $\mathcal{A}_P$ and can not be represented as a disjoint P-combination of $\mathcal{A}'_i \equiv \mathcal{A}_i$, $i \in I$. At the same time, there are E-combinations such that all $\mathcal{A}' \equiv \mathcal{A}_E$ can be represented as E-combinations of some $\mathcal{A}'_j \equiv \mathcal{A}_i$. We call this representability of $\mathcal{A}'$ to be the *E-representability*.

If there is $\mathcal{A}' \equiv \mathcal{A}_E$ which is not E-representable, we have the E'-representability replacing E by E' such that E' is obtained from E adding equivalence classes with models for all theories T, where T is a theory of a restriction $\mathcal{B}$ of a structure $\mathcal{A}' \equiv \mathcal{A}_E$ to some E-class and $\mathcal{B}$ is not elementary equivalent to the structures $\mathcal{A}_i$. The resulting structure $\mathcal{A}_{E'}$ (with the E'-representability) is a *e-completion*, or a *e-saturation*, of $\mathcal{A}_E$. The structure $\mathcal{A}_{E'}$ itself is called *e-complete*, or *e-saturated*, or *e-universal*, or *e-largest*.

For a structure $\mathcal{A}_E$ the number of *new* structures with respect to the structures $\mathcal{A}_i$, i. e., of the structures $\mathcal{B}$ which are pairwise elementary non-equivalent and elementary non-equivalent to the structures $\mathcal{A}_i$, is called the *e-spectrum* of $\mathcal{A}_E$ and denoted by e-$\mathrm{Sp}(\mathcal{A}_E)$. The value $\sup\{e\text{-}\mathrm{Sp}(\mathcal{A}') \mid \mathcal{A}' \equiv \mathcal{A}_E\}$ is called the *e-spectrum* of the theory $\mathrm{Th}(\mathcal{A}_E)$ and denoted by e-$\mathrm{Sp}(\mathrm{Th}(\mathcal{A}_E))$.

If $\mathcal{A}_E$ does not have E-classes $\mathcal{A}_i$, which can be removed, with all E-classes $\mathcal{A}_j \equiv \mathcal{A}_i$, preserving the theory $\mathrm{Th}(\mathcal{A}_E)$, then $\mathcal{A}_E$ is called *e-prime*, or *e-minimal*.

For a structure $\mathcal{A}' \equiv \mathcal{A}_E$ we denote by $\mathrm{TH}(\mathcal{A}')$ the set of all theories $\mathrm{Th}(\mathcal{A}_i)$ of E-classes $\mathcal{A}_i$ in $\mathcal{A}'$.

By the definition, an e-minimal structure $\mathcal{A}'$ consists of E-classes with a minimal set $\mathrm{TH}(\mathcal{A}')$. If $\mathrm{TH}(\mathcal{A}')$ is the least for models of $\mathrm{Th}(\mathcal{A}')$ then $\mathcal{A}'$ is called *e-least*.

Definition [4]. Let $\overline{\mathcal{T}}$ be the class of all complete elementary theories of relational languages. For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ we denote by $\mathrm{Cl}_E(\mathcal{T})$ the set of all theories $\mathrm{Th}(\mathcal{A})$, where $\mathcal{A}$ is a structure of some E-class in $\mathcal{A}' \equiv \mathcal{A}_E$, $\mathcal{A}_E = \mathrm{Comb}_E(\mathcal{A}_i)_{i \in I}$, $\mathrm{Th}(\mathcal{A}_i) \in \mathcal{T}$. As usual, if $\mathcal{T} = \mathrm{Cl}_E(\mathcal{T})$ then $\mathcal{T}$ is said to be *E-closed*.

The operator Cl_E of E-closure can be naturally extended to the classes $\mathcal{T} \subset \overline{\mathcal{T}}$ as follows: $\mathrm{Cl}_E(\mathcal{T})$ is the union of all $\mathrm{Cl}_E(\mathcal{T}_0)$ for subsets $\mathcal{T}_0 \subseteq \mathcal{T}$.

For a set $\mathcal{T} \subset \overline{\mathcal{T}}$ of theories in a language Σ and for a sentence φ with $\Sigma(\varphi) \subseteq \Sigma$ we denote by $\mathcal{T}_\varphi$ the set $\{T \in \mathcal{T} \mid \varphi \in T\}$.

Proposition 1.1 [4]. *If $\mathcal{T} \subset \overline{\mathcal{T}}$ is an infinite set and $T \in \overline{\mathcal{T}} \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an accumulation point for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any formula $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.*

Theorem 1.2 [4]. *For any sets $\mathcal{T}_0, \mathcal{T}_1 \subset \overline{\mathcal{T}}$, $\mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1) = \mathrm{Cl}_E(\mathcal{T}_0) \cup \mathrm{Cl}_E(\mathcal{T}_1)$.*

Definition [4]. Let $\mathcal{T}_0$ be a closed set in a topological space $(\mathcal{T}, \mathcal{O}_E(\mathcal{T}))$, where $\mathcal{O}_E(\mathcal{T}) = \{\mathcal{T} \setminus \mathrm{Cl}_E(\mathcal{T}') \mid \mathcal{T}' \subseteq \mathcal{T}\}$. A subset $\mathcal{T}'_0 \subseteq \mathcal{T}_0$ is said to be *generating* if $\mathcal{T}_0 = \mathrm{Cl}_E(\mathcal{T}'_0)$. The generating set $\mathcal{T}'_0$ (for $\mathcal{T}_0$) is *minimal* if $\mathcal{T}'_0$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}'_0$ is *least* if $\mathcal{T}'_0$ is contained in each generating set for $\mathcal{T}_0$.

Theorem 1.3 [4]. *If $\mathcal{T}'_0$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:*

(1) *$\mathcal{T}'_0$ is the least generating set for $\mathcal{T}_0$;*

(2) *$\mathcal{T}'_0$ is a minimal generating set for $\mathcal{T}_0$;*

(3) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}'_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}'_0)_\varphi = \{T\}$;*

(4) *any theory in $\mathcal{T}'_0$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}'_0$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.*

Definition [6]. Two theories T_1 and T_2 of a language Σ are *disjoint* modulo Σ_0, where $\Sigma_0 \subseteq \Sigma$, or Σ_0-*disjoint* if T_1 and T_2 are do not have common nonempty predicates for $\Sigma \setminus \Sigma_0$. If T_1 and T_2 are $\varnothing$-disjoint, these theories are called simply *disjoint*.

2. Semilattices and lattices for families of theories

Definition. Let X be a nonempty set of E-closed families $\mathcal{T} \subset \overline{\mathcal{T}}$. Operations $\mathcal{T}_1 \wedge \mathcal{T}_2 \rightleftharpoons \mathcal{T}_1 \cap \mathcal{T}_2$ and $\mathcal{T}_1 \vee \mathcal{T}_2 \rightleftharpoons \mathrm{Cl}_E(\mathcal{T}_1 \cup \mathcal{T}_2)$, for E-closed $\mathcal{T}_1, \mathcal{T}_2 \subset \overline{\mathcal{T}}$, generate a set Y and form the structure $\langle Y; \wedge, \vee \rangle$ denoted by $L(X)$.

It is well known [1] that any $L(X)$ is a lattice extensible to a complete lattice $\mathrm{CL}(X)$ with

$$\bigwedge_{j \in J} \mathrm{Cl}_E(\mathcal{T}_j) = \bigcap_{j \in J} \mathrm{Cl}_E(\mathcal{T}_j)$$

and

$$\bigvee_{j \in J} \mathrm{Cl}_E(\mathcal{T}_j) = \mathrm{Cl}_E\left(\bigcup_{j \in J} \mathcal{T}_j\right).$$

By Theorem 1.2, for E-closed $\mathcal{T}_1, \mathcal{T}_2$, $\mathrm{Cl}_E(\mathcal{T}_0 \cup \mathcal{T}_1) = \mathcal{T}_0 \cup \mathcal{T}_1$, i.e., the operation $\vee$ is the set-theoretic union. At the same time, in general case, for E-closed $\mathcal{T}_j$, $\bigvee\limits_{j \in J} \mathcal{T}_j \neq \bigcup\limits_{j \in J} \mathcal{T}_j$, since, for instance, the union of infinite set of singletons can generate new theories. Thus, $L(X)$ is just a standard algebra with usual set-theoretic unions and intersections (but can be without even relative complements since these complements can be not E-closed), whereas $\mathrm{CL}(X)$ is its natural extension.

Now we consider restrictions of $L(X)$ in the following way.

For a nonempty set X of E-closed families with the least generating sets, the operation $\vee$ generates a set $Z \subseteq Y$ and forms a upper semilattice $\mathrm{SLLGS}(X) = \langle Z; \vee \rangle$ restricting the universe and the language of $L(X)$.

Below we will show that $\mathrm{SLLGS}(X)$ always consists of families with the least generating sets whereas the operation $\wedge$ can generate a family without the least generating sets.

Proposition 2.1. *If E-closed sets $\mathcal{T}_1$ and $\mathcal{T}_2$, in a language Σ, have the least generating sets, then $\mathcal{T}_1 \cup \mathcal{T}_2$, being E-closed, has the least generating set.*

Proof. Let $\mathcal{T}'_1$ and $\mathcal{T}'_2$ be the least generating sets for $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively, and $\mathcal{T}'_0$ be a subset of $\mathcal{T}'_1 \cup \mathcal{T}'_2$ consisting of all isolated points with respect to $\mathcal{T}'_1 \cup \mathcal{T}'_2$, i.e., of elements $T \in \mathcal{T}'_1 \cup \mathcal{T}'_2$ with formulas $\varphi \in T$ such that $(\mathcal{T}'_1 \cup \mathcal{T}'_2)_\varphi$ is a singleton.

Now we assume on contrary that $\mathcal{T}_1 \cup \mathcal{T}_2$ does not have the least generating set. Then there is a theory $T_1 \in \mathcal{T}'_1 \cup \mathcal{T}'_2$ such that $T_1 \notin \mathrm{Cl}_E(\mathcal{T}'_0)$. Without loss of generality we assume that $T_1 \in \mathcal{T}'_1$. Since T_1 is isolated with respect to $\mathcal{T}'_1$ and not isolated with respect to $\mathcal{T}'_1 \cup \mathcal{T}'_2$ there is a formula $\varphi \in T_1$ such that for any $\psi \in T_1$ forcing φ, $(\mathcal{T}'_1)_\psi = \{T_1\}$ and $(\mathcal{T}'_1 \cup \mathcal{T}'_2)_\psi$ is infinite. Since $T_1 \notin \mathrm{Cl}_E(\mathcal{T}'_0)$, there are infinitely many theories $T \in (\mathcal{T}'_2)_\psi$ which are not isolated with respect to $\mathcal{T}'_1 \cup \mathcal{T}'_2$. It implies that for any formula $\chi \in T$ forcing ψ there are infinitely many theories in $(\mathcal{T}'_1)_\chi$. But since $\chi \vdash \psi$, $(\mathcal{T}'_1)_\psi$ is infinite contradicting $|(\mathcal{T}'_1)_\psi| = 1$. □

Remark 2.2. Arguments for [6, Proposition 3.9] show that the converse for Proposition 2.1 is not true, since there is $\mathcal{T}_1 \cup \mathcal{T}_2$ with the least generating set such that $\mathcal{T}_1$ has the least generating set (for $\mathcal{F}_q$ in terms of [5]) and $\mathcal{T}_2$ does not have the least generating set (for $\{J_q \mid q \in \mathbb{Q}\}$ in terms of [5]).

If we denote by Σ_0 the set of nonempty predicates for $\{J_q \mid q \in \mathbb{Q}\}$ and take a Σ_0-disjoint copy $\mathcal{F}'_q$ for $\mathcal{F}_q$, which also generates $\{J_q \mid q \in \mathbb{Q}\}$ with $J_q = \underline{\lim}\, F_q = \underline{\lim}\, F'_q$, we get families $\mathcal{T}$ and $\mathcal{T}'$ for $\{J_q \mid q \in \mathbb{Q}\} \cup \mathcal{F}_q$ and $\{J_q \mid q \in \mathbb{Q}\} \cup \mathcal{F}'_q$ respectively such that $\mathcal{T} \cap \mathcal{T}'$ is a family of theories for $\{J_q \mid q \in \mathbb{Q}\}$, which does not have the least generating set.

Remark 2.3. The infinite semilattices $\mathrm{SLLGS}(X)$ can be both complete and incomplete, and in the incomplete case $\mathrm{SLLGS}(X)$ can not be extended to a complete semilattice consisting of families with the least generating sets.

Indeed, taking infinitely many Σ_0-disjoint copies $\mathcal{F}^\mu_q$ of $\mathcal{F}_q$ [5], $\mu < \lambda$, and forming the set X by E-closed families of theories for $\{J_q \mid q \in \mathbb{Q}\} \cup \mathcal{F}^\mu_q$ [5] we can freely unite elements of X obtaining E-closed families with the least generating sets corresponding to unions of $\mathcal{F}^\mu_q$.

At the same time, each singleton $\{T\}$, for $T \in \overline{\mathcal{T}}$ is E-closed and with the least generating set. Then taking a set X of singletons we generate the semilattice $\mathrm{SLLGS}(X)$ (which is in fact a distributive lattice with related complements) consisting of all finite subsets of $\cup X$. As there are E-closed families $\mathcal{T}$ without the least generating sets, taking a (infinite) union of singletons $\{T\}$ for $T \in \mathcal{T}$ we form the family $\mathcal{T}$. Thus, infinite unions of families with the least generating sets can be without the least generating sets, and in this case SLLGS(X) can not be extended to a complete semilattice consisting of families with the least generating sets.

Summarizing Proposition 2.1 and Remarks 2.2, 2.3 we have

Theorem 2.3. 1. *For any nonempty set X of E-closed families with the least generating sets the structure* SLLGS(X) *is a upper semilattice.*

2. *There is a upper semilattice* SLLGS(X) *with elements $x_1, x_2 \in X$ having the least generating sets and such that $x_1 \cap x_2$ does not have the least generating set.*

3. *There is a upper semilattice* SLLGS(X) *which can not be extended to a complete semilattice consisting of families with the least generating sets.*

Now we take a nonempty set X of E-closed families with the least generating sets and $\mathcal{T}_1, \mathcal{T}_2 \in X$ with the least generating sets $\mathcal{T}'_1$ and $\mathcal{T}'_2$ respectively. We denote by $\mathcal{T}_1 \wedge' \mathcal{T}_2$ the family $\mathcal{T}_0 \subseteq \mathcal{T}_1 \cap \mathcal{T}_2$ with the greatest generating set $\mathcal{T}'_0$ consisting of all isolated points for $\mathcal{T}_1 \cap \mathcal{T}_2$.

Remark 2.4. By the definition, $\mathcal{T}'_0 \supseteq (\mathcal{T}'_1 \cap \mathcal{T}_2) \cup (\mathcal{T}_1 \cap \mathcal{T}'_2)$. In particular, $\mathcal{T}'_0 \supseteq \mathcal{T}'_1 \cap \mathcal{T}'_2$. These inclusions can be strict.

Indeed, take Σ_0-disjoint families $\mathcal{T}_1$ and $\mathcal{T}_2$ of theories for $F_q \cup \{J_q\}$ and $F'_q \cup \{J_q\}$ [5], where $J_q = \underline{\lim}\, F_q = \underline{\lim}\, F'_q$ and Σ_0 is the set of predicate symbols interpreted by nonempty relations for J_q. For the theory T_0 corresponding to J_q, we have $\{T_0\} = \mathcal{T}_1 \wedge' \mathcal{T}_2$, whereas $\mathcal{T}'_1 \cap \mathcal{T}'_2 = \varnothing$.

For a set X the operations $\wedge'$ and $\vee$ generate the set $U \supseteq X$ with a structure LLGS(X) $\rightleftharpoons \langle U; \wedge', \vee\rangle$.

Directly checking we have

Proposition 2.5. *Any structure* LLGS(X) *is a lattice.*

By the definition for every $\mathcal{T}_1, \mathcal{T}_2 \in U$ with the least generating sets $\mathcal{T}'_1$ and $\mathcal{T}'_2$ respectively, we have, in LLGS(X), that $\mathcal{T}_1 \leq \mathcal{T}_2$ if and only if $\mathcal{T}'_2$ consists of three disjoint parts $\mathcal{T}'_{2,1}$, $\mathcal{T}'_{2,2}$, $\mathcal{T}'_{2,3}$ such that:

1) $\mathcal{T}'_{2,1} \subseteq \mathcal{T}'_1$,
2) $(\mathcal{T}'_{2,2} \cup \mathcal{T}'_{2,3}) \cap \mathcal{T}'_1 = \varnothing$,
3) $\mathcal{T}'_{2,2}$ is used for generations of elements in $\mathcal{T}'_1 \setminus \mathcal{T}'_2$;
4) $\mathcal{T}'_{2,3}$ is not used for generations of elements in $\mathcal{T}'_1 \setminus \mathcal{T}'_2$.

The following proposition is obvious.

Proposition 2.6. *If $\mathcal{T}'_2 \setminus \mathcal{T}'_1$ is finite then $\mathcal{T}'_2 = \mathcal{T}'_1 \cup \mathcal{T}'_{2,3}$ and, moreover, $\mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{T}'_{2,3}$.*

Remark 2.7. The set $\mathcal{T}'_{2,3}$ can have an arbitrary cardinality whereas for each element in $\mathcal{T}'_1 \setminus \mathcal{T}'_2$ being isolated by some $(\mathcal{T}'_1)_\varphi$, the neighbourhood $(\mathcal{T}'_{2,2})_\varphi$ should contain infinitely many elements and the cardinality of $\mathcal{T}'_{2,2}$ is at least $\lambda \cdot \omega$, where λ is the number of disjoint $\mathcal{T}'_{2,2}$-neighbourhoods for the elements in $\mathcal{T}'_1 \setminus \mathcal{T}'_2$.

Theorem 2.8. *Any lattice* LLGS(X) *is distributive.*

Proof. We have to show two identities:

$$\mathcal{T}_1 \wedge' (\mathcal{T}_2 \cup \mathcal{T}_3) = (\mathcal{T}_1 \wedge' \mathcal{T}_2) \cup (\mathcal{T}_1 \wedge' \mathcal{T}_3), \tag{1}$$

$$\mathcal{T}_1 \cup (\mathcal{T}_2 \wedge' \mathcal{T}_3) = (\mathcal{T}_1 \cup \mathcal{T}_2) \wedge' (\mathcal{T}_1 \cup \mathcal{T}_3) \tag{2}$$

for any $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in U$ with the least generating sets $\mathcal{T}'_1, \mathcal{T}'_2, \mathcal{T}'_3$ respectively.

For the proof of (1) we note that the least generating set for $\mathcal{T}_1\wedge'(\mathcal{T}_2\cup\mathcal{T}_3)$ consists of isolated points T belonging both to $\mathcal{T}_1$ and to $\mathcal{T}_2\cup\mathcal{T}_3$. But then T belongs to $\mathcal{T}_2$ or to $\mathcal{T}_3$. In the first case, T is an isolated point for $\mathcal{T}_1\wedge'\mathcal{T}_2$ and, in the second case, — an isolated point for $\mathcal{T}_1\wedge'\mathcal{T}_3$. Therefore, $T\in(\mathcal{T}_1\wedge'\mathcal{T}_2)\cup(\mathcal{T}_1\wedge'\mathcal{T}_3)$ and thus $\mathcal{T}_1\wedge'(\mathcal{T}_2\cup\mathcal{T}_3)\subseteq(\mathcal{T}_1\wedge'\mathcal{T}_2)\cup(\mathcal{T}_1\wedge'\mathcal{T}_3)$.

Conversely, if an isolated point T belongs to $(\mathcal{T}_1\wedge'\mathcal{T}_2)\cup(\mathcal{T}_1\wedge'\mathcal{T}_3)$ then T belongs to $\mathcal{T}_1\wedge'\mathcal{T}_2$ or to $\mathcal{T}_1\wedge'\mathcal{T}_3$. If $T\in\mathcal{T}_1\wedge'\mathcal{T}_2$ then either T is an isolated point for $\mathcal{T}_1\wedge'\mathcal{T}_2$ or belong to the E-closure of isolated points in $\mathcal{T}_1\wedge'\mathcal{T}_2$. Anyway, $T\in\mathcal{T}_1\wedge'(\mathcal{T}_2\cup\mathcal{T}_3)$. Similarly we obtain $T\in\mathcal{T}_1\wedge'(\mathcal{T}_2\cup\mathcal{T}_3)$ for any isolated $T\in\mathcal{T}_1\wedge'\mathcal{T}_2$. Thus, $(\mathcal{T}_1\wedge'\mathcal{T}_2)\cup(\mathcal{T}_1\wedge'\mathcal{T}_3)\subseteq\mathcal{T}_1\wedge'(\mathcal{T}_2\cup\mathcal{T}_3)$ and the identity (1) holds.

For the proof of (2) we note that the least generating set for $\mathcal{T}_1\cup(\mathcal{T}_2\wedge'\mathcal{T}_3)$ consists of isolated points T belonging to $\mathcal{T}_1$ and being an isolated point for $\mathcal{T}_1$, or belonging to $\mathcal{T}_2\wedge'\mathcal{T}_3$, being an isolated point for $\mathcal{T}_2\wedge'\mathcal{T}_3$, and then belonging to $\mathcal{T}_2$ and to $\mathcal{T}_3$. If $T\in\mathcal{T}_1$ then $T\in\mathcal{T}_1\cup\mathcal{T}_2$ and $T\in\mathcal{T}_1\cup\mathcal{T}_3$, whence $T\in(\mathcal{T}_1\cup\mathcal{T}_2)\wedge'(\mathcal{T}_1\cup\mathcal{T}_3)$. If $T\in\mathcal{T}_2\wedge'\mathcal{T}_3$ then again $T\in\mathcal{T}_1\cup\mathcal{T}_2$ and $T\in\mathcal{T}_1\cup\mathcal{T}_3$ implying $T\in(\mathcal{T}_1\cup\mathcal{T}_2)\wedge'(\mathcal{T}_1\cup\mathcal{T}_3)$.

Conversely, if an isolated point T belongs to $(\mathcal{T}_1\cup\mathcal{T}_2)\wedge'(\mathcal{T}_1\cup\mathcal{T}_3)$ then T belongs to $\mathcal{T}_1\cup\mathcal{T}_2$ and to $\mathcal{T}_1\cup\mathcal{T}_3$. So $T\in\mathcal{T}_1$ or $T\in\mathcal{T}_2$, and $T\in\mathcal{T}_1$ or $T\in\mathcal{T}_2$. If $T\in\mathcal{T}_1$ then $T\in\mathcal{T}_1\cup(\mathcal{T}_2\wedge'\mathcal{T}_3)$. Otherwise, $T\in\mathcal{T}_2\wedge'\mathcal{T}_3$ and so again $T\in\mathcal{T}_1\cup(\mathcal{T}_2\wedge'\mathcal{T}_3)$.

Thus, the identity (2) holds. □

Remark 2.9. For every E-closed family $\mathcal{T}$ with the least generating set $\mathcal{T}'$ there is a superatomic Boolean algebra $\mathcal{B}(\mathcal{T})$ [2] consisting of all subsets of $\mathcal{T}$ generated by arbitrary subsets of $\mathcal{T}'$. If $\mathcal{T}_1\leq\mathcal{T}_2$ in $\mathcal{B}(\mathcal{T})$ and $\mathcal{T}_1,\mathcal{T}_2$ have the least generating sets $\mathcal{T}'_1$ and $\mathcal{T}'_2$, respectively, then $\mathcal{T}'_1\subseteq\mathcal{T}'_2$ and vice versa. Thus, $\mathcal{B}(\mathcal{T})$ is isomorphic to the Boolean algebra $\mathcal{B}(\mathcal{T}')$ with the natural relation $\subseteq$ and consisting of all subsets of $\mathcal{T}'$.

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