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# ON THE UNIQUE DETERMINATION OF DOMAINS BY THE CONDITION OF THE LOCAL ISOMETRY OF THE BOUNDARIES IN THE RELATIVE METRICS. II

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Abstract. We prove the theorem on the unique determination of a strictly convex domain in $\mathbb{R}^n$, where $n \geq 2$, in the class of all $n$-dimensional domains by the condition of the local isometry of the Hausdorff boundaries in the relative metrics, which is a generalization of A. D. Aleksandrov's theorem on the unique determination of a strictly convex domain by the condition of the (global) isometry of the boundaries in the relative metrics.

We also prove that, in the cases of a plane domain $U$ with nonsmooth boundary and of a three-dimensional domain $A$ with smooth boundary, the convexity of the domain is no longer necessary for its unique determination by the condition of the local isometry of the boundaries in the relative metrics.

**Keywords:** intrinsic metric, relative metric of the boundary, local isometry of the boundaries, strict convexity.

## 1. Introduction

Let $\mathcal{U}$ be a class of domains (i.e., open connected sets) in the real Euclidean $n$-dimensional space $\mathbb{R}^n$, where $n \geq 2$. We say (see, e.g., [1]) that a domain $U \in \mathcal{U}$ *is uniquely determined in the class* $\mathcal{U}$ *by the relative metric of its* (*Hausdorff*) *boundary* if each domain $V \in \mathcal{U}$ whose Hausdorff boundary is isometric to the Hausdorff boundary of $U$ in the relative metrics is itself isometric to $U$ (in the Euclidean metric).

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**Remark 1.1.** Let $U$ be a domain in $\mathbb{R}^n$ $(n \geq 2)$ and let $\rho_U$ be its intrinsic metric. Consider the Hausdorff completion of the metric space $(U, \rho_U)$, i.e., the completion of this space in the intrinsic metric $\rho_U$. Identifying the points of this completion that correspond to the points of $U$ with these points themselves and removing them from the completion, we obtain a metric space $(\mathrm{fr}_H U, \rho_{\mathrm{fr}_H U,U})$; the set $\mathrm{fr}_H U$ of its elements is called the *Hausdorff boundary* of the domain $U$, and $\rho_{\mathrm{fr}_H U,U}$ is the relative metric on this Hausdorff boundary. The isometry of the Hausdorff boundaries of domains $U$ and $V$ with respect to their relative metrics means the existence of a surjective isometry $f : (\mathrm{fr}_H U, \rho_{\mathrm{fr}_H U,U}) \to (\mathrm{fr}_H V, \rho_{\mathrm{fr}_H V,V})$ between these boundaries.

The results of [2], [3], [4] imply in particular that any bounded domain in $\mathbb{R}^n$ is uniquely determined by the condition of the isometry of the boundaries in the relative metrics. At the same time, in accoirdance with the results of [5], a bounded polygonal plane domain $U$ is uniquely determined by the condition of the local isometry of the boundaries in the relative metrics in the class of all such domains if and only if $U$ is convex.

**Remark 1.2.** Let $\mathcal{M}$ be a class of domains in $\mathbb{R}^n$ with $n \geq 2$. Following [1], we say that a domain $U \in \mathcal{M}$ is uniquely determined in the class $\mathcal{M}$ by the condition of the local isometry of the (Hausdorff) boundaries of domains in the relative metrics if, for any domain $V$ in $\mathcal{M}$, the local isometry of its Hausdorff boundary to the Hausdorff boundary of $U$ in the relative metrics implies the isometry of $U$ and $V$ (in the Euclidean metric). The local isometry in the relative metrics between the Hausdorff boundaries $\mathrm{fr}_H U$ and $\mathrm{fr}_H V$ of domains $U$ and $V$ means the existence of a bijective mapping $f : \mathrm{fr}_H U \to \mathrm{fr}_H V$ of these boundaries that is a local isometry in their relative metrics, i.e., a mapping such that, for every $y \in \mathrm{fr}_H U$, there exists $\varepsilon > 0$ satisfying the following condition: The equality $\rho_{\mathrm{fr}_H U,U}(a,b) = \rho_{\mathrm{fr}_H V,V}(f(a), f(b))$ holds for any two elements $a$ and $b$ in the $\varepsilon$-neighborhood $Z(y) = \{z \in \mathrm{fr}_H U : \rho_{\mathrm{fr}_H U,U}(z,y) < \varepsilon\}$ of $y$. It is clear that $f^{-1}$ is also a local isometry in the relative metrics of the boundaries.

**Remark 1.3.** Let $U$ be a domain in $\mathbb{R}^n$. As in [1], we say that $U$ has *smooth boundary* (respectively, *Lipschitz boundary*) if the Euclidean boundary $\mathrm{fr}\, U$ of this domain is an $(n-1)$-submanifold of class $C^1$ (a Lipschitz submanifold) without boundary in $\mathbb{R}^n$. In the case of a domain $U$ with Lipschitz boundary, its Hausdorff boundary $\mathrm{fr}_H U$ is naturally identified with the Euclidean boundary and the metric $\rho_{\mathrm{fr}\, U,U}$ corresponding to the Hausdorff metric can be defined as follows:

$$\rho_{\mathrm{fr}\, U,U}(x,y) = \liminf_{x' \to x, y' \to y;\, x', y' \in U} \{\inf[l(\gamma_{x',y',U})]\},$$

where $x, y \in \mathrm{fr}\, U$ and $\inf[l(\gamma_{x',y',U})]$ is the infimum of the lengths $l(\gamma_{x',y',U})$ of smooth paths $\gamma_{x',y',U} : [0,1] \to U$ joining $x'$ and $y'$ in $U$. Recall also that a domain $U$ is said to be *strictly convex* if it is convex and the interior of the interval joining any two points in its closure $\mathrm{cl}\, U$ is contained in $U$.

**Lemma 1.1.** *Let $U$ and $V$ be two plane domains with smooth boundaries and let $f : \mathrm{fr}\, U \to \mathrm{fr}\, V$ be a bijective mapping that is a local isometry of the boundaries of these domains in the relative metrics. Then $f$ is a* (*global*) *isometry of the boundaries* $\mathrm{fr}\, U$ *and* $\mathrm{fr}\, V$ *in their intrinsic metrics.*

**Lemma 1.2.** *Suppose that domains $U$ and $V$ and a mapping $f : \mathrm{fr}\, U \to \mathrm{fr}\, V$ satisfy to the hypothesis of Lemma* 1.1 *and, moreover,* $\mathrm{fr}\, U$ *is bounded. Then* $\mathrm{fr}\, V$

*is also bounded and $f$ has the following property: There exists $\varepsilon > 0$ such that $\rho_{\mathrm{fr}\,U,U}(a,b) = \rho_{\mathrm{fr}\,V,V}(f(a),f(b))$ if $a,b \in \mathrm{fr}\,U$ and $\rho_{\mathrm{fr}\,U,U}(a,b) < \varepsilon$.*

**Lemma 1.3.** *Under the hypothesis of Lemma 1.1 and the additional assumption that $\mathrm{fr}\,U$ is connected, the $\mathrm{fr}\,V$ is also connected.*

In this paper, we continue the study of the unique determination of domains by the condition of the local isometry of their boundaries in the relative metrics.

In the paper, we obtain some new assertions on the unique determination of space domains with smooth boundaries by the condition considered in the article. All these results emphasize the specific nature of our approach to the rigidity problems of domains in $\mathbb{R}^n$.

Note that below $[a,b] = \{bt + (1-t)a \in \mathbb{R}^n : 0 \le t \le 1\}$, $[a,b[= \{bt + (1-t)a \in \mathbb{R}^n : 0 \le t < 1\}$ ($]a,b] = \{bt + (1-t)a \in \mathbb{R}^n : 0 < t \le 1\}$) and $]a,b[= \{bt + (1-t)a \in \mathbb{R}^n : 0 < t < 1\}$ are the segment (closed interval), the half-open interval, and the interval in $\mathbb{R}^n$ with endpoints $a,b \in \mathbb{R}^n$, $a \neq b$; $\mathrm{Int}\,I$ is the interior of the segment (of the half-open interval) $I$, $\mathrm{Int}]a,b[=]a,b[$; $B(x_0,r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ is the open ball in $\mathbb{R}^n$ of radius $r$ ($0 < r < \infty$) centered at $x_0 \in \mathbb{R}^n$; $\mathrm{Id}_E$ is the identity mapping of a set $E$: $\mathrm{Id}_E(x) = x$ for $x \in E$.

In what follows, paths $\gamma : [\alpha,\beta] \to \mathbb{R}^n$, where $\alpha,\beta \in \mathbb{R}$, are assumed continuous.

## 2. Unique Determination of Space Domains

Consider the case of space domains. But first recall some notions and facts from [1] which we will use below.

**Definition 2.1.** The support of an element $a$ of the Hausdorff boundary $\mathrm{fr}_H\,U$ of a domain $U \subset \mathbb{R}^n$ ($n \ge 2$) is a point $a' = a'_a$ of the Euclidean boundary $\mathrm{fr}\,U$ to which a Cauchy sequence $\{x_j\}_{j\in\mathbb{N}}$ of points $x_j \in U$ representing $a$ converges in the intrinsic metric $\rho_U$ of $U$.

**Lemma 2.1.** *Every element $a \in \mathrm{fr}_H\,U$ has a unique support $a'_a$.*

**Lemma 2.2.** *The set of the supports $a'_a$ of elements $a \in \mathrm{fr}_H\,U$ is everywhere dense (in the Euclidean metric) on the Euclidean boundary $\mathrm{fr}\,U$ of $U$.*

The mapping $p_U : \mathrm{fr}_H\,U \to \mathrm{fr}\,U$ denotes the transformation of points of the Hausdorff boundary $\mathrm{fr}_H\,U$ assigning to each element $a \in \mathrm{fr}_H\,U$ its support $a' = a'_a$.

Below we will use the following assertion, which is a generalization of Lemma 3.1 in [3] to the case of local isometries of the boundaries of domains.

**Lemma 2.3.** *Let $U,V$ be domains in $\mathbb{R}^n$ ($n \ge 2$) such that there exists a bijective mapping $f : \mathrm{fr}_H\,U \to \mathrm{fr}_H\,V$ that is a local isometry in the relative metrics of the Hausdorff boundaries of $U$ and $V$. Then for every $w \in \mathrm{fr}_H\,U$, there exists a number $\varepsilon = \varepsilon_w > 0$ satisfying the following condition: If $a',b'$ are any two elements in $\mathrm{fr}\,U$ with $]a',b'[\subset U$ and the elements $a,b \in \mathrm{fr}_H\,U$ generated by the path $\gamma(t) = tb' + (1-t)a'$, $t \in [0,1]$ (i.e., generated respectively by the Cauchy sequences $\{\gamma(1/n)\}_{n=3,4,\dots}$ and $\{\gamma(1-1/n)\}_{n=3,4,\dots}$ in the intrinsic metric $\rho_U$ of $U$) belong to the $\varepsilon$-neighborhood $Z(w) = \{z \in \mathrm{fr}_H\,U : \rho_{\mathrm{fr}_H\,U,U}(z,w) < \varepsilon\}$ of $w$ then $]p_V f(a), p_V f(b)[\subset V$.*

The proof of Lemma 2.3 differs from the proof of Lemma 3.1 in [3] by negligible modifications and is therefore omitted.

Suppose now that the domain under consideration is strictly convex. Then we have

**Theorem 2.1.** *Let $n \geq 2$. If a domain $U \subset \mathbb{R}^n$ is strictly convex then it is uniquely determined in the class of all domains in $\mathbb{R}^n$ by the condition of the local isometry of the boundaries in the relative metrics.*

*Proof.* Let $V$ be a domain such that there exists a bijective mapping $f : \mathrm{fr}_H U \to \mathrm{fr}_H V$ that is a local isometry in the relative metrics of the Hausdorff boundaries $\mathrm{fr}_H U$ and $\mathrm{fr}_H V$ of $U$ and $V$. Assume that $x$ and $y$ are points of the Euclidean boundary $\mathrm{fr}\, U$ of $U$ (by the strict convexity of $U$ and Remark 1.3, we may assume that $x$ and $y$ both belong to $\mathrm{fr}_H U$). By Lemma 2.3, each element $w \in \mathrm{fr}_H U$ has an $\varepsilon_w$-neighborhood $Z(w) = \{z \in \mathrm{fr}_H U : \rho_{\mathrm{fr}_H U, U}(z, w) < \varepsilon\}$ with the property: If $a, b \in Z(w)$ are arbitrary points then $]p_V f(a), p_V f(b)[ \subset V$ (see Lemma 2.3 concerning $Z(w)$). This implies that the mapping $\bar{f} : \mathrm{fr}\, U \to \mathrm{fr}\, V$ such that $\bar{f}(x) = p_V f(x)$ for $x \in \mathrm{fr}\, U$ is a locally isometry in the Euclidean metric (i.e., if $w \in \mathrm{fr}\, U$ then for every $z \in Z(w)$ there exist a ball $B_x = B(x, r_x) \subset \mathbb{R}^n$ and an isometry $F_x : \mathbb{R}^n \to \mathbb{R}^n$ in the Euclidean metric such that $F_x|_{(\mathrm{fr}\, U) \cup B_x} = \bar{f}|_{(\mathrm{fr}\, U) \cup B_x}$.

Let $\bar{f}(\mathrm{fr}\, U) = T \subset \mathrm{fr}\, V$. We assert that the closure $\mathrm{cl}\, T$ of the set $T$ coincides with the Euclidean boundary $\mathrm{fr}\, V$ of $V$. Assuming that $M = ((\mathrm{fr}\, V) \setminus \mathrm{cl}\, T) \neq \varnothing$, consider a point $z \in M$. Since $\mathrm{cl}\, T$ is closed, $\mathrm{dist}\{z, T\} = \mathrm{dist}\{z, \mathrm{cl}\, T\} > 0$. Taking also into account that, by Lemma 2.2, the set of the supports of the Hausdorff boundary of a domain is dense on its Euclidean boundary, we assert that there exists $a \in \mathrm{fr}_H V$ whose support $a' = p_V a$ satisfies the condition $\mathrm{dist}\{a', T\} = \mathrm{dist}\{a', \mathrm{cl}\, T\} > 0$. Put $\widetilde{a} = f^{-1}(a)$. We have $\bar{f}(\widetilde{a}) = p_V f(\widetilde{a}) = p_V(f(f^{-1}(a) = p_V a = a' \in T$. Therefore, $\mathrm{cl}\, T = \mathrm{fr}\, V$.

Further, show that $\bar{f}$ can be extended to a Euclidean isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ of the whole space $\mathbb{R}^n$. Indeed, let $a$ and $b$ be any two points on the Euclidean boundary $\mathrm{fr}\, U$ of $U$. We will demonstrate that

$$|\bar{f}(a) - \bar{f}(b)| = |a - b|. \tag{2.1}$$

To this end, consider a path $\gamma : [0, 1] \to \mathrm{fr}\, U$ with endpoints $\gamma(0) = a$ and $\gamma(1) = b$. Since $\bar{f}$ is a local isometry in the Euclidean metric, for every $t \in [0, 1]$ there is a ball $B_t = B(\bar{f}(\gamma(t)), r_t) \subset \mathbb{R}^n$ such that there exists a Euclidean isometry $F_t : \mathbb{R}^n \to \mathbb{R}^n$ with $F_t|_{(\mathrm{fr}\, U) \cap B_t} = \bar{f}|_{(\mathrm{fr}\, U) \cap B_t}$. Since $\gamma$ is continuous, the sets $\gamma^{-1}((\mathrm{fr}\, U) \cap B_t)$, where $t \in [0, 1]$, form an open covering of $[0, 1]$. But then we can extract a finite subcovering $\{E_s = \gamma^{-1}((\mathrm{fr}\, U) \cap B_{t_s}), s = 1, \dots, k\}$. If $E_{s_1} \cap E_{s_2} \neq \varnothing$, where $1 \leq s_1, s_2 \leq k$, then $(\mathrm{fr}\, U) \cap B_{t_{s_1}} \cap B_{t_{s_2}} \neq \varnothing$. Since $U$ is strictly convex, we easily conclude that $F_{t_{s_1}} = F_{t_{s_2}}$. Hence we can assert that there exists a unique Euclidean isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F_s = F$ for all $s = 1, \dots, k$ and, consequently, $\bar{f}|_{\mathrm{Im}\, \gamma} = F|_{\mathrm{Im}\, \gamma}$. This implies the desired equality (2.1). And from it, by its turn (with regard for the above-stated), the assertion of the theorem follows.

**Remark 2.1.** Theorem 2.1 is a generalization of a theorem by A. D. Aleksandrov about the unique determination of the boundary $\mathrm{fr}\, U$ of a strictly convex domain $U \subset \mathbb{R}^n$ $(n \geq 2)$ by the relative metric $\rho_{\mathrm{fr}\, U, U}$ (A. D. Aleksandrov's theorem was first published (with his consent) by V. A. Aleksandrov in [6]). An important particular case of this theorem is given by the following

**Theorem 2.2.** *Let $U_1$ be a strictly convex domain in $\mathbb{R}^n$. Suppose that $U_2 \subset \mathbb{R}^n$ is a domain whose closure is a Lipschitz manifold* (*such that* $\mathrm{fr}(\mathrm{cl}\, U_2) = \mathrm{fr}\, U_2 \neq \varnothing$);

*moreover*, $\operatorname{fr} U_1$ *and* $\operatorname{fr} U_2$ *are* (*globally*) *isomteric in their relative metrics* $\rho_{\operatorname{fr} U_1, U_1}$ *and* $\rho_{\operatorname{fr} U_2, U_2}$. *Then* $\operatorname{fr} U_1$ *and* $\operatorname{fr} U_2$ *are isometric in the Euclidean metric of* $\mathbb{R}^n$.

Let us now pass to the proof of [7, Theorem 2.2], which was announced in [7].

Assume that $U$ is a bounded nonconvex domain in $\mathbb{R}^2$ with Lipschitz boundary $\operatorname{fr} U$ and there exists a point $P \in \operatorname{fr} U$ such that $U$ is locally strictly convex on $\operatorname{fr} U \setminus \{P\}$ towards the complement $cU$ of $U$. We assert that $U$ is uniquely determined by the condition of the local isometry of the boundaries in the relative metrics. The proof of this assertion is carried out by the same scheme and with the use of the same tools as the proof of Theorem 2.1 with certain insignificant modifications. Let us briefly dwell on them.

Suppose that $V$ is another domain in $\mathbb{R}^2$ whose Hausdorff boundary is locally isometric to the Hausdorff boundary of $U$, $f : \operatorname{fr}_H U \to \operatorname{fr}_H V$ is a bijection that is a local isometry in the relative metrics of the Hausdorff boundaries $\operatorname{fr}_H U$ and $\operatorname{fr}_H V$ of $U$ and $V$, and $T = \bar{f}((\operatorname{fr} U) \setminus \{P\})$ (since $\operatorname{fr} U$ is Lipschitz, we, taking into account Remark 1.3, identify $\operatorname{fr}_H U$ with $\operatorname{fr} U$). We assert that, in this case, just as in the proof of Theorem 2.1, $\operatorname{cl} T = \operatorname{fr} V$. One can prove this basing on the arguments from the proof of Theorem 2.1. Nevertheless, by Lemma 2.2 and the infinity of the part of the set of supports of the Hausdorff boundary $\operatorname{fr}_H V$ contained in the set $M = (\operatorname{fr} V) \setminus \operatorname{cl} T$, the point $a'$ mentioned there can be chosen so that $\alpha = \bar{f}^{-1}(a)\ (\neq \varnothing) \subset (\operatorname{fr} U) \setminus \{P\}$.

We omit the rest of the proof since it repeats the arguments used in the proof of Theorem 2.1 almost verbatim. Thus, Theorem 2.2 in [7] is proved.

As opposed to what happens in the case of domains in $\mathbb{R}^2$ (see Theorem 2.1 in [7]), for space domains, as in Theorem 2.2 of [7], convexity is no longer necessary in solving the problems on the unique determination of domains by the condition of the local isometry of the boundaries in the relative metrics. In fact, the following holds:

**Theorem 2.3.** *There exists a* nonconvex *domain* $U \subset \mathbb{R}^3$ *with smooth boundary that is uniquely determined in the class of all three-dimensional domains with smooth boundary by the condition of the local isometry of the boundaries in the relative metrics.*

*Proof.* Construct a desired domain $U$ as follows:

Consider the cardioid arc

$$\theta = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 - \sqrt{x^2 + z^2} + z = 0,\ x^2 + z^2 > 0,\ x \geq 0,\ y = 0\}.$$

Leaving it fixed except for the part $\theta_1$, which is cut out from it by the disk $\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq \frac{1}{9},\ y = 0\}$, replace the cardioid arc $\theta_1$ by the circular arc $\{(x, y, z) \in \mathbb{R}^3 : z = 1 - \sqrt{\frac{2}{3} - x^2},\ 0 \leq x \leq \frac{\sqrt{5}}{9},\ y = 0\}$. It is not hard to verify that, after the rotation of the curve obtained in this way around the axis $Oz$ (up to a complete rotation), we obtain a closed smooth surface that is the boundary of a three-dimensional nonconvex Jordan domain, which we will take as a desired domain $U$, proving below that it is uniquely determined in the class of all domains in $\mathbb{R}^3$ with smooth boundaries by the condition of the local isometry of the boundaries in the relative metrics.

So, let $V \subset \mathbb{R}^3$ be another domain with smooth boundary and let $f : \operatorname{fr} U \to \operatorname{fr} V$ be a bijective mapping of $\operatorname{fr} U$ onto $\operatorname{fr} V$ that is a local isometry of the boundaries $\operatorname{fr} U$

and $\operatorname{fr} V$ in their relative metrics. Consider the curve $\theta_0 = \theta \setminus \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \le 1/4,\ y = 0\}$. After the rotation around the axis $Oz$, this part of the cardioid forms a region $S$ of $\operatorname{fr} U$ locally strictly convex towards the complement $cU$ of $U$. Applying the same technique as in the proof of Theorem 2.1 and basing on Lemma 2.3 in addition, we first see that there exists an isometry $F : \mathbb{R}^3 \to \mathbb{R}^3$ in the Euclidean metric such that $f|_S = F|_S$.

Without loss of generality, we may assume that $F = \mathrm{Id}_{\mathbb{R}^3}$. Denote by $S^*$ the part of $\operatorname{fr} U$ obtained by the rotation of the arc $\theta^* = \operatorname{cl}(\theta \setminus (\theta_1 \cup \theta_0))$, and consider the intersection of $S^*$ with a closed half-plane bounded by the axis $Oz$. We may also assume that this intersection is the curve $\theta^*$. Now, we show that any two sufficiently close points $a$ and $b$ of this curve (note that the degree of proximity of these points is determined by Lemma 2.2 applied to $f$) cut out an arc $ab$ from $\theta^*$ whose image under $f$ is a plane curve. Indeed, considering a third point $c$ of the arc $ab$, reckoning with the local strict convexity of $\theta^*$ (with respect to the plane domain $U_{x,z} = U \cap \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$; moreover, its convexity in the plane $\tau_{x,y} = \{(x, y, z) \in \mathbb{R}^3 : y = 0\}$ is directed towards the complement $\tau_{x,y} \setminus U_{x,z}$ of $U_{x,z}$) and applying Lemma 2.3 to each pair in the triple of points $a$, $c$, and $b$, we conclude that the point $f(c)$ is on the surface $\widetilde{S}$ formed by the rotation of the points of the arc $f(ab) = f(a)f(b)$ around the straight line $\zeta$ passing through the points $f(a)$ and $f(b)$, and the intersection of $\widetilde{S}$ with each half-plane bounded by $\zeta$ has the same length as the arc $ab$. If the arc $f(ab)$ were not plane then its length would be greater than the length of the arc $ab$. This would contradict Lemma 1.1. Hence, the arc $f(ab)$ is plane. Applying arguments close to those used in the proof of the first part of item $(ii)$ of Theorem 2.1 in [7], we further establish the existence of an isometry $F : \mathbb{R}^3 \to \mathbb{R}^3$ in the Euclidean metric with $F|_{ab} = f|_{ab}$. Therefore, the arc $f(ab)$ (together with the arc $ab$) is strictly convex and hence if two planes contain the arc $f(ab)$ then they coincide. Applying our last considerations to the arc $\theta^* \cup \theta_0$, taking into account what was said above, and using induction, it is not difficult to prove that the curve $f(\theta^* \cup \theta_0)$ is contained in the plane $\tau_{x,y}$, i.e., in the same plane that the curve $\theta^* \cup \theta_0$. Once again using the arguments from the proof of Theorem 2.1 in [7], we infer that $f|_{\theta^* \cup \theta_0} = \mathrm{Id}_{\theta^* \cup \theta_0}$. Considering the remaining intersections of $U$ with half-planes bounded by the axis $Oz$ and involving all that was stated above, we finally have

$$f|_W = \mathrm{Id}_W$$

where $W$ is the part of $\operatorname{fr} U$ obtained by the rotation of the arc $\theta^* \cup \theta_0$ around the axis $Oz$.

Put $M = f((\operatorname{fr} U) \setminus W) \cap cV \cap \{(x, y, z) \in \mathbb{R}^3 : z \ge 2/9\} \ne \varnothing$. Let $\alpha > 2/9$ be such that

$$M_\alpha = M \cap \{(x, y, z) \in \mathbb{R}^3 : z = \alpha\} \ne \varnothing$$

and

$$M \cap \{(x, y, z) \in \mathbb{R}^3 : z > \alpha\} = \varnothing.$$

Suppose that $M_\alpha$ contains a point $(\bar{x}, \bar{y}, \alpha)$ with $\bar{x}^2 + \bar{y}^2 > 0$. Without loss of generality, we may assume that $\bar{x}^2 + \bar{y}^2 = \max\limits_{(x,y,z) \in M_\alpha} (x^2 + y^2)$. Moreover, since $M_\alpha \cap f(W) = \varnothing$, we have $(\bar{x}, \bar{y}, \alpha) \notin f(W)$. Next, let $\chi = \{\bar{x}(1 + \lambda/\sqrt{\bar{x}^2 + \bar{y}^2})e_1 + \bar{y}(1 + \lambda/\sqrt{\bar{x}^2 + \bar{y}^2})e_2 + (\alpha - \lambda t)e_3 : \lambda \ge 0\}$ be a ray going out from the point $P_0 = (\bar{x}, \bar{y}, \alpha)$ and let the value of $t$ $(> 0)$ be so small that this ray intersects $f((\operatorname{fr} U) \setminus W) \setminus \{P_0\}$ and the distance between $P_0$ and the nearest point $P$ of the set

$(f((\mathrm{fr}\,U)\setminus W)\setminus\{P_0\})\cap\chi$ to it is less than the number $\varepsilon=\varepsilon_{P_0}$ of Lemma 2.3 for the mapping $f^{-1}$ (in this connection, note that the plane $\tau_\alpha=\{(x,y,z)\in\mathbb{R}^3 : z=\alpha\}$ is supporting to the surface $f((\mathrm{fr}\,U)\setminus W)$ and hence is the tangent plane to it at all points $R\in M_\alpha$). Consequently, by the lemma and the fact that the interval $]P,P_0[$ is contained in $V$, the interval $]f^{-1}(P),f^{-1}(P_0)[$ must be contained in $U$. But this is impossible. Therefore, it remains to consider the case of $\bar{x}=\bar{y}=0$. However, this case is also contradictory; to see that it suffices to consider the ray $\{\lambda e_1+(\alpha-\lambda t)e_3 : \lambda\ge 0\}$ as a desired one and then repeat the arguments used in the previous case.

We must yet discuss the case of $\alpha=2/9$. If $\mathrm{dist}(M\cap\tau_{2/9},W)>0$ then, using the arguments from the previous two cases, we see that this situation is also impossible. Now, let $\mathrm{dist}(M\cap\tau_{2/9},W)=0$. The above-stated facts and the smoothness of the boundaries $\mathrm{fr}\,U$ and $\mathrm{fr}\,V$ of $U$ and $V$ imply the following circumstance: For every point $z^0\in M_{2/9}$ $(=M\cap\tau_{2/9})$, there exists a number $\varkappa_0>0$ such that any ray starting from $z^0$ and intersecting the cone $K=\{(x,y,z)\in\mathbb{R}^3 : z=\frac{1}{7}+\frac{5}{63}\sqrt{x^2+y^2},\ \frac{1}{7}\le z\le\frac{2}{9}\}$ at a point lying between the planes $\tau_{2/9}$ and $\tau_{2/9-\varkappa_0}$ has common points with the surface $(f((\mathrm{fr}\,U)\setminus W))\setminus\{z^0\}$ (here we take into account that the generatrices of the cone $K$ pass through the points of the boundary of the manifold $\mathrm{cl}((\mathrm{fr}\,U)\setminus W)$ being tangent to $\mathrm{fr}\,V$ at these points.) Choosing as $z^0$ a point that is so close to $W$ that the closed interval $[z^0,\widetilde{z}]$ (where $\widetilde{z}\in K\cap\tau_{2/9-\varkappa_0/2}$) of the ray $\chi$ starting from it and intersecting the circle $K\cap\tau_{2/9-\varkappa_0/2}$ has the least possible length of such segments, consider the nearest point $P\in(f((\mathrm{fr}\,U)\setminus W)\setminus\{z^0\})\cap\chi$ to $z^0$. Assuming in addition that $|P-z^0|<\varepsilon_{z^0}$ (where $\varepsilon_{z^0}$ is taken for $f^{-1}$ as in Lemma 2.3), we can apply the above arguments to make sure that this case is also impossible. We finally arrive at the inequality

$$f_3(x,y,z)<\frac{2}{9} \tag{2.2}$$

(where $f=(f_1,f_2,f_3):\mathrm{fr}\,U\to\mathrm{fr}\,V$), which holds for all points $(x,y,z)\in(\mathrm{fr}\,U)\setminus W$.

Consider the bounded open set $A\in\mathbb{R}^3$ whose boundary is composed of the sets $f((\mathrm{fr}\,U)\setminus W)$ and $\Xi=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\le\frac{5}{81},\ z=\frac{2}{9}\}$. It is a three-dimensional Jordan domain contained in the complement to $V$. Prove that $A$ is convex. Supposing the contrary and using the proof of the Leja–Wilkosz theorem of [8] exposed in [9], we conclude that there are three points $X\in\mathrm{Int}\,A$, $Y\in\mathrm{Int}\,A$ and $Z\in\mathrm{Int}\,A$ such that $[X,Y]\subset\mathrm{Int}\,A$, $[Y,Z]\subset\mathrm{Int}\,A$, $[X,Z]\not\subset\mathrm{Int}\,A$. Fixing the location of the plane $\tau$ containing these points, in this plane, we can construct, for instance, a concave elliptical arc $\gamma$ locally supporting $A$ outwards. Then, changing the location of $Z$ in its small spherical neighborhood, we can obtain a continual family of concave elliptical arcs locally supporting $A$ outwards. The plane measure of each part of $\mathrm{fr}\,V$ situated in one of the indicated plane intersections cannot be positive since $\mathrm{fr}\,V$ is a smooth bounded surface and hence has finite area. Therefore, there exist segments $[a,b]$ of arbitrarily small linear sizes such that $]a,b[\subset cA$ and $a,b\in\mathrm{fr}\,A$. Moreover, we may also assume that $a,b\notin\Xi$. Hence, we again get into the situation described in proving (2.2), which implies that $A$ is convex.

Thus, the surfaces $\mathrm{cl}((\mathrm{fr}\,U)\setminus W)$ and $f(\mathrm{cl}((\mathrm{fr}\,U)\setminus W))$ satisfy the conditions of Theorem 2 in [10, Chapter 3, Section 7]. Using this theorem, we see that these surfaces are equal. Granted this and the above, Theorem 2.3 is completely proved.

In conclusion, note that the main results of our article were earlier announced in [11].

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