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ON RECOGNITION OF ALTERNATING GROUPS BY PRIME GRAPH

A.M.STAROLETOV

ABSTRACT. The *prime graph* $GK(G)$ of a finite group G is the graph whose vertex set is the set of prime divisors of $|G|$ and in which two distinct vertices r and s are adjacent if and only if there exists an element of G of order rs. Let Alt_n denote the alternating group of degree n. Assume that $p \geq 13$ is a prime and n is an integer such that $p \leq n \leq p+3$. We prove that if G is a finite group such that $GK(G) = GK(Alt_n)$, then G has a unique nonabelian composition factor, and this factor is isomorphic to Alt_t, where $p \leq t \leq p+3$.

Keywords: alternating group, prime graph, simple groups.

1. INTRODUCTION

Suppose that G is a finite group. The set of prime divisors of $|G|$ is denoted by $\pi(G)$. The *spectrum* $\omega(G)$ of G is the set of its element orders. The spectrum defines the *prime graph* (or the *Gruenberg — Kegel graph*) $GK(G)$ of G: the set of vertices is $\pi(G)$, and two distinct vertices r and s are adjacent if and only if $rs \in \omega(G)$.

Throughout the paper we use notations for simple groups from [1], as well as the standard abbreviation $L_n^{\varepsilon}(q)$, where $\varepsilon \in \{+,-\}$, $L_n^+(q) = L_n(q)$, and $L_n^-(q) = U_n(q)$. A nonabelian simple group L is called *recognizable by spectrum* if the spectrum characterizes L in the class of all finite groups, i.e. if for a finite group H, the equality $\omega(H) = \omega(G)$ implies $H \simeq G$. Similarly, L is said to be *recognizable by prime graph* if the prime graph characterizes L in the class of all finite groups.

It is known that Alt_5 is recognizable by spectrum [2]. However, $GK(Alt_5) = GK(Alt_6)$, and so Alt_5 is not recognizable by its prime graph. More generally, I.B. Gorshkov proved that Alt_n with $n \geq 5$ is recognizable by spectrum iff

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$n \neq 6, 10$ [3]. If $L \in \{Alt_6, Alt_{10}\}$ then there exist infinitely many pairwise nonisomorphic groups whose spectrum is equal to $\omega(L)$ [4, 5]; nevertheless, if $\omega(G) = \omega(L)$ then G has a unique nonabelian composition factor S, and $S \in \{Alt_5, L\}$ [6, 7].

There are few results about finite groups whose prime graph coincides with $GK(Alt_n)$, where $n \geq 5$. A.S. Kondratiev [8] described finite groups with prime graph equal to $GK(Alt_{10})$. In particular, he proved that there exist soluble groups G such that $GK(G) = GK(Alt_{10})$. Moreover, if S is a finite nonabelian simple group such that $\pi(S) \subseteq \{2, 3, 5, 7\}$ and $S \not\simeq L_2(8), U_4(2), S_4(7)$, then there exists a finite group G such that S is a composition factor of G and $GK(G) = G(Alt_{10})$. B. Khosravi and A.Z. Moghanjoghi in [9] proved that if G is a finite group such that $GK(G) = GK(Alt_p)$, where $p > 19$ is a prime and $p-2$ is also prime, then G has a unique nonabelian composition factor, and this factor is isomorphic to Alt_p. M.A. Zvezdina [10] described all finite simple groups S such that $GK(S) = GK(Alt_n)$ for every $n \geq 5$. In particular, if $n > 9$ and $GK(S) = GK(Alt_n)$, then S is an alternating group. M.R. Zinov′eva and V.D. Mazurov in [11] determined all finite nonabelian simple groups S whose prime graph is equal to the prime graph of a Frobenius or 2-Frobenius group F. In particular, if S is an alternating group, then F is soluble and either $S \simeq Alt_9$ or $S \simeq Alt_{12}$.

Suppose that p is the largest prime less than or equal to n. Note that if $n-p \geq 3$, then $GK(Alt_n) = GK(Alt_n \times Z_3)$. Therefore, if Alt_n is recognizable by prime graph, then $n-p \leq 2$. It is easy to see that if $n-p \geq 5$, then $GK(Alt_n) = GK(Alt_n \times Alt_5^k)$ for any natural integer k. So in this case a finite group with prime graph equal to $GK(Alt_n)$ can have any number of nonabelian composition factors. Moreover, if S is a finite nonabelian simple group and n is an integer greater than any element of $\pi(S)$, then $GK(Alt(n!-2)) = GK(Alt(n!-2) \times S)$ because all $n!-2$, $n!-3, \ldots$, $n!-n$ are composite. Thus any finite nonabelian simple group can be a composition factor of a finite group with prime graph equal to $GK(Alt_m)$ for a suitable m.

The goal of the present paper is to describe nonabelian composition factors of a finite group G such that $GK(G) = GK(Alt_n)$, where $p \leq n \leq p+3$ for some prime $p \geq 11$. It turns out that in general these factors are alternating groups.

Theorem 1. *Let $n \geq 5$ be an integer such that $p \leq n \leq p+3$ for the largest prime p less than or equal to n. Let G be a finite group such that $GK(G) = GK(Alt_n)$ and suppose that G is insoluble if $n = 9, 10, 12$. Then the following hold.*
(i) G has a unique nonabelian composition factor S.
(ii) If $p \geq 13$ then $S \simeq Alt_t$, where $p \leq t \leq p+3$.
(iii) If $p \geq 13$ and $p-2$ is a prime, then $S \simeq Alt_t$, where $p \leq t \leq n$.

Theorem 2. *If Alt_n, where $n \geq 5$, is recognizable by prime graph, then $n = p$ or $n = p+1$ for a prime p such that $p-2$ is also a prime.*

We also describe all possibilities for S in Theorem 1 for $11 \leq n \leq 21$ and show that all of them are realizable, i.e. there exists a finite group G such that $GK(G) = GK(Alt_n)$ and G has a composition factor isomorphic to S. In particular, we prove that Alt_{13} is recognizable by prime graph.

Theorem 3. *Let n, G, and S be as in Theorem 1. Then the following hold.*
(i) If $n = 11$ then $S = Alt_{11}$. If $n = 12$ then $S \in \{L_2(11), M_{11}, U_5(2), Alt_{11}, Alt_{12}\}$, and each of these possibilities is realizable.
(ii) If $n = 13$ then $G \simeq Alt_{13}$.

(iii) If $14 \leq n \leq 21$ *then* $S = Alt_t$ *is realizable if and only if* $t = n$ *or the pair* (n, t) *is one of the following:* $(14, 13)$, $(15, 13)$, $(15, 14)$, $(16, 13)$, $(21, 20)$.

Based on Theorems 2 and 3, we propose the following.

Conjecture 1. *If* p *and* $p - 2$ *are primes greater than* 7, *then* Alt_p *is recognizable by prime graph.*

2. Preliminaries

2.1. **Some auxiliary results.** Let G be a finite group and π be a set of primes. Then G is a *π-group* if $\pi(G) \subseteq \pi$ and a *π'-group* if $\pi(G) \cap \pi = \varnothing$. A π-subgroup H of G is called a *Hall π-subgroup* if $|G : H|$ is not divided by any element of π. If $\pi = \{p\}$ for a prime p, we write *p-subgroup* or *p'-subgroup* instead of $\{p\}$-subgroup or $\{p\}'$-subgroup, respectively. A semidirect product of a group H acting on a group N is denoted by $N \rtimes H$.

Let r be a prime. Following [12], denote by $\mathfrak{S}_r$ the set of all finite nonabelian simple groups S such that $r \in \pi(S)$ and all elements of $\pi(S)$ do not exceed r.

The *soluble radical* of a finite group G is the maximal normal soluble subgroup of G.

Lemma 1. *Let* N *be a normal elementary abelian subgroup of* G *and* $H = G/N$. *Let* $G_1 = N \rtimes H$ *be the natural semidirect product of* H *and* N. *Then* $GK(G) = GK(G_1)$.

Proof. It was proved in [13, Lemma 10] that $\omega(G_1) \subseteq \omega(G)$, and hence if distinct primes s and t are adjacent in $GK(G_1)$, then s and t are adjacent in $GK(G)$. Let now distinct primes s and t are adjacent in $GK(G)$. Then there exist $x, y \in G$ such that $|x| = s$, $|y| = r$, and $xy = yx$. If neither $x \in N$ nor $y \in N$, then $rs \in \omega(H)$, and so s and t are adjacent in $GK(G_1)$. Therefore, we can suppose that $x \in N$. Let $\overline{y}$ be the image of y in H. Then the order of the element $x\overline{y}$ of G_1 is equal to st. Thus $GK(G) = GK(G_1)$. □

Lemma 2. *Let* G *be a finite group and* K *be the soluble radical of* G. *Assume that* $p \in \pi(G) \setminus \pi(K)$. *If* $\pi \subset \pi(G) \setminus \{p\}$ *is the set of vertices that are not adjacent with* p *in* $GK(G)$, *then a Hall* π*-subgroup of* K *is nilpotent.*

Proof. Let H be a Hall π-subgroup of K. By the Frattini argument, we have that p divides $|N_G(H)|$. Let g be an element of $N_G(H)$ of order p. Then g acts on H and $C_H(g) = 1$, so H is nilpotent by the Thomson theorem. □

Lemma 3 ([14], Lemma 3.6)**.** *Let* s *and* v *be distinct primes, a group* H *be a semidirect product of a normal* v*-subgroup* T *and a cyclic subgroup* $C = \langle g \rangle$ *of order* s, *and let* $[T, g] \neq 1$. *Suppose that* H *acts faithfully on a vector space* V *of positive characteristic* t *not equal to* v. *If the minimal polynomial of* g *on* V *does not equal* $x^s - 1$, *then*
(i) $C_T(g) \neq 1$;
(ii) T *is nonabelian*;
(iii) $v = 2$ *and* s *is a Fermat prime.*

Lemma 4. *Let* G *be a finite group and* $U \leq V$ *be normal subgroups of* G *such that* $S = G/V$ *is a nonabelian simple group,* V/U *is an* s*-group, and* U *is an elementary abelian* r*-subgroup, where* s *and* r *are distinct primes. If* $C_V(U) \leq U$ *and* $rt \notin \omega(G)$

for some $t \in \pi(S) \setminus \{s, r\}$, then G/U contains a subgroup isomorphic to either S or a perfect central extension of S by an s-group.

Proof. Since there exists $t \in \pi(S) \setminus \{r\}$ such that $rt \notin \omega(G)$ and S is simple, we have that $C_G(U) \leq U$. Let $\overline{G} = G/U$, $\overline{V} = V/U$, and g be an element of $\overline{G}$ of order t. If $[\overline{V}, \langle g\rangle] \neq 1$ then by Lemma 3, we obtain that $rt \in \omega(G)$; a contradiction. So $[\overline{V}, \langle g\rangle] = 1$. Since S is simple, we obtain that $\overline{V}C_{\overline{G}}(\overline{V}) = \overline{G}$. Thus $C_{\overline{G}}(\overline{V})/C_{\overline{G}}(\overline{V}) \cap \overline{V} \simeq \overline{G}/\overline{V} \simeq S$. The derived series of $C_{\overline{G}}(\overline{V})$ contains a perfect group. This group is the required group. □

Let $a < b$ be real numbers. Denote by $\pi(a, b)$ the number of primes s such that $a < s \leq b$.

Lemma 5 ([15], Corollary 3)**.** *If $x \geq 20\frac{1}{2}$ then $\pi(x, 2x) > 3x/(5 \ln x)$, and if $x > 1$ then $\pi(x, 2x) < 7x/(5 \ln x)$.*

Lemma 6 ([16], Proposition 5.4)**.** *For all $x \geq 89693$, there exists a prime p such that $x < p \leq x(1 + \frac{1}{\ln^3 x})$.*

2.2. **Zsigmondy primes.** If n is a nonzero integer and r is an odd prime with $(r, n) = 1$, then $e(r, n)$ denotes the multiplicative order of n modulo r. Given an odd integer n, we put $e(2, n) = 1$ if $n \equiv 1 \pmod 4$, and $e(2, n) = 2$ otherwise.

Fix an integer a with $|a| > 1$. A prime r is said to be a *primitive prime divisor* of $a^i - 1$ if $e(r, a) = i$. We write $r_i(a)$ to denote some primitive prime divisor of $a^i - 1$, if such a prime exists, and $R_i(a)$ to denote the set of all such divisors. Zsigmondy in [17] proved that primitive prime divisors exist for almost all pairs (a, i).

Lemma 7 ([17] Zsigmondy)**.** *Let a be an integer and $|a| > 1$. For every natural number i the set $R_i(a)$ is nonempty except for the pairs $(a, i) \in \{(2, 1), (2, 6), (-2, 2), (-2, 3), (3, 1), (-3, 2)\}$.*

Lemma 8. *Let a and i be integers with $|a| > 1$ and $i > 0$. If i is odd then $R_i(-a) = R_{2i}(a)$, and if i is a multiple of 4 then $R_i(-a) = R_i(a)$.*

Proof. It is a consequence of [14, Lemma 1.3]. □

Lemma 9 ([18], Lemma 6)**.** *Let q, m, and k be natural numbers. Then $R_{mk}(q) \subseteq R_m(q^k)$. If, in addition, $(m, k) = 1$ then $R_m(q) \subseteq R_m(q^k)$. In particular, $|R_m(q^k)| > 1$ for all coprime $m \geq 4$ and $k \geq 2$ except for the case $m = 6$ and $q = 2$.*

2.3. **Connection between $GK(G)$ and the structure of G.** Let G be a finite group. Recall that a subset of vertices of a graph is called a *coclique*, if every two vertices of this subset are nonadjacent. Denote by $t(G)$ the greatest size of a coclique in $GK(G)$. We refer to a coclique containing r as an $\{r\}$-coclique. If $r \in \pi(G)$ then $t(r, G)$ denotes the greatest size of $\{r\}$-cocliques. Let $s(G)$ denotes the number of connected components in $GK(G)$. If $|G|$ is even then π_1 stands for the set of primes in the connected component of $GK(G)$ containing 2.

Lemma 10 ([19], Proposition 2), ([20], Theorem 1)**.** *Let G be a finite group with $t(G) \geqslant 3$ and $t(2, G) \geqslant 2$. Then the following hold:*
(i) There exists a nonabelian simple group S such that $S \leqslant \overline{G} = G/K \leqslant \mathrm{Aut}\, S$, where K is the soluble radical of G.
(ii) For every coclique ρ of $GK(G)$ of size at least 3, at most one prime of ρ divides the product $|K| \cdot |\overline{G}/S|$. In particular, $t(S) \geqslant t(G) - 1$.

(iii) *One of the following holds:*

(a) *every prime* $r \in \pi(G)$ *nonadjacent to* 2 *in* $GK(G)$ *does not divide the product* $|K| \cdot |\overline{G}/S|$; *in particular,* $t(2, S) \geqslant t(2, G)$;

(b) *there exists a prime* $r \in \pi(K)$ *nonadjacent to* 2 *in* $GK(G)$; *in which case* $t(G) = 3$, $t(2, G) = 2$, *and* $S \simeq Alt_7$ *or* $L_2(q)$ *for some odd* q.

Lemma 11. *Let* G *be a finite group and* $s(G) > 1$. *Then one of the following holds.*
(i) G *is a Frobenius group;*
(ii) G *is a* 2*-Frobenius group;*
(iii) *There exists a normal nilpotent* π_1*-subgroup* K *of* G *and a nonabelian simple group* S *such that* $S \leq \overline{G} \leq \mathrm{Aut}(S)$, *where* $\overline{G} = G/K$. *Moreover, the group* $\overline{G}/S$ *is a* π_1*-group.*

Proof. Follows from [21, Theorem A]. □

Lemma 12. *Let* p *be a prime and* n *be an integer such that* $p \leq n \leq p + 3$. *If* $p \geq 31$ *then* $t(Alt_n) \geq 5$, *and if* $p \geq 157$ *then* $t(Alt_n) \geq 16$.

Proof. Put $x = [n/2]$. Then $t(Alt_n) \geq \pi(x, 2x)$. Assume first that $n \geq 60$, in particular $x \geq 30$. Lemma 5 implies that $\pi(x, 2x) \geq 5$, and so $t(Alt_n) \geq 5$. It is easy to verify that if $31 \leq p < 60$ then $t(Alt_n) \geq 5$. So $t(Alt_n) \geq 5$ for all $p \geq 31$. Assume that $n \geq 400$. Then $x \geq 200$. Lemma 5 implies that $t(Alt_n) > 16$. When $157 \leq p < 400$, the assertion can be verified using a table of primes less than 400 (e.g., see [22]). □

Lemma 13. *Let* G *be a finite group and* $GK(G) = GK(Alt_n)$, *where* $n \geq 13$. *Assume that* $Alt_n \in \mathfrak{S}_p$ *and* $n \leq p+3$, *where* p *is a prime. Then there exists a finite nonabelian simple group* $S \in \mathfrak{S}_p$ *such that* $t(S) \geq t(G) - 1$ *and* $S \leq G/K \leq \mathrm{Aut}\, S$, *where* K *is the soluble radical of* G.

Proof. Since 2 and p are nonadjacent in $GK(Alt_n)$, we have that $t(2, G) = t(2, Alt_n) \geq 2$. If $p \geq 31$ then by Lemma 12 we obtain that $t(Alt_n) \geq 5$. It is easy to verify that if $13 \leq p \leq 29$ then $t(Alt_n) \geq 3$. Lemma 10 yields that there exists a finite nonabelian simple group S such that $S \leq G/K \leq \mathrm{Aut}\, S$, where K is the soluble radical of G. Moreover, $t(S) \geq t(G) - 1$. It remains to prove that $p \in \pi(S)$. If $p \geq 31$ then by Lemma 12 $t(S) \geq t(G) - 1 \geq 4$, and Lemma 10 implies that $p \in \pi(S)$. Let $p \leq 29$. Assume that $p \notin \pi(S)$. Lemma 10 and [12, Table 1] imply that $p \in \pi(K)$ and either $S \simeq Alt_7$ or $S \simeq L_2(u)$, where u is an odd prime power. Let r be the greatest element of $\pi(S)$. Then $S \in \mathfrak{S}_r$ and $r \leq 23$. Elements of $\mathfrak{S}_r$ for $r \leq 23$ are listed in [12, Table 1].

Suppose that $17 \leq p \leq 29$. Then either $\{11, 13, p\}$ or $\{17, 19, p\}$ is a coclique of size three in $GK(Alt_n)$. Lemma 10 implies that either $11, 13 \in \pi(S)$ or $17, 19 \in \pi(S)$. In particular, $r \geq 13$ and $S \not\simeq Alt_7$. According to [12, Table 1], there are no groups $L_2(u)$ with such property either; a contradiction.

Let $p = 13$. If $S = Alt_7$ then $11, 13 \in \pi(K)$. Since $\{7, 11, 13\}$ is a coclique in $GK(Alt_n)$, this contradicts Lemma 10. So $S = L_2(u)$. If $11 \notin \pi(S)$ then we obtain a contradiction as in the case $S = Alt_7$. Therefore $S \in \mathfrak{S}_{11}$. According to [12, Table 1], we obtain that $S = L_2(11)$. Then $7 \notin \pi(S)$, and hence $|\{7, 11, 13\} \cap \pi(K)| \geq 2$ which contradicts Lemma 10.

□

2.4. **Properties of simple classical groups.** For a simple classical group S, we write $\mathrm{prk}(S)$ to denote its dimension if S is a linear or unitary group, and its Lie rank if S is a symplectic or orthogonal group.

Following [23], we define the next function on integer numbers.

$$\eta(i) = \begin{cases} i, & \text{if } i \text{ is odd,} \\ i/2, & \text{if } i \text{ is even.} \end{cases}$$

Following [14], we introduce one more function to formulate adjacency criteria in the prime graph of a simple classical group. Namely, given a simple classical group S over a field of order u and a prime r coprime to u, we put

$$\varphi(r, S) = \begin{cases} e(r, \varepsilon u), & \text{if } L = L_m^\varepsilon(u), \\ \eta(e(r, u)), & \text{if } S \text{ is symplectic or orthogonal.} \end{cases}$$

Lemma 14. *Let S be a finite simple group of Lie type over a field of characteristic v and $m = \mathrm{prk}(S)$. Let r, s be odd primes with r, $s \in \pi(G) \setminus \{v\}$. Put $k = e(r, -u)$ and $l = e(s, -u)$ if $S \simeq L_m^-(u)$, and $k = e(r, u)$ and $l = e(s, u)$ otherwise. Suppose that $2 \leq \varphi(r, S) \leq \varphi(s, S)$. Then the following hold.*

(i) If $S = L_m^\varepsilon(u)$ then r and s are nonadjacent in $GK(S)$ if and only if $\varphi(r, S) + \varphi(s, S) > m$, and $\frac{l}{k}$ is not a natural number.

(ii) If $S \in \{O_{2m+1}(u), S_{2m}(u)\}$ then r and s are nonadjacent in $GK(S)$ if and only if $\varphi(r, S) + \varphi(s, S) > m$, and $\frac{l}{k}$ is not an odd natural number;

(iii) If $S = O_{2m}^\varepsilon(u)$ then r and s are nonadjacent in $GK(S)$ if and only if $2\varphi(r, S) + 2\varphi(s, S) > 2m - (1 - \varepsilon(-1)^{k+l})$, $\frac{l}{k}$ is not an odd natural number, and, if $\varepsilon = +$, then the chain of equalities: $m = l = 2\varphi(s, S) = 2\varphi(r, S) = 2k$ is not true.

Proof. It follows from [23, Propositions 2.1, 2.2] and [24, Propositions 2.4, 2.5]. □

Lemma 15. *Let S be a classical simple group and $m = \mathrm{prk}(S)$. If $t(S) \geq 15$ then there exist $r_1, r_2, r_3, r_4 \in \pi(S) \setminus \{v\}$ such that the following hold.*
(i) r_1 is adjacent with r_2 and nonadjacent with r_4 in $GK(S)$;
(ii) r_3 is adjacent with r_4 and nonadjacent with r_2 in $GK(S)$;
(iii) if S is unitary then $e(r_i, -u) \geq \max(7, (m-5)/3)$ for $1 \leq i \leq 4$, otherwise $e(r_i, u) \geq \max(7, (m-5)/3)$ for $1 \leq i \leq 4$.

Proof. Let $\varepsilon = -$ if S is unitary and $\varepsilon = +$ otherwise. Since $t(S) \geq 15$, according to [24, Tables 2, 3], we obtain that $m \geq 29$ if S is a linear or unitary group, and $m \geq 19$ if S is a symplectic or orthogonal group. Assume first that $m > 23$. Note that there exists an integer j such that $0 \leq j \leq 2$ and $m - j$ is divisible by 3. If $S = O_{2m}^-(u)$ and $j = 0$, take $r_1 \in R_{2m}(u)$ and $r_2 \in R_{(2m)/3}(u)$, otherwise take $r_1 \in R_{m-j}(\varepsilon u)$ and $r_2 \in R_{(m-j)/3}(\varepsilon u)$. Let $r_3 \in R_{m-j-3}(\varepsilon u)$, $r_4 \in R_{(m-j-3)/3}(\varepsilon u)$. Since $m > 23$, we have that $e(r_i, \varepsilon u) \geq (m-5)/3 > 6$, where $1 \leq i \leq 4$, and hence r_1, r_2, r_3, and r_4 exist by Lemma 7. Note that $e(r_1, \varepsilon u)/e(r_2, \varepsilon u) = e(r_3, \varepsilon u)/e(r_4, \varepsilon u) = 3$, so r_1 is adjacent with r_2, and r_3 is adjacent with r_4 in $GK(S)$ by Lemma 14. Observe that $e(r_1, \varepsilon u) > e(r_4, \varepsilon u)$, $e(r_3, \varepsilon u) > e(r_2, \varepsilon u)$. To prove that r_2 is nonadjacent with r_4 and r_1 is nonadjacent with r_3, it is sufficient to show that $e(r_1, \varepsilon u) + e(r_4, \varepsilon u) > m$, $e(r_3, \varepsilon u) + e(r_2, \varepsilon u) > m$ and that $e(r_1, \varepsilon u)$ is not divisible by $e(r_4, \varepsilon u)$, $e(r_3, \varepsilon u)$ is not divisible by $e(r_2, \varepsilon u)$. Since $m > 23$, we have that $e(r_1, \varepsilon u) + e(r_4, \varepsilon u) \geq m - 2 + (m-5)/3 > m$ and $e(r_2, \varepsilon u) + e(r_3, \varepsilon u) \geq m - 5 + (m-2)/3 > m$. Suppose that $e(r_1, \varepsilon u)$ is divisible by $e(r_4, \varepsilon u)$. Then $(m-j-3)/3$ divides $2(m-j)$, so $\frac{6(m-j)}{m-j-3}$ is an integer. Obviously, $(m-j, m-j-3) \leq 3$, so $m-j-3 \leq 18$, and hence $m \leq 23$; a contradiction. Suppose that $e(r_3, \varepsilon u)$ is divisible by $e(r_2, \varepsilon u)$. Then $(m-j)/3$ divides $m-j-3$, so $\frac{3(m-j-3)}{m-j}$ is an integer. As above, we obtain a contradiction with $m > 23$.

Assume now that $m \leq 23$. In this case, S is an orthogonal or symplectic group. If $m = 22, 23$, or $m = 21$ and $S \not\simeq O^-_{42}(u)$, then we put $(r_1, r_2, r_3, r_4) = (r_{21}(u), r_7(u), r_{40}(u), r_8(u))$. When $S = O^-_{42}(u)$, we put $(r_1, r_2, r_3, r_4) = (r_{42}(u), r_{14}(u), r_{40}(u), r_8(u))$. Finally, for $m = 19, 20$ we put $(r_1, r_2, r_3, r_4) = (r_{36}(u), r_{12}(u), r_{30}(u), r_{10}(u))$. Using Lemma 14, we obtain that r_1, r_2, r_3, r_4 satisfy the conditions (i) and (ii) of the lemma. Since $m \leq 23$, we have that $(m-5)/3 \leq 6$, and hence $e(r_i, u) \geq \max(7, (m-5)/3)$ for $1 \leq i \leq 4$, as required. □

Lemma 16 ([14], Lemma 3.8)**.** *For a simple classical group S over a field of order u and characteristic v with* $\mathrm{prk}(S) = m \geqslant 4$*, put*

$$j = \begin{cases} m, & \text{if } S \simeq L_m(u); \\ 2m-2, & \text{if either } S \simeq O^+_{2m}(u) \text{ or } S \simeq U_m(u) \text{ and } m \text{ is even}, \\ 2m, & \text{otherwise.} \end{cases}$$

Then $(r_j(u), |P|) = 1$ for every proper parabolic subgroup P of S. If $i \neq j$ and a primitive prime divisor $r_i(u)$ lies in $\pi(S)$, then there is a proper parabolic subgroup P of S such that $r_i(u)$ lies in $\omega(P)$. In particular, if two distinct primes $r, s \in \pi(S)$ do not divide the order of any proper parabolic subgroup of S, then r and s are adjacent in $GK(S)$.

Lemma 17 ([14], Lemma 3.7)**.** *Let S be a simple classical group over a field of order u and characteristic v, and* $\mathrm{prk}(S) > 4$*. Suppose that a prime s divides the order of a proper parabolic subgroup of S, and $(s, v(u^2-1)) = 1$. Then S includes a subgroup H such that H is a semidirect product of a normal v-subgroup T and a cyclic subgroup $C = \langle g \rangle$ of order s with $[T, g] \neq 1$, and at least one of three assertions from the conclusion of Lemma 3 does not hold for H.*

3. Proof of Theorem 1

In this section we prove Theorem 1. We split the proof into several parts, and Theorem 1 will follow from the next lemma and four propositions. Throughout this section $n \geq 5$ is an integer such that $p \leq n \leq p+3$ for the largest prime p less than or equal to n and G is a finite group such that $GK(G) = GK(Alt_n)$.

Lemma 18. *Suppose that $5 \leq p \leq 11$ and if $n = 9, 10, 12$ then G is insoluble. Then G has a unique a nonabelian composition factor S. Moreover, if $n \neq 10$ then $S \in \mathfrak{S}_p$.*

Proof. Note that if $n \neq 10$ then $GK(Alt_n)$ is disconnected. Therefore, at least one of the statements $(i) - (iii)$ of Lemma 11 holds true for G. M. R. Zinov′eva and V. D. Mazurov in [11] described all finite nonabelian simple groups L whose prime graph is equal to the prime graph of a Frobenius or 2-Frobenius group F. In particular, if L is an alternating group, then F is soluble and $L \in \{Alt_9, Alt_{12}\}$. So, if $n \neq 10$, then G is as in the case (iii) of Lemma 11, and hence G has a unique nonabelian simple group S and $S \in \mathfrak{S}_p$. Finite groups G such that $GK(G) = GK(Alt_{10})$ were described in [8], in particular, it was proved that if G is insoluble then G has a unique nonabelian composition factor. □

Further we suppose that $p \geq 13$. By Lemma 13, we have that there exists a nonabelian simple group $S \in \mathfrak{S}_p$ such that $S \leq G/K \leq \mathrm{Aut}(S)$, where K is the soluble radical of S. Moreover, $t(S) \geq t(G) - 1$.

In order to use Lemma 15, it is convenient to define $\varepsilon = -$ if S is an unitary group and $\varepsilon = +$ otherwise.

Proposition 1. *Suppose that $p \geq 773$. Then $S \simeq Alt_t$, where $p \leq t \leq p + 3$. Moreover, if $p - 2$ is a prime then $t \leq n$.*

Proof. Lemma 12 implies that $t(Alt_n) \geq 16$, and hence $t(S) \geq 15$. Values of $t(S)$ for all nonabelian simple groups S are obtained in [24], in particular, according to [24, Tables 1, 4], we have that if S is a sporadic simple group or exceptional simple group of Lie type, then $t(S) \leq 12$. So S is either an alternating group of degree at least p or a simple classical group.

Suppose that $S \simeq Alt_t$. If $t \geq p + 4$ then $2p \in \omega(G) \setminus \omega(Alt_n)$; a contradiction. Moreover, if $p - 2$ is a prime then $3(p - 2) \in \omega(Alt_{p+1}) \setminus \omega(Alt_p)$, $2(p - 2) \in \omega(Alt_{p+2}) \setminus \omega(Alt_{p+1})$, and $5(p - 2) \in \omega(Alt_{p+3}) \setminus \omega(Alt_{p+2})$. It remains to prove that S is not a classical group. Assume that this is false, and let u be the order of the base field of S, $m = \mathrm{prk}(S)$, and $u = v^k$, where v is a prime, k is an integer. Since p and 2 are nonadjacent in $GK(S)$ and $t(S) \geq 15$, by [23, Propositions 3.1, 4.1–4.4] we obtain that $p \neq v$ and $l = e(p, u) > 6$.

Lemma 19. *$\pi(k) \subseteq \pi(l)$.*

Proof. Assume that there exists prime $r \in \pi(k) \setminus \pi(l)$. Then $u = u_0^r$ for some integer u_0. Since $(l, r) = 1$, we have that $|R_l(u)| = |R_l(u_0^r)| > 1$ by Lemma 9. Therefore, there exists $p' \in R_l(u)$ such that $p' \neq p$, and hence p is adjacent with p' in $GK(S)$ due to Lemma 14. Since p can only be adjacent with 3 in $GK(S)$, we get a contradiction. □

Lemma 20. *Let $r \in R_i(\varepsilon u)$ and $i \geq \max((m - 5)/3, 7)$. If r divides $|\overline{G}/S|$ then there exists $r' \in R_i(\varepsilon u)$ such that r' does not divide $|\overline{G}/S|$.*

Proof. Assume that any element of $R_i(\varepsilon u)$ divides $|\overline{G}/S|$. Since $i \geq 7$, we have that r divides the order of some field automorphism of S and so r divides k. Therefore $u = u_0^r$, where u_0 is an integer. Since $e(r, \varepsilon u) = i$, we have that $r - 1$ is divisible by i, and hence $(i, r) = 1$. If $\varepsilon = +$ then by Lemma 9 we obtain that $|R_i(\varepsilon u)| = |R_i(u_0^r)| > 1$. If $\varepsilon = -$ and $i \not\equiv 2 \pmod 4$, then either $R_i(-u) = R_i(u)$ or $R_i(-u) = R_{2i}(u)$ by Lemma 8. Since $(2i, r) = 1$, Lemma 9 yields that $|R_i(-u)| > 1$. When $\varepsilon = -$ and $i \equiv 2 \pmod 4$, we have that $R_i(-u) = R_{i/2}(u)$. Since $i \geq 7$, we have that $i/2 \geq 4$, and since $i \equiv 2 \pmod 4$, we have that $i/2 \neq 6$. Now Lemma 9 implies that $|R_{i/2}(u)| > 1$. Therefore, there exists $r' \in R_i(\varepsilon u) \setminus \{r\}$.

By the assumption, we have that r' divides $|\overline{G}/S|$. As above, we obtain that r' divides k. By Lemma 19, it follows that l is divisible by rr', and so $l \geq rr'$. Since $r - 1$ and $r' - 1$ are divisible by i, we have that $l \geq (i + 1)^2$. On the other hand, it is obvious that $2m \geq l$, and so $2m \geq (i + 1)^2$. Since $i \geq 7$, we have that $m \geq 32$. Moreover, the inequality $i + 1 \geq (m - 2)/3$ holds true. Whence $2m \geq (m - 2)^2/9$. So $22m - 4 \geq m^2$. This inequality contradicts $m \geq 32$. The lemma is proved. □

Lemma 21. *Let $r \in R_i(\varepsilon u)$ and $i \geq \max(7, (m-5)/3)$. Then r does not divide $|K|$.*

Proof. Assume to the contrary that r divides $|K|$, in particular $r \neq p$. Denote the j-th prime number by p_j and let $p = p_y$, where y is an integer. Suppose that r

and p_{y-1} are nonadjacent in $GK(G)$. Then $\{r, p_{y-1}, p\}$ is a coclique in $GK(G)$, and hence $p_{y-1} \in \pi(S)$ by Lemma 10. Now Lemma 16 yields that either p_{y-1} or p divides the order of some proper parabolic subgroup P of S. Denote this prime by s. Observe that $s \notin \pi(v(u^2-1))$. Indeed, $t(s, S) \geq t(G)-1 \geq 15$. However, $t(v, S) \leq 4$ according to [23, Tables 4, 6] and $t(r_1(u), S), t(r_2(u), S) \leq 4$ by Lemma 14. Let R be a Hall $3'$-subgroup of K. By Lemma 2, we have that R is nilpotent. By the Frattini argument, we have that $N_G(R)/N_K(R) = G/K$, so we may assume that R is a r-group, R is normal in both K and G. Since S is simple, $p \in \pi(S)$, and p is nonadjacent with r in $GK(G)$, we obtain that $C_G(R) \leq K$. Since $O_{r'}(C_G(R))$ is normal in G, we can assume that $C_G(R) \leq R$. Lemma 17 implies that there exists subgroup H of S such that $H = T \rtimes C$, where T is a v-group, C is a cyclic group of order s, $[T, C] \neq 1$, and at least one of three assertion of Lemma 3 does not hold for H. Now we prove that such subgroup exists in G/R. If $R = K$ then H is the required group. If $R \neq K$ then $\widetilde{K} = K/R$ is a 3-group. Obviously $\widetilde{K}$ is normal in $\widetilde{G} = G/R$, and hence, if $v \neq 3$, then there exists a subgroup of $\widetilde{G}$ isomorphic to H in the full preimage of H by the Schur-Zassenhaus theorem. If $v = 3$ then we can use the full preimage H_1 of H. Indeed, $H_1 = T_1 \rtimes C_1$, where $C_1 \simeq C$ and $\widetilde{T_1} = T$. Since $[T, C] \neq 1$, we have that $[T_1, C_1] \neq 1$. It is obvious that assertion (iii) of Lemma 3 does not hold for H_1, so H_1 is the required group in this case. By the Schur-Zassenhaus theorem, G contains a subgroup isomorphic to H. So we can assume that H acts faithfully on R by conjugations, and hence H acts faithfully on $R/\Phi(R)$, where $\Phi(R)$ is the Frattini subgroup of R. Since $R/\Phi(R)$ is an elementary abelian group, we obtain a faithfully action of H on a vector space of characteristic r. By Lemma 3, we derive that $rs \in \omega(G)$; a contradiction. Thus either $r = p_{y-1}$ or r is adjacent with p_{y-1}. If $r = p_{y-1}$ then, since $t(G) \geq 16$, we have that $\{r, p_{y-2}, p\}$ is a coclique in $GK(G)$ and we get a contradiction as above. So r is adjacent with p_{y-1}, in particular $r + p_{y-1} \leq p + 3$.

According to [14, Table 1], we obtain that $(3m+5)/4 \geq t(S)$. Since $t(G) \geq \pi(p/2, p)$, Lemma 5 implies that $t(G) \geq 3p/(10\ln(p/2))$. By Lemma 10, we have that $t(S) \geq t(G) - 1$, whence

$$(3m+5)/4 \geq 3p/(10\ln(p/2)) - 1. \tag{1}$$

Now

$$p - p_{y-1} \geq r - 3 \geq 1 + (m-5)/3 - 3 = (m-11)/3. \tag{2}$$

Notice that

$$\frac{m-11}{3} = \frac{3m-33}{9} = \frac{4}{9} \cdot \frac{3m-33}{4} = \frac{4}{9} \cdot \frac{3m+5-38}{4}. \tag{3}$$

Combining (2) with (3) and (1), we obtain that

$$p - p_{y-1} \geq \frac{4}{9} \cdot \frac{3p}{10\ln(p/2)} - \frac{4}{9} - \frac{38}{9} = \frac{2p}{15\ln(p/2)} - \frac{42}{9} > \frac{2p}{15\ln(p)} - 5.$$

Suppose that $p > 90000$. Since 89989 is a prime, it is true that $p_{y-1} \geq 89989$. Lemma 6 implies that $p \leq p_{y-1}(1 + 1/\ln^3(p_{y-1}))$. Therefore, $p - p_{y-1} \leq p_{y-1}/\ln^3(p_{y-1}) < p/\ln^3(p)$. So $\frac{p}{\ln^3(p)} > \frac{2p}{15\ln(p)} - 5$. On the other hand, since $p > 90000$, we have that $\ln^2(p) > 30$, and hence $\frac{p}{\ln^3(p)} < \frac{p}{30\ln(p)} = \frac{2p}{15\ln(p)} - \frac{p}{10\ln(p)}$. The function $\frac{p}{\ln(p)}$ increases as p tends to the infinity, so if $p > 90000$ then $\frac{p}{\ln(p)} > 50$, and hence $\frac{p}{10\ln(p)} > 5$. Therefore, $\frac{p}{\ln^3(p)} < \frac{2p}{15\ln(p)} - 5$; a contradiction.

Suppose that $5000 < p \leq 90000$. Then, according to [28, Table 1], we have that $p - p_{y-1} \leq 72$. By (1), we conclude that $(3m + 5)/4 > \frac{3\cdot 5000}{10\cdot 8} - 1 > 186$. Whence $m > 246$. Applying (2), we obtain that $p - p_{y-1} > 78$; a contradiction.

Suppose that $1361 \leq p \leq 5000$. Then we have that $p - p_{y-1} \leq 34$. By (2), $34 \geq \frac{m-11}{3}$, and hence $m \leq 113$. On the other hand, as was mentioned above $(3m + 5)/4 \geq \pi(p/2, p) - 1$. Using a table of primes (e.g., see [22]) or GAP [25], we conclude that $\pi(p/2, p) \geq 95$. Whence $m \geq 123$; a contradiction.

Suppose that $1151 \leq p < 1361$. Then we have that $p - p_{y-1} \leq 22$. As above, $p - p_{y-1} \geq \frac{m-11}{3}$, and hence $m \leq 77$. However, if $1151 \leq p < 1361$ then $(3m+5)/4 \geq \pi(p/2, p) - 1 \geq 84$. Whence $m \geq 110$; a contradiction.

Suppose that $773 \leq p < 1151$. Then we have that $p - p_{y-1} \leq 20$. Therefore, $p - p_{y-1} \geq \frac{m-11}{3}$, and hence $m \leq 71$. On the other hand, $(3m+5)/4 \geq \pi(p/2, p) - 1 \geq 60$. Whence $m \geq 78$; a contradiction.

□

Now we are ready to get a contradiction. By Lemma 15, there exist $r_1, r_2, r_3, r_4 \in \pi(S) \setminus \{v\}$ such that $e(r_i, \varepsilon u) \geq \max(7, (m - 5)/3)$ for $1 \leq i \leq 4$, $r_1 r_2 \in \omega(S)$, $r_3 r_4 \in \omega(S)$, $r_1 r_4 \notin \omega(S)$, and $r_2 r_3 \notin \omega(S)$. Lemmas 20, 21 guarantee that we may choose r_1, r_2, r_3, r_4 in such a way that $(r_i, |\overline{G}/S| \cdot |K|) = 1$ for $1 \leq i \leq 4$. Therefore $r_1 r_2 \in \omega(Alt_n)$, $r_3 r_4 \in \omega(Alt_n)$, $r_1 r_4 \notin \omega(Alt_n)$, $r_2 r_3 \notin \omega(Alt_n)$. Whence, $r_1 + r_2 \leq n$, $r_1 + r_4 > n$, $r_3 + r_4 \leq n$, $r_3 + r_2 > n$. The first and second inequalities yield that $r_4 > r_2$, and the third and fourth inequalities yield that $r_4 < r_2$; a contradiction. The proposition is proved. □

Proposition 2. *If* $19 \leq p < 773$ *then* $S \simeq Alt_t$, *where* $p \leq t \leq p + 3$. *Moreover, if* $p - 2$ *is a prime then* $t \leq n$.

Proof. Suppose that the assertion is false. Arguing as in the proof of Proposition 1, we may assume that S is not an alternating group.

Suppose that $19 \leq p \leq 100$. In this case, all elements of $\mathfrak{S}_p$ and prime factorization of their orders are listed in [12, Table 1]. Let $p_1 < p_2 < p_3$ be the three largest primes less than p. It is not hard to verify that p_1, p_2, p_3, and p are pairwise nonadjacent in $GK(Alt_n)$. Lemma 10 implies that $|\{p_1, p_2, p_3\} \cap \pi(S)| \geq 2$. Verifying this inequality for groups in $\mathfrak{S}_p$ using [12, Table 1], we obtain that $p \in \{19, 23, 29, 43\}$ and $S \in \{^2E_6(2), Fi_{23}, Fi'_{24}, U_3(37), J_4\}$. We consider each possible p in turn.

Let $p = 19$. Then $S \simeq {}^2E_6(2)$. According to [1], we have that $7 \cdot 11 \in \omega(Alt_n) \setminus \omega(S)$. So either 7 or 11 divides $|K|$. Let U be a Sylow r-subgroup of K, where $r \in \{7, 11\}$. By the Frattini argument and Lemma 2, we may assume that U is a normal elementary abelian r-subgroup of K, K/U is a 3-group. Since $19r \notin \omega(G)$, we have that $C_G(U) \leq K$. Now $K = U \rtimes R$, where $R \in Syl_3(K)$. Therefore $O_3(C_K(U))$ is normal in G, and hence we can assume that $C_K(U) = U$. By [1], we have that ${}^2E_6(2)$ contains a subgroup isomorphic to $O_8^-(2)$. Since the order of the Schur multiplier of $O_8^-(2)$ is not divisible by 3, Lemma 4 implies that G/U contains a subgroup isomorphic to $O_8^-(2)$. By Lemma 1, we may suppose that the extension of $O_8^-(2)$ by U splits. Using [25], from the character table of $O_8^-(2)$ in the case $r = 11$ and the table of 7-modular characters of $O_8^-(2)$, when $r = 7$, we derive that $17r \in \omega(G)$; a contradiction.

Let $p = 23$. Then $S \simeq Fi_{23}$. Since $19 \in \pi(G) \setminus \pi(S)$ and $|Out(S)| = 1$, we obtain that $19 \in \pi(K)$. If $n \geq 24$ then 11 and 13 are adjacent in $GK(G)$ and nonadjacent in $GK(S)$. Therefore, we have that either $11 \in \pi(K)$ or $13 \in \pi(K)$. By Lemma 2, we obtain that $11 \cdot 19 \in \omega(G)$ or $13 \cdot 19 \in \omega(G)$; a contradiction. So $n = 23$. Since 7 and 11 are adjacent in $GK(G)$ and nonadjacent in $GK(S)$, we obtain that either $7 \in \pi(K)$ or $11 \in \pi(K)$. Lemma 2 implies $7 \cdot 19 \in \omega(G)$ or $11 \cdot 19 \in \omega(G)$; a contradiction.

Let $p = 29$. Then $n \in \{29, 30\}$ and $S \simeq Fi'_{24}$. Since $19 \notin \pi(S)$, we have that $19 \in \pi(K)$. Suppose that $n = 29$. Then 11 and 17 are adjacent in $GK(G)$ and nonadjacent in $GK(S)$. Therefore, we obtain that either $11 \in \pi(K)$ or $17 \in \pi(K)$. By Lemma 2, we obtain that $11 \cdot 19 \in \omega(G)$ or $17 \cdot 19 \in \omega(G)$; a contradiction. So $n = 30$. Then 13 and 17 are adjacent in $GK(G)$ and nonadjacent in $GK(S)$. Arguing as above, we obtain that $13 \cdot 19 \in \omega(G)$ or $17 \cdot 19 \in \omega(G)$; a contradiction.

Let $p = 43$. Then $S \in \{U_3(37), J_4\}$, and hence $17, 41 \notin \pi(S)$. Since $\{17, 41, 43\}$ is a coclique in $GK(G)$, this contradicts Lemma 10.

Suppose that $100 < p < 157$. Then according to [12], either $S \simeq L_2(p)$ or S is one of the groups from [12, Table 3] corresponding to p. Now we prove that $t(S) \leq 8$. Values of $t(S)$ for all nonabelian simple groups were obtained in [23, 24]. If $S = L_2(p)$ then according to [24, Table 2], we have that $t(S) = 3$. Let now S be a group from [12, Table 3] for some p with $100 < p < 157$, in particular, S is not a sporadic group. If S is a linear or unitary group then $\mathrm{prk}(S) \leq 12$, and hence by [24, Table 2] we obtain that $t(S) \leq 6$. If S is a symplectic or orthogonal group then $\mathrm{prk}(S) \leq 8$, and hence by [24, Table 3] we obtain that $t(S) \leq 7$. If S is an exceptional simple group of Lie type, then S is not of type E_8 according to [12, Table 3]. Now [24, Table 4] implies that $t(S) \leq 8$. So we have in all cases that $t(S) \leq 8$, and hence $t(Alt_n) \leq 9$ by Lemma 10. However, it is easy to verify that $t(Alt_n) \geq 11$, when $p > 100$; a contradiction.

Suppose finally that $157 \leq p < 773$. Then S is not a sporadic group. Now we prove that $t(S) \leq 12$. If S is a linear or unitary group then according to [12, Table 4] $\mathrm{prk}(S) \leq 12$, and hence by [24, Table 2] we obtain that $t(S) \leq 6$. If S is a symplectic or orthogonal group then $\mathrm{prk}(S) \leq 12$, and hence by [24, Table 3] we obtain that $t(S) \leq 10$. If S is an exceptional simple group of Lie type then $t(S) \leq 12$ according to [24, Table 4]. In all cases, we obtain that $t(S) \leq 12$, and hence $t(Alt_n) \leq 13$ by Lemma 10. On the other hand, Lemma 12 implies that $t(Alt_n) \geq 16$ as $p \geq 157$; a contradiction.

□

Proposition 3. *If* $p = 17$ *then* $S \simeq Alt_n$.

Proof. Assume that S is not an alternating group. By Lemma 10, we have that either $11 \in \pi(S)$ or $13 \in \pi(S)$. According to [12, Table 1], we have that $S \in \{U_4(4), U_3(17), O^-_{10}(2), L_2(13^2), S_4(13), L_3(16), S_6(4), O^+_8(4), F_4(2)\}$. Description of the spectra of simple linear and unitary groups is contained in [26, Corollary 3]. Using this description, we have that $(4^4-1)/5 = 17 \cdot 3 \in \omega(U_4(4))$, $(17^3+1)/54 = 7 \cdot 13 \in \omega(U_3(17))$, $13^2+1 = 10 \cdot 17 \in \omega(L_2(13^2))$, $(16^3-1)/45 = 7 \cdot 13 \in \omega(L_3(16))$. So in the cases we get a contradiction with $GK(G) = GK(Alt_n)$. Description of the spectra of simple symplectic and orthogonal groups is contained in [27, Corollaries 2–4, 6, 8, 9]. Using this description we obtain that $(13^2+1)/2 = 5 \cdot 17 \in \omega(S_4(13))$, $2 \cdot 17 \in \omega(S_6(4))$, $2 \cdot 17 \in \omega(O^+_8(4))$. Therefore $S \in \{O^-_{10}(2), F_4(2)\}$. Since $13 \notin \pi(O^-_{10}(2))$, $11 \notin \pi(F_4(2))$, we have that either 13 or 11 divides $|K|$. According to [1], both groups $O^-_{10}(2)$ and $F_4(2)$ contain a subgroup isomorphic to

$O_8^-(2)$. Therefore, we may assume that $G/K \simeq O_8^-(2)$. Lemma 2 yields that K is nilpotent, and hence we may assume that K is a 13-subgroup or 11-subgroup. Using [25], from the character table of $O_8^-(2)$, we obtain that either $11 \cdot 17 \in \omega(Alt_n)$ or $13 \cdot 17 \in \omega(Alt_n)$; a contradiction.

Now we prove that $S \simeq Alt_n$. Suppose on the contrary that $S \simeq Alt_t$, where $t \neq n$ and $t \geq 17$. If $n = 17$ then $t \geq 18$. Since $7 \cdot 11 \in \omega(Alt_t) \setminus \omega(Alt_n)$, we get a contradiction. So $n = 18$. If $t \geq 19$ then $19 \in \pi(Alt_n) \setminus \pi(Alt_t)$; a contradiction. Therefore $t = 17$. Since $7 \cdot 11 \in \omega(Alt_{18}) \setminus \omega(Alt_{17})$, we have that $\{7, 11\} \cap \pi(K) \neq \varnothing$. By Lemma 2, K is nilpotent, so we may assume that K is an elementary abelian r-subgroup, where $r \in \{7, 11\}$. From the tables of 7-modular and 11-modular characters for Alt_{17} (see [25]), we obtain that either $7 \cdot 13 \in \omega(G)$ or $11 \cdot 13 \in \omega(G)$; a contradiction. The proposition is proved. □

Proposition 4. *If $p = 13$ then $S \simeq Alt_t$, where either $t = n$ or $(n, t) \in \{(14, 13), (15, 13), (15, 14), (16, 13)\}$.*

Proof. Assume that S is not an alternating group. Suppose that $11 \notin \pi(S)$. Then $7 \in \omega(S)$. According to [12, Table 1], we obtain that $S \in \{L_2(13), L_2(27), G_2(3), {}^3D_4(2), Sz(8), L_2(64), U_4(5), L_3(9), S_6(3), O_7(3), G_2(4), S_4(8), O_8^+(3)\}$. By [26, Corollary 3] and [27, Corollary 3], we have that $5^2 + 1 = 2 \cdot 13 \in \omega(U_4(5))$, $64+1 = 5 \cdot 13 \in \omega(L_2(64))$, $(9^3 - 1)/8 = 7 \cdot 13 \in \omega(L_3(9))$, $8^2 + 1 = 5 \cdot 13 \in \omega(S_4(8))$. According to [1], the groups $G_2(3)$, $G_2(4)$, $S_6(3)$ contain maximal subgroups isomorphic to $L_2(13)$; $O_7(3)$ has a maximal subgroup isomorphic to $G_2(3)$; $O_8^+(3)$ has a maximal subgroup isomorphic to $O_7(3)$. So it remains to consider the cases when $S \in \{L_2(13), L_2(27), {}^3D_4(2), Sz(8)\}$. Since $11 \notin \pi(S)$, we have that $11 \in \pi(K)$. Let H be a Hall $3'$-subgroup of K. By Lemma 2, we obtain that H is nilpotent. So we may assume that H is a normal 11-subgroup of K, and K/H is a 3-group. Note that according to [1], the order of the Schur multiplier of S is not divisible by 3. Lemma 4 implies that G/H contains a subgroup isomorphic to S. So we may assume that S acts on H. From the character tables of $L_2(13)$, $L_2(27)$, ${}^3D_4(2)$, $Sz(8)$ (see [1] or [25]), we obtain that $11 \cdot 7 \in \omega(G) \setminus \omega(Alt_n)$; a contradiction.

Let now $11 \in \pi(S)$. According to [12, Table 1], we obtain that $S \in \{L_5(3), L_6(3), Suz, Fi_{22}\}$. By [26, Corollary 3], primes 2 and 13 are adjacent in both $GK(L_5(3))$ and $GK(L_6(3))$, so $S \not\simeq L_5(3), L_6(3)$. If $S \in \{Suz, Fi_{22}\}$ then $35 \in \omega(Alt_n) \setminus \omega(S)$, so either $5 \in \omega(K)$ or $7 \in \omega(K)$. As above, we may assume that K is an extension of a 3-group by an elementary abelian r-subgroup H, where $r \in \{5, 7\}$. Lemma 4 implies that G/H contains a subgroup isomorphic to either S or $3.S$, where $3.S$ is the perfect central extension of S by the group of order 3. From the tables of 5-modular and 7-modular characters of S and $3.S$, we obtain that an element of order 13 fixes some nonidentity element of H (see [25]). Therefore $13 \cdot 5 \in \omega(G) \setminus \omega(Alt_n)$ or $13 \cdot 7 \in \omega(G) \setminus \omega(Alt_n)$; a contradiction. Thus $S \simeq Alt_t$, where $t \geq 13$.

Assume that $n = 13$. If $t \geq 14$ then $3 \cdot 11 \in \omega(Alt_t) \setminus \omega(Alt_n)$. So $t = 13$. Let $n = 14$. If $t \geq 15$ then $2 \cdot 11 \in \omega(Alt_t) \setminus \omega(Alt_n)$. So $t = 13$ or $t = 14$.

Let $n = 15$. If $t \geq 16$ then $5 \cdot 11 \in \omega(Alt_t) \setminus \omega(Alt_n)$. So $13 \leq t \leq 15$.

Let $n = 16$. If $t \geq 17$ then $17 \in \pi(Alt_n) \setminus \pi(Alt_t)$. So $t \leq 16$. If $t = 14, 15$ then $5 \cdot 11 \in \omega(Alt_n) \setminus \omega(Alt_t)$, so either 5 or 11 divides $|K|$. As above, we may assume that K has a normal elementary abelian r-subgroup H such that K/H is a 3-group, $C_K(H) \leq H$, and $r \in \{5, 11\}$. Since the order of the Schur multiplier of S is not divisible by 3, Lemma 4 implies that G/H contains a subgroup isomorphic to S. From the tables of 5-modular and 11-modular characters of S, we obtain that either $5 \cdot 13 \in \omega(G)$ or $11 \cdot 13 \in \omega(G)$; a contradiction. Thus $t = 13$. □

4. Proof of Theorem 2

In this section we prove Theorem 2. Let $n \geq 5$ be an integer and p is the largest prime less than or equal to n. Suppose that Alt_n is recognizable. As mentioned in Introduction, if $n - p \geq 3$ then $GK(Alt_{p+3}) = GK(Alt_{p+3} \times Z_3)$. So $n - p \leq 2$. Theorem 2 will follow from the next lemma, where we construct a finite group $G \not\simeq Alt_n$ such that $GK(G) = GK(Alt_n)$, where n either equals $p + 2$ and is composite, or $n = p, p + 1$ and $p - 2$ is composite.

Lemma 22. *Let $L = Alt_n$, where $n = p$ or $n = p + 1$ for a prime $p \geq 5$. Let $\rho : L \to GL_{p-1}(V)$ be the standard irreducible 2-modular representation of L of degree $p - 1$, and G be the natural semidirect product $V \rtimes \rho(L)$. Then*
(i) $2p \notin \omega(G)$ and if $p - 2$ is a prime, then $2(p - 2) \in \omega(G)$.
(ii) $GK(L) = GK(G)$ if and only if $p - 2$ is composite.
(iii) If $n = p + 1$ then $GK(G) = GK(Alt_{p+2})$ if and only if $p + 2$ is composite.

Proof. Let $e = (e_1, e_2, \ldots, e_n)$ be a basis of an n-dimensional vector space over the field of order 2. If $n = p$ then V can be chosen as $\langle e_1+e_2, e_2+e_3, \ldots, e_{p+1}+e_p\rangle$, and we take $f_1 = e_1+e_2$, $f_2 = e_2+e_3$,…, $f_{p-1} = e_{p-1}+e_p$ as a basis f of V. If $n = p+1$ then V can be chosen as the quotient space of $U = \langle e_1+e_{p+1}, e_2+e_{p+1}, \ldots, e_p+e_{p+1}\rangle$ by the line $\langle e_1 + e_2 + \ldots + e_{p+1}\rangle$. Let $\overline{x}$ denotes the image of a vector $x \in U$ in V. Then we can take $f_1 = \overline{e_1} + \overline{e_{p+1}}$, $f_2 = \overline{e_2} + \overline{e_{p+1}}$,…, $f_{p-1} = \overline{e_{p-1}} + \overline{e_{p+1}}$ as a basis f of V. Let $g = (1, 2, 3, \ldots, p)$ be the cycle of order p in L. Then $(e_1)g = e_2$, $(e_2)g = e_3$,…, $(e_{p-1})g = e_p$, $(e_p)g = e_1$, and if $n = p + 1$ then $(e_{p+1})g = e_{p+1}$. So g transforms f in the following way $(f_1)g = f_2$, $(f_2)g = f_3$,…, $(f_{p-2})g = f_{p-1}$. If $n = p$ then $(f_{p-1})g = e_1 + e_p = f_1 + f_2 + \ldots + f_{p-1}$. If $n = p + 1$ then $(f_{p-1})g = \overline{e_p} + \overline{e_{p+1}} = \overline{e_1} + \overline{e_2} + \ldots + \overline{e_{p-1}} = f_1 + f_2 + \ldots + f_{p-1}$. Therefore g has the following matrix with respect to the basis f:

$$[g]_f = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$$

Denote such matrix of size $k \times k$ for $k \geq 2$ by A_k, and let E_k denote the identity matrix of size k. Then $det(A_k - \lambda E_k) = \lambda^k + \lambda^{k-1} + \ldots + \lambda + 1$. In our case $\dim V = p - 1$ is even, so 1 is not an eigenvalue for $[g]_f$. Since all elements of order p are conjugated in Sym_p and Sym_{p+1}, we have that all elements of order p in L have trivial centralizers in V, in particular $2p \notin \omega(G)$. Let $h = (1, 2, \ldots, (p - 2))$ be the cycle of order $(p - 2)$ in L. Then obviously $(f_{p-1})h = f_{p-1}$. So (i) is proved.

If $p - 2$ is composite then $GK(G) = GK(L)$, since the prime graphs of these groups may differ only in the edge between 2 and p. On the other hand, if $p - 2$ is prime, by (i) we obtain that $2(p - 2) \in \omega(G) \setminus \omega(L)$.

If $p + 2$ is a prime then $p + 2 \in \pi(Alt_{p+2}) \setminus \pi(G)$. Let $p + 2$ be composite. Then $GK(Alt_{p+2})$ may differ from $GK(Alt_{p+1})$ only if $p - 2$ is prime. In this case, 2 and $p - 2$ are adjacent in $GK(Alt_{p+2})$ and nonadjacent in $GK(Alt_{p+1})$. However, as we proved above, 2 and $p - 2 = |h|$ are adjacent in $GK(G)$, and since 2 and p are nonadjacent in $GK(G)$, we obtain that $GK(G) = GK(Alt_{p+2})$, as required. If $p-2$ is composite then $GK(G) = GK(Alt_{p+1}) = GK(Alt_{p+2})$. The lemma and theorem are proved. □

5. Proof of Theorem 3

In this section we prove Theorem 3. By Propositions 3, 4, it remains to prove that Alt_{13} is recognizable by prime graph, consider the cases $p = 11, 19$, and show that each possibility for S from the statement of Theorem 3 is realizable. The following lemmas complete the proof of Theorem 3.

Lemma 23. *Suppose that $p = 11$. If $n = 11$ then $S \simeq Alt_{11}$. If $n = 12$ then $S \in \{L_2(11), M_{11}, U_5(2), Alt_{11}, Alt_{12}\}$ and for each possible S there exists a finite group G such that $GK(G) = GK(Alt_{12})$ and G has a composition factor isomorphic to S.*

Proof. Let $7 \in \pi(S)$ and S is not an alternating group. Then by [12, Table 1], we obtain that $S \in \{M_{22}, McL, HS, U_6(2)\}$. Using [1], we obtain that $3 \cdot 7 \notin \omega(S)$. Note that 7 does not divide $|Out(S)|$, and if $S \neq U_6(2)$, then 3 does not divide $|Out(S)|$. If $S = U_6(2)$ and $|\overline{G}/S|$ is divisible by 3, then according to [1], the group $\overline{G}$ contains a subgroup isomorphic to $Z_3 \times U_5(2)$, and hence $3 \cdot 11 \in \omega(G)$; a contradiction. So $3 \in \pi(K)$ or $7 \in \pi(K)$. We may assume that K is an elementary abelian 3-group or 7-group. Using [25], from the tables of 3-modular and 7-modular characters of S, we derive that either $3 \cdot 11 \in \omega(G)$ or $7 \cdot 11 \in \omega(G)$; a contradiction.

Let now $7 \notin \pi(S)$ and S is not an alternating group. Then according to [12, Table 1], we have that $S \in \{L_2(11), M_{11}, M_{12}, U_5(2)\}$. Since 7 does not divide $|\overline{G}/S|$, we conclude that $7 \in \pi(K)$. We may assume that K is an elementary abelian 7-group. Let $n = 11$. Then $5 \cdot 7 \notin \omega(Alt_n)$. From the table of 7-modular characters for S, we obtain that $5 \cdot 7 \in \omega(G)$; a contradiction. So $n = 12$. If $S = M_{12}$ then from the table of 7-modular characters of S, we obtain that $7 \cdot 11 \in \omega(G)$; a contradiction.

Suppose that S is an alternating group, in particular $S \in \{Alt_{11}, Alt_{12}\}$. Notice that $5 \cdot 7 \in \omega(Alt_{12}) \setminus \omega(Alt_{11})$, so if $n = 11$ then $S = Alt_{11}$.

Let $S \in \{L_2(11), M_{11}, U_5(2)\}$. Notice that $2 \cdot 3 \in \omega(S)$ in these cases. According to [25], the group S has irreducible 5-modular and ordinary representations on some 10-dimensional vector spaces V_1 and V_2, respectively, such that all elements of order 11 do not fix nontrivial vectors in V_1 and V_2, and every element of order 2 or 3 fixes some nontrivial vectors in V_1 and V_2.

We may assume that V_2 is a vector space over a field of characteristic 7. Put $G = (V_1 \times V_2) \rtimes S$. It is easy to verify that $GK(G) = GK(Alt_{12})$.

It remains to construct the group G in the case $n = 12$ and $S = Alt_{11}$. Consider the standard irreducible 5-modular representation of Alt_{11} on a vector space V of dimension 10. According to [25], all elements of order 11 in the image of Alt_{11} under this representation have trivial centralizer in V, however, some element of order 7 has nontrivial centralizer in V. So if $G = V \rtimes Alt_{11}$ is the natural semidirect product of V and Alt_{11}, then $GK(G) = GK(Alt_{12})$. The lemma is proved.

□

Lemma 24. *Alt_{13} is recognizable by prime graph.*

Proof. Suppose that G is a finite group and $GK(G) = GK(Alt_{13})$. By Theorem 1, we obtain that $Alt_{13} \leq G/K \leq Sym_{13}$, where K is the soluble radical of G

and Sym_{13} is the symmetric group of degree 13. Suppose that $K \neq 1$. Then we may assume that K is an elementary abelian r-group, where r is less than 13 by Lemma 10. If $r \neq 11$, then using [25], from the table of r-modular characters of Alt_{13}, we obtain that $r \cdot 11 \in \omega(G)$; a contradiction. If $r = 11$ then from the table of 11-modular characters of Alt_{13}, we derive that $7 \cdot 11 \in \omega(G)$; a contradiction. Therefore $K = 1$. Since $22 \in \omega(Sym_{13})$, we conclude that $G = Alt_{13}$. □

Remark. Notice that Alt_{13} is the smallest alternating group recognizable by prime graph. Indeed, $GK(Alt_5) = GK(Alt_6)$, $GK(Alt_7) = GK(L_3(4).2_1)$ (the notation $L_3(4).2_1$ is taken from GAP [25]), $GK(Alt_8) = GK(V \rtimes Alt_7)$, where $V \rtimes Alt_7$ is the natural semidirect of Alt_7 and a vector space V of dimensional 6 over the field of order 3 (see [25]). As mentioned in Introduction, there exist soluble groups with prime graphs $GK(Alt_9)$, $GK(Alt_{10})$, $GK(Alt_{12})$. The group Alt_{11} is not recognizable by prime graph due to Theorem 2.

Lemma 25. *If $p = 19$ and $n \leq 21$, then $S \simeq Alt_t$, where either $n = t$ or $(n,t) = (21, 20)$.*

Proof. By Proposition 2, we have that $S \simeq Alt_t$, where $19 \leq t \leq n$. So if $n = 19$ then $t = n$. Now we prove that $t \neq 19$ if $20 \leq n \leq 21$. Assume the contrary. Since $13 \cdot 7 \in \omega(Alt_n) \setminus \omega(Alt_{19})$, we have that 7 or 13 divides $|K|$. By Lemma 2, the group K is nilpotent. So we may suppose that K is an elementary abelian r-subgroup, where $r = 7$ or $r = 13$. Note that S contains subgroups isomorphic to Alt_{17} and Alt_{18}. From 13-modular and 7-modular character tables of Alt_{17} and Alt_{18}, respectively, we obtain that either $13 \cdot 11 \in \omega(G)$ or $7 \cdot 17 \in \omega(G)$; a contradiction. □

Lemma 26. *Let n be an integer and $14 \leq n \leq 21$. Suppose that t is an integer such that the pair (n,t) is one of the following: $(14, 13)$, $(15, 13)$, $(15, 14)$, $(16, 13)$, $(21, 20)$. Then there exists a finite group G such that $GK(G) = GK(Alt_n)$ and G has a composition factor isomorphic to Alt_t.*

Proof. If $(n,t) \in \{(15, 14), (21, 20)\}$ then the required groups G are constructed in Lemma 22. So it remains to consider the cases $(n,t) \in \{(14, 13), (15, 13), (16, 13)\}$.

Assume that $n = 14$ and $t = 13$. Obviously the graph $GK(Alt_{14})$ differs from $GK(Alt_{13})$ only in the edge between 3 and 11. Consider the standard irreducible 3-modular representation of Alt_{13} on a vector space V of dimension 12. By the values of the corresponding character (see [25]), all elements of order 13 in Alt_{13} have trivial centralizers in V, and elements of order 11 have nontrivial centralizers. Let $G = V \rtimes Alt_{13}$ be the natural semidirect product of V and Alt_{13}. Then $GK(G) = GK(Alt_{14})$, and hence G is the required group.

Assume that $n = 15$ and $t = 13$ then the graph $GK(Alt_{15})$ differs from $GK(Alt_t)$ only in two edges between 2 and 11, 3 and 11. Consider the standard irreducible 2-modular and 3-modular representations of Alt_{13} on vector spaces V_1 and V_2, respectively. By the character tables of these representations (see [25]), all elements of order 13 in Alt_{13} have trivial centralizers in V_1, V_2 and elements of order 11 have nontrivial centralizers. Let $G = (V_1 \times V_2) \rtimes Alt_{13}$. Then $GK(G) = GK(Alt_{15})$.

Assume that $n = 16$ and $t = 13$. Note that $GK(Alt_{16})$ differs from $GK(Alt_{13})$ in exactly four edges: between 3 and 13, 2 and 11, 3 and 11, 5 and 11. Consider the standard irreducible 2-modular and 5-modular representation of Alt_{13} on vector spaces V_1 and V_2, respectively. Using [25], we obtain that the elements of order 13 have trivial centralizers in V_1 and V_2, at the same time, the elements of order

11 have nontrivial centralizers. According to [25], the group Alt_{13} has a 3-modular irreducible representation on a vector space V_3 of dimension 64 such that elements of orders 11 and 13 have nontrivial centralizers in V_3. If G is the natural semidirect product of the groups $V_1 \times V_2 \times V_3$ and Alt_{13}, then $GK(Alt_{16}) = GK(G)$. The lemma and theorem are both proved.

□

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References

[1] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985. MR0827219

[2] W.J. Shi, *A characteristic property of* A_5, J. Southwest-China Teach. Univ., **11** (1986), 11–14.

[3] I.B. Gorshkov, *Recognizability of alternating groups by spectrum*, Algebra Logic, **52**:1 (2013), 41–45. MR3113477

[4] R. Brandl, W. Shi, *Finite groups whose element orders are consecutive integers*, J. Algebra, **1 43**:2 (1991), 388–400. MR1132578

[5] V.D. Mazurov, *Recognition of finite groups by a set of orders of their elements*, Algebra Logic, **37**:6 (1998), 371–379. MR1680412

[6] Y.V. Lytkin, *On groups critical with respect to a set of natural numbers*, Sib. Electron. Math. Rep., **10** (2013), 666–675. MR3262316

[7] A.M. Staroletov, *Groups isospectral to the degree 10 alternating group*, Siberian Math. J., **51**:3 (2010), 507–514. MR2683105

[8] A.S. Kondrat′ev, *Finite groups with the same prime graph as the group* A_{10}, Proc. Steklov Inst. Math. **285** (suppl. 1) (2014), 99–107. Zbl 1307.20021

[9] B. Khosravi, A. Zarea Moghanjoghi, *Quasirecognition by prime graph of some alternating groups*, Int. J. Contemp. Math. Sci., **2**:25–28 (2007), 1351–1358. MR2380198

[10] M.A. Zvezdina, *On nonabelian simple groups with the same prime graph as an alternating group*, Sib. Math. J., **54**:1 (2013), 47–55. MR3089327

[11] M.R. Zinov′eva, V.D. Mazurov, *On finite groups with disconnected prime graph*, Proc. Steklov Inst. Math., **283**:1 (2013), 139–145. MR3476387

[12] A.V. Zavarnitsine, *Finite simple groups with narrow prime spectrum*, Sib. Electr. Math. Rep., **6** (2009), 1–12. MR2586673

[13] V.D. Mazurov, A.V. Zavarnitsine, *Element orders in coverings of symmetric and alternating groups*, Algebra and Logic, **38**:3 (1999), 159–170. MR1766731

[14] A. V. Vasil′ev, *On finite groups isospectral to simple classical groups*, J. Algebra, **423** (2015), 318–374. MR3283720

[15] J.B. Rosser, L.Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math., **6**:1 (1962), 64–94. MR0137689

[16] P. Dusart, *Explicit estimates of some functions over primes*, Ramanujan J., (2016) doi:10.1007/s11139-016-9839-4.

[17] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. Phys., **3** (1892), 265–284. JFM 24.0176.02

[18] M.A. Grechkoseeva, D.V. Lytkin, *Almost recognizability by spectrum of finite simple linear groups of prime dimension*, Sib. Math. J., **53**:4 (2012), 645–655. MR3013528

[19] A.V. Vasil′ev, *On a relation between the structure of a finite group and the properties of its prime graph*, Siberian Math. J., **46**:3 (2005), 396–404. MR2164556

[20] A.V. Vasil′ev, I B Gorshkov, *On the recognition of finite simple groups with a connected prime graph*, Siberian Math. J., **50**:2 (2009), 233–238. MR2531755

[21] J. S. Williams, *Prime graph components of finite groups*, J. Algebra, **69**:2 (1981), 487–513. MR0617092

[22] D.N. Lehmer, *List of prime numbers from 1 to 10006721*, Washington, DC: Carnegie Institution, 1914.

[23] A.V. Vasil′ev, E.P. Vdovin, *An adjacency criterion for the prime graph of a finite simple group*, Algebra Logic, **44**:6 (2005), 381–406. MR2213302

[24] A.V. Vasil′ev, E. P. Vdovin, *Cocliques of maximal size in the prime graph of a finite simple group*, Algebra Logic, **50**:4 (2011), 291–322. MR2893582

[25] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.5 (2016), (http://www.gap-system.org).

[26] A.A. Buturlakin, *Spectra of finite linear and unitary groups*, Algebra Logic, **47**:2 (2008), 91–99. MR2438007

[27] A.A. Buturlakin, *Spectra of finite symplectic and orthogonal groups*, Siberian Adv. in Math., **21**:3 (2011), 176–210.

[28] L.J. Lander, T.R. Parkin, *On first appearance of prime differences*, Math. Comp., **21** (1967), 483–488. MR0230677

Alexey Mikhailovich Staroletov
Sobolev Institute of Mathematics,
4 Acad. Koptyug avenue,
630090, Novosibirsk, Russia

Novosibirsk State University,
2 Pirogova Str.,
630090, Novosibirsk, Russia
E-mail address: staroletov@math.nsc.ru