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EXPANDING 1-INDISCERNIBLE COUNTABLY CATEGORICAL WEAKLY O-MINIMAL THEORIES BY EQUIVALENCE RELATIONS

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Abstract. Here we investigate expansions of 1-indiscernible countably categorical weakly o-minimal structures by an equivalence relation. We present a criterion when such an expansion preserves both countable categoricity and weak o-minimality.

Keywords: weak o-minimality, countable categoricity, convexity rank, expansion of models.

1. Introduction

Let L be a countable first-order language. Throughout this article we consider L-structures and assume that L contains a binary relation symbol $<$, which is interpreted as a linear ordering in these structures. The present paper deals with the notion of *weak o-minimality*, originally deeply investigated by D. Macpherson, D. Marker, and C. Steinhorn in [1]. A subset A of a linearly ordered structure M is said to be *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of M is a union of finitely many convex sets in M. Recall that such a structure $\mathcal{M}$ is called *o-minimal* if every definable (with parameters) subset of the structure $\mathcal{M}$ is a union of finitely many intervals and points in $\mathcal{M}$. Thus, weak o-minimality is a generalization of o-minimality. Real closed fields with a proper convex valuation ring provide an important class of examples of weakly o-minimal (not o-minimal) structures.

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Let A, B be arbitrary subsets of a linearly ordered structure M. Then the expression $A < B$ means that $a < b$ whenever $a \in A$ and $b \in B$. The expression $A < b$ means that $A < \{b\}$. We denote by A^+ (and, respectively, A^-) the set of elements b of the considered structure M with the condition $A < b$ ($b < A$).

Definition 1. [2] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of the theory T, and let $\phi(x)$ be an arbitrary M-definable formula with one free variable.

The *rank of convexity of the formula* $\phi(x)$ ($RC(\phi(x))$) is defined as follows:

1) $RC(\phi(x)) \geq 1$, if $\phi(M)$ is infinite.

2) $RC(\phi(x)) \geq \alpha+1$, if there exist a parametrically definable equivalence relation $E(x, y)$ and an infinite number of elements $b_i, i \in \omega$, such that:

- For any $i, j \in \omega$, whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$
- For each $i \in \omega$ $\quad RC(E(x, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of $\phi(M)$

3) $RC(\phi(x)) \geq \delta$, if $RC(\phi(x)) \geq \alpha$ for all $\alpha \leq \delta$ (δ is limit).

We say $RC(\phi(x)) = \alpha$ if $RC(\phi(x)) \geq \alpha$ holds and $RC(\phi(x)) \geq \alpha + 1$ doesn't hold.

If $RC(\phi(x)) = \alpha$ for some α, then we say that $RC(\phi(x))$ is defined. Otherwise (i.e. if $RC(\phi(x)) \geq \alpha$ for all α), we assume $RC(\phi(x)) = \infty$.

Particularly, a theory has the convexity rank 1 if there is no definable (with parameters) equivalence relation with an infinite number of convex infinite classes.

In this paper we investigate the problem of preserving properties at expanding models of countably categorical weakly o-minimal theories by binary predicates. Earlier in the works [3] – [5] we have studied the problem of preserving properties for expansions of models of countably categorical weakly o-minimal theories by unary predicates. As it is known, in work [6] B.S. Baizhanov proved that the expansion of a model of a weakly o-minimal theory by a unary predicate that distinguishes a finite number of convex sets preserves weak o-minimality of the expanded theory. However, in the case of expanding a model of a weakly o-minimal theory by a binary predicate that distinguishes a finite number of convex sets for each fixed both the first and the second parameter, the expanded theory can lose weak o-minimality (Example 1).

Recall some notions originally introduced in [1].

Let $Y \subset M^{n+1}$ be an $\emptyset$–definable set, let $\pi : M^{n+1} \to M^n$ be the projection which drops the last coordinate, and let $Z := \pi(Y)$. For each $\bar{a} \in Z$ let $Y_{\bar{a}} := \{y : (\bar{a}, y) \in Y\}$. Suppose that for every $\bar{a} \in Z$ the set $Y_{\bar{a}}$ is bounded above, but does not have a supremum in M. We let $\sim$ be the $\emptyset$–definable equivalence relation on M^n, given by

$$\bar{a} \sim \bar{b} \text{ for all } \bar{a}, \bar{b} \in M^n \setminus Z, \text{ and } \bar{a} \sim \bar{b} \Leftrightarrow \sup Y_{\bar{a}} = \sup Y_{\bar{b}}, \text{ if } \bar{a}, \bar{b} \in Z.$$

Let $\overline{Z} := Z/\sim$, and for each tuple $\bar{a} \in Z$ we denote by $[\bar{a}]$ the $\sim$-class of the tuple $\bar{a}$. There exists a natural $\emptyset$–definable total order on $M \cup \overline{Z}$, defined as follows. Let $\bar{a} \in Z$ and $c \in M$. Then $[\bar{a}] < c$ if and only if $w < c$ for all $w \in Y_{\bar{a}}$. If $\bar{a} \not\sim \bar{b}$, then there exists some $x \in M$ such that $[\bar{a}] < x < [\bar{b}]$ or $[\bar{b}] < x < [\bar{a}]$, and therefore $<$ induces a total order on $M \cup \overline{Z}$. We call such a set $\overline{Z}$ a *sort* (in this case, $\emptyset$–definable sort) in $\overline{M}$, where $\overline{M}$ is the Dedekind completion of the structure M, and consider

$\overline{Z}$ as naturally embedded in $\overline{M}$. Similarly, we can obtain a sort in $\overline{M}$, considering infima instead of the suprema.

Definition 2. [1] Let M be a linearly ordered structure, $D \subseteq M$ be an infinite set, $K \subseteq \overline{M}$, $f : D \to K$ be a function. We say that f is *locally increasing (locally decreasing, locally constant)* on D if for any $x \in D$ there exists an infinite interval $J \subseteq D$, containing $\{x\}$, so that f is strictly increasing (strictly decreasing, constant) on J.

We also say that a function f is *locally monotone* on D if f is either locally increasing or locally decreasing on D.

Proposition 1. [7] *Let M be a weakly o-minimal structure, $A \subseteq M$, $p \in S_1(A)$ be a non-algebraic type. Then any function in the A-definable sort whose domain contains the set $p(M)$ is locally monotone or locally constant on $p(M)$.*

Example 1. Let $M := \langle \mathbb{R}, < \rangle$ be a linearly ordered structure on the set of real numbers $\mathbb{R}$. It is obvious that M is a model of a countably categorical o-minimal theory. Expand the model M by a new binary relation $S(x, y)$ as follows: let $M' := \langle \mathbb{R}, <, S^2 \rangle$ be such that $S(x, y)$ is the graph of the following unary function f, defined as $f(b) = 2b$ for each $b \in \mathbb{Q}$ and $f(c) = -c$ for each $c \in \mathbb{R} \setminus \mathbb{Q}$. It is obvious that $S(a, M)$ and $S(M, a)$ for each $a \in M$ are singleton sets, i.e. convex sets. Nevertheless, note that M' is not weakly o-minimal, since there is no partition of the set $\mathbb{R}$ into a finite number of convex sets, on each of which the definable function f is locally monotone or locally constant.

We say M is an *1-indiscernible* structure if for any $a, b \in M$ $tp(a/\emptyset) = tp(b/\emptyset)$.

Example 2. [8] Let $M_n := \langle \mathbb{Q}^n; <, E_1^2, E_2^2, \ldots, E_{n-1}^2 \rangle$, where $\mathbb{Q}^n$ is the set of n-tuples $x = (x_0, \ldots, x_{n-1})$ of rational numbers, ordered lexicographically by $<$, and for each $i = 1, \ldots, n-1$ let the equivalence relation E_i be given by $E_i(x, y) \Leftrightarrow x_j = y_j$ for all $j < n - i$. Then for each i the equivalence classes of E_i are convex subsets of $\mathbb{Q}^n$. Moreover, E_{n-1} refines E_i for each $2 \leq i \leq n - 1$.

It is easy to show that M_n is an 1-indiscernible countably categorical weakly o-minimal structure and $Th(M_n)$ has convexity rank n.

Proposition 2. [9] *Let M be an 1-indiscernible countably categorical weakly o-minimal structure of finite convexity rank. Then there is $n \in \omega$ such that M is isomorphic to $M_n := \langle \mathbb{Q}^n; <, E_1^2, E_2^2, \ldots, E_{n-1}^2 \rangle$ (Example 2).*

Here we confine ourselves to an investigation of the problem of preserving both countable categoricity and weak o-minimality for expansions of models of 1-indiscernible countably categorical weakly o-minimal theories of finite convexity rank by an equivalence relation partitioning the universe of the model into infinitely many infinite convex classes.

2. Results

Example 3. Let $M := \langle \mathbb{Q}, < \rangle$ be a linearly ordered structure on the set of rational numbers $\mathbb{Q}$. It is obvious that M is a countably categorical o-minimal structure. Expand the model M by a new binary relation $E(x, y)$ as follows:

let $M' := \langle \mathbb{Q}, <, E^2 \rangle$ be such that for any $a, b \in \mathbb{Q}$

$$E(a, b) \Leftrightarrow (2n - 1)\sqrt{2} < a, b < (2n + 1)\sqrt{2}$$

for some $n \in \mathbb{Z}$.

Then it is not difficult to understand that $E(x, y)$ is an equivalence relation that partitions $\mathbb{Q}$ into infinitely many infinite convex classes, and the E-classes are ordered by the type $\omega^* + \omega$.

By simple quantifier elimination it is a routing to show that M' is a weakly o-minimal structure. Observe that $Th(M')$ is not countably categorical because the ordered set of integers is interpretable as M'/E.

Example 4. Let $M := \langle \mathbb{Q} \times \mathbb{Q}, <, E^2 \rangle$ be a linearly ordered structure on the set $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically. The relation $E(x, y)$ is defined as follows:

$$\text{for any } a = (m_1, n_1), b = (m_2, n_2) \in \mathbb{Q} \times \mathbb{Q} \quad E(a, b) \Leftrightarrow m_1 = m_2.$$

It is obvious that $E(x, y)$ is an equivalence relation that partitions $\mathbb{Q} \times \mathbb{Q}$ into infinitely many infinite convex classes, and the E-classes are ordered by the type $\mathbb{Q}$.

Extend the universe $\mathbb{Q} \times \mathbb{Q}$ of the structure M by adding two elements to each E-class, which are the left and the right endpoints of the E-class. As a result, we get a new structure $M' := \langle M', <, E^2 \rangle$. Consider the reduct of the structure M' to the structure $M'' := \langle M', < \rangle$. It is obvious that M'' is a countably categorical o-minimal structure. Its expansion $M' := \langle M', <, E^2 \rangle$ is a countably categorical linearly ordered structure.

Consider the following formula:

$$\phi(x) := \exists y_1 \exists y_2 [y_1 < x < y_2 \wedge \forall z \forall t (y_1 \leq z < x \wedge x < t \leq y_2 \rightarrow \neg E(z, t))]$$

This formula says that x is an endpoint of some E-class. Observe that $\phi(M')$ is a union of infinitely many convex sets. Thus, $Th(M')$ is not weakly o-minimal.

Proposition 3. *Let M be an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank 1, M' be an expansion of the model M by an equivalence relation $E(x, y)$ partitioning M into infinitely many infinite convex classes. Then $Th(M')$ is a countably categorical weakly o-minimal theory iff the following conditions hold:*

(1) There exist only finitely many E-classes having at least one endpoint;

(2) There exist only finitely many E-classes having an immediate predecessor or an immediate successor in the induced ordering on M/E.

Proof. ($\Rightarrow$) Consider the formula $\phi(x)$ from Example 4. Understand that $\phi(M)$ is finite. If $\phi(M)$ is infinite then by weak o-minimality it contains an infinite interval, but endpoints of infinite convex E-classes cannot form an infinite interval. Thus, $\phi(M)$ is finite, which implies that there are only finitely many E-classes having at least one endpoint.

We now understand that condition (2) holds. Assume the contrary: there are infinitely many E-classes having an immediate predecessor or an immediate successor.

Case 1. For each $m < \omega$ there exists a discretely ordered chain of E-classes of length n_1 for some $m \leq n_1 < \omega$.

Then by compactness there exists a model N' of the theory $Th(M')$, in which there is an infinite discretely ordered chain of E-classes. Without loss of generality, suppose that such a chain is ordered by the type ω. Consider the following formulas:

$$F_1(x, y) := E(x, y)$$

$$F_2(x,y) := F_1(x,y) \vee \forall t(x \leq t \leq y \wedge \neg F_1(x,t) \rightarrow E(t,y))$$

$$\ldots\ldots\ldots$$

$$F_n(x,y) := F_{n-1}(x,y) \vee \forall t(x \leq x \leq y \wedge \neg F_{n-1}(x,t) \rightarrow E(t,y)), \quad n < \omega$$

Observe that $F_1(a, N')$ defines the class $E(a, N')$, and $F_n(a, N')$ for each $n < \omega$ defines the class $E(a, N')$ and the $n-1$ E-classes immediately following it.

Then we obtain that there exists $a \in N'$ such that

$$F_1(a, N') \subset F_2(a, N') \subset \ldots \subset F_n(a, N') \subset \ldots$$

which contradicts countable categoricity of $Th(M')$.

Case 2. There exists $m < \omega$ with a discretely ordered chain of E-classes of length m and m is maximal with this property.

Then there exists $k < \omega$ such that $2 \leq k \leq m$ and there is an infinite number of chains of length k. Then there exists $a \in M'$ such that $F_k(a, M')$ is a union of infinitely many convex sets, which contradicts weak o-minimality of $Th(M')$.

($\Leftarrow$) By (1) there are only finitely many E-classes that have at least one endpoint. Then, by the linear ordering of M' each such endpoint is definable. By (2) we can also define by a formula each E-class having an immediate predecessor or an immediate successor, as well as possible nonempty intervals between some of these classes (those intervals where E-classes are densely ordered without endpoints); Moreover, E-classes that are minimal (the leftmost E-class) or maximal (the rightmost E-class) in intervals with dense ordering of E-classes are distinguished. Thus, we obtain a finite number of $\emptyset$-definable formulas $\theta_i(x), 1 \leq j \leq n$, so that for any $1 \leq i < j \leq n$

$$\theta_i(M') \cap \theta_j(M') = \emptyset.$$

Each of these formulas defines some 1-type over $\emptyset$. By standard methods it is easy to show that $Th(M')$ admits quantifier elimination up to atomic formulas and the formulas $\theta_1(x), \theta_2(x), \ldots, \theta_n(x)$ (the last formulas define convex sets in M'), hence we get that $Th(M')$ is a countably categorical weakly o-minimal theory. □

Corollary 1. *Let M be an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank 1, M' be an expansion of the model M by an equivalence relation $E(x,y)$ partitioning M into infinitely many infinite convex classes. Then M' is an 1-indiscernible countably categorical weakly o-minimal structure iff the following conditions hold:*

(1) Each E-class has no endpoints in M';

(2) The induced order on E-classes is a dense linear order without endpoints.

Example 5. Let $M' := \langle \mathbb{Q}, <, E^2 \rangle$ be the structure from Example 3. We replace each point $a \in \mathbb{Q}$ by copy of rational numbers and define a new structure $M'' := \langle \mathbb{Q} \times \mathbb{Q}, <, E^2, E_0^2 \rangle$, where the relation $E_0(x,y)$ defines as follows:

$$E_0(a,b) \Leftrightarrow m_1 = m_2 \text{ for any } a = (m_1, n_1), b = (m_2, n_2) \in \mathbb{Q} \times \mathbb{Q}$$

Then it is not difficult to understand that $E_0(x,y)$ is an equivalence relation partitioning every E-class into infinitely many infinite convex classes such that the E_0-subclasses of each E-class are densely ordered without endpoints.

It can be proved that M'' is a weakly o-minimal structure, but the theory $Th(M'')$ is not countably categorical.

Example 6. Take a countable number of copies of the structure $M' := \langle \mathbb{Q}, <, E^2 \rangle$ from Example 3, ordered by type $\mathbb{Q}$. Then we get a new structure $M'' := \langle \mathbb{Q}\times\mathbb{Q}, < E^2, E_1^2 \rangle$, where the relation $E_1(x, y)$ is defined as follows:

$$E_1(a, b) \Leftrightarrow m_1 = m_2 \text{ for all } a = (m_1, n_1), b = (m_2, n_2) \in \mathbb{Q}\times\mathbb{Q}$$

Then $E(a, M'') \subset E_1(a, M'')$ for any $a \in M''$ and $E_1(x, y)$ is an equivalence relation partitioning M'' into infinitely many infinite convex classes, ordered by type $\mathbb{Q}$. Note that E-subclasses of each E_1-class are ordered by type $\omega^* + \omega$.

We can also understand that M'' is a weakly o-minimal structure, but $Th(M'')$ is not countably categorical.

Theorem 1. *Let M be an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank n, and let $E_1(x, y), E_2(x, y), \ldots, E_{n-1}(x, y)$ be $\emptyset$-definable equivalence relations partitioning M into infinitely many infinite convex classes, such that for any $a \in M$*

$$E_1(a, M) \subset E_2(a, M) \subset \ldots \subset E_{n-1}(a, M).$$

Let M' be an expansion of the model M by a new equivalence relation $E^*(x, y)$ partitioning M' into infinitely many infinite convex classes. Then $Th(M')$ is a countably categorical weakly o-minimal theory if and only if the following conditions hold:

(A) There exist only finitely many E^*-classes having at least one endpoint;

(B) There exist only finitely many E^*-classes having an immediate predecessor or an immediate successor in the induced ordering on M/E^*;

(C) M' is partitioned into finitely many infinite convex sets $X_1, \ldots, X_m$ such that for each $1 \le i \le m$ exactly one of the following items holds:

$(1)_s$ *there is $1 \le s \le n-1$ such that $E^*(a, M) = E_s(a, M)$ for all $a \in X_i$;*

$(2)_1$ *$E^*(a, M) \subset E_1(a, M)$, $\sup E^*(a, M) < \sup E_1(a, M)$ and $\inf E_1(a, M) < \inf E^*(a, M)$ for all $a \in X_i$;*

$(2)_{j+1}$ *there is $1 \le j \le n-2$ such that $E_j(a, M) \subset E^*(a, M) \subset E_{j+1}(a, M)$, $\sup E_j(a, M) < \sup E^*(a, M)$, $\inf E_j(a, M) < \inf E^*(a, M)$, $\sup E^*(a, M) < \sup E_{j+1}(a, M)$ and $\inf E_{j+1}(a, M) < \inf E^*(a, M)$ for all $a \in X_i$;*

$(2)_n$ *$E_{n-1}(a, M) \subset E^*(a, M)$, $\sup E_{n-1}(a, M) < \sup E^*(a, M)$ and $\inf E^*(a, M) < \inf E_{n-1}(a, M)$ for all $a \in X_i$;*

$(3)_s$ *there is $1 \le s \le n-1$ such that $X_i = E^*(a, M)$ for some $a \in M$, $E^*(a, M) \subset E_s(a, M)$ and either $\sup E^*(a, M) = \sup E_s(a, M)$ or $\inf E^*(a, M) = \inf E_s(a, M)$;*

$(4)_s$ *there is $1 \le s \le n-1$ such that $X_i = E_s(a, M)$ for some $a \in M$, $E_s(a, M) \subset E^*(a, M)$ and either $\sup E_s(a, M) = \sup E^*(a, M)$ or $\inf E_s(a, M) = \inf E^*(a, M)$;*

$(5)_s$ *there is $1 \le s \le n-1$ such that $X_i = E^*(a, M) \cap E_s(a, M)$ for some $a \in M$, $E^*(a, M) \setminus E_s(a, M) \neq \emptyset$ and $E_s(a, M) \setminus E^*(a, M) \neq \emptyset$.*

Proof. ($\Rightarrow$) (A) and (B) follows from the proof of Proposition 3. Next, we consider the following formulas for all $1 \le s \le n-1$ and $1 \le j \le n-2$:

$$\phi_{1s}^{\emptyset}(x) := \forall y[E_s(x, y) \leftrightarrow E^*(x, y)]$$

i.e. $\phi_{1s}^{\emptyset}(x)$ defines the set of elements $a \in M$ such that $E^*(a, M) = E_s(a, M)$.

$$\phi_{21}^{\emptyset}(x) := \forall y[E^*(x, y) \to E_1(x, y)] \wedge \exists z_1 \exists z_2 (z_1 < x < z_2 \wedge E_1(z_1, z_2) \wedge$$

$$\wedge \neg E^*(z_1, x) \wedge \neg E^*(x, z_2))]$$

i.e. $\phi^{\emptyset}_{21}(x)$ defines the set of elements $a \in M$ such that $E^*(a, M) \subset E_1(a, M)$, $\sup E^*(a, M) < \sup E_1(a, M)$ and $\inf E_1(a, M) < \inf E^*(a, M)$.

$$\phi^{\emptyset}_{2,j+1}(x) := \forall y[(E_j(x, y) \to E^*(x, y)) \wedge (E^*(x, y) \to E_{j+1}(x, y))]\wedge$$
$$\wedge\exists t_1\exists z_1\exists z_2\exists t_2(t_1 < z_1 < x < z_2 < t_2 \wedge E_{j+1}(t_1, t_2) \wedge E^*(z_1, z_2)\wedge$$
$$\wedge\neg E^*(t_1, z_1) \wedge \neg E^*(z_2, t_2) \wedge \neg E_j(z_1, x) \wedge \neg E_j(x, z_2))]$$

i.e. $\phi^{\emptyset}_{2,j+1}(x)$ defines the set of elements $a \in M$ such that $E_j(a, M) \subset E^*(a, M) \subset E_{j+1}(a, M)$, $\sup E_j(a, M) < \sup E^*(a, M)$, $\inf E_j(a, M) < \inf E^*(a, M)$, $\sup E^*(a, M) < \sup E_{j+1}(a, M)$ and $\inf E_{j+1}(a, M) < \inf E^*(a, M)$.

$$\phi^{\emptyset}_{2n}(x) := \forall y[E_{n-1}(x, y) \to E^*(x, y)]\wedge$$
$$\wedge\exists z_1\exists z_2(z_1 < x < z_2 \wedge E^*(z_1, z_2) \wedge \neg E_{n-1}(z_1, x) \wedge \neg E_{n-1}(x, z_2)]$$

i.e. $\phi^{\emptyset}_{2n}(x)$ defines the set of elements $a \in M$ such that $E_{n-1}(a, M) \subset E^*(a, M)$, $\sup E_{n-1}(a, M) < \sup E^*(a, M)$ and $\inf E^*(a, M) < \inf E_{n-1}(a, M)$.

$$\phi^{r}_{3s}(x) := \forall y[E^*(x, y) \to E_s(x, y)] \wedge \forall y_1[x < y_1 \wedge \neg E^*(x, y_1) \to \neg E_s(x, y_1)]$$

i.e. $\phi^{r}_{3s}(x)$ defines classes $E^*(a, M)$ for some $a \in M$ such that $E^*(a, M) \subset E_s(a, M)$ and $\sup E^*(a, M) = \sup E_s(a, M)$.

$$\phi^{l}_{3s}(x) := \forall y[E^*(x, y) \to E_s(x, y)] \wedge \forall y_1[x > y_1 \wedge \neg E^*(x, y_1) \to \neg E_s(x, y_1)]$$

i.e. $\phi^{l}_{3s}(x)$ defines classes $E^*(a, M)$ for some $a \in M$ such that $E^*(a, M) \subset E_s(a, M)$ and $\inf E^*(a, M) = \inf E_s(a, M)$.

$$\phi^{r}_{4s}(x) := \forall y[E_s(x, y) \to E^*(x, y)] \wedge \forall y_1[x < y_1 \wedge \neg E_s(x, y_1) \to \neg E^*(x, y_1)]$$

i.e. $\phi^{r}_{4s}(x)$ defines classes $E_s(a, M)$ for some $a \in M$ such that $E_s(a, M) \subset E^*(a, M)$ and $\sup E_s(a, M) = \sup E^*(a, M)$.

$$\phi^{l}_{4s}(x) := \forall y[E_s(x, y) \to E^*(x, y)] \wedge \forall y_1[x > y_1 \wedge \neg E_s(x, y_1) \to \neg E^*(x, y_1)]$$

i.e. $\phi^{l}_{4s}(x)$ defines classes $E_s(a, M)$ for some $a \in M$ such that $E_s(a, M) \subset E^*(a, M)$ and $\inf E_s(a, M) = \inf E^*(a, M)$.

$$\phi^{r}_{5s}(x) := \exists y_1\exists y_2[y_1 < x < y_2 \wedge E_s(y_1, x) \wedge E^*(x, y_2) \wedge \neg E_s(x, y_2) \wedge \neg E^*(y_1, x)]$$

i.e. $\phi^{r}_{5s}(x)$ defines non-empty intersections $E^*(a, M) \cap E_s(a, M)$ for some $a \in M$ such that there exist $b_1 \in E_s(a, M) \setminus E^*(a, M)$ and $b_2 \in E^*(a, M) \setminus E_s(a, M)$ with $b_1 < b_2$.

$$\phi^{l}_{5s}(x) := \exists y_1\exists y_2[y_1 < x < y_2 \wedge E^*(y_1, x) \wedge E_s(x, y_2) \wedge \neg E^*(x, y_2) \wedge \neg E_s(y_1, x)]$$

i.e. $\phi^{l}_{5s}(x)$ defines non-empty intersections $E^*(a, M) \cap E_s(a, M)$ for some $a \in M$ such that there exist $b_1 \in E_s(a, M) \setminus E^*(a, M)$ and $b_2 \in E^*(a, M) \setminus E_s(a, M)$ with $b_1 > b_2$.

It is easy to see that for any $a \in M'$ there exist i, s, ε such that $1 \le i \le 5$, $1 \le s \le n-1$, $\varepsilon \in \{\emptyset, r, l\}$ and $M' \models \phi^{\varepsilon}_{is}(a)$, and also that

$$M' \models \neg\exists x[\phi^{\varepsilon_1}_{is}(x) \wedge \phi^{\varepsilon_2}_{jk}(x)]$$

for any $i, j, s, k, \varepsilon_1, \varepsilon_2$ such that $1 \le i, j \le 5$, $1 \le s, k \le n-1$, $\varepsilon_1, \varepsilon_2 \in \{\emptyset, r, l\}$ provided that $i \ne j$, $s \ne k$ or $\varepsilon_1 \ne \varepsilon_2$.

By weak o-minimality of $Th(M')$, each of these formulas defines a set that is the union of a finite number of convex sets, which implies (C).

($\Leftarrow$) Fulfillment of conditions (A) and (B) according to the proof of Proposition 3 define by a formula the available endpoints of E-classes; each E-class, having an immediate predecessor or an immediate successor, as well as gaps where E-classes are densely ordered without endpoints; in addition, minimal or maximal E-classes are distinguished in the intervals of dense ordering of E-classes having the leftmost or rightmost E-class. Thus, we obtain a finite number of $\emptyset$-definable formulas $\theta_i(x), 1 \leq j \leq n$, such that for any $1 \leq i < j \leq n$

$$\theta_i(M') \cap \theta_j(M') = \emptyset.$$

The fulfillment of condition (C) ensures that each of the formulas $\phi^{\varepsilon}_{is}(x)$, $1 \leq i \leq 6$, $1 \leq s \leq n-1$, $\varepsilon \in \{\emptyset, r, l\}$, defines a set that is the union of a finite number of convex sets. Then by linear ordering of M' each of the formulas $\phi^{\varepsilon}_{is}(x)$ decomposes into a finite number of convex $\emptyset$-definable formulas $\phi^{\varepsilon 1}_{is}(x), \phi^{\varepsilon 2}_{is}(x), \ldots, \phi^{\varepsilon n^{\varepsilon}_{is}}_{is}(x)$ for some $n^{\varepsilon}_{is} < \omega$.

We understand that there are only finitely many E^*-classes (and, consequently, E_s-classes for some $1 \leq s \leq n-1$) defined by the following formulas:

$$\phi^r_{6s}(x) := \exists z(x < z \wedge \neg E_s(x,z) \wedge \forall t_1(E^*(z,t_1) \rightarrow \neg E_s(x,t_1)) \wedge$$
$$\wedge \forall t_2(E_s(x,t_2) \rightarrow \neg E^*(z,t_2)) \wedge \forall u[x \leq u \leq z \rightarrow E_s(x,u) \vee E^*(u,z)])$$

i.e. $\phi^r_{6s}(x)$ defines classes $E_s(a, M)$ for some $a \in M$ which are immediately followed by an E^*-class.

$$\phi^l_{6s}(x) := \exists z(x < z \wedge \neg E^*(x,z) \wedge \forall t_1(E_s(z,t_1) \rightarrow \neg E^*(x,t_1)) \wedge$$
$$\wedge \forall t_2(E^*(x,t_2) \rightarrow \neg E_s(z,t_2)) \wedge \forall u[x \leq u \leq z \rightarrow E^*(x,u) \vee E_s(u,z)])$$

i.e. $\phi^r_{6s}(x)$ defines classes $E^*(a, M)$ for some $a \in M$ which are immediately followed by an E_s-class.

Assume the contrary: there are an infinite number of E^*-classes and E_s-classes defined by the formula $\phi^l_{6s}(x)$. Then we assert that $\phi^l_{6s}(M')$ is the union of an infinite number of disjoint convex sets. In fact, if $b \in \phi^l_{6s}(M')$, then there exists $a \in M'$ such that

$$E^*(b, M') \cap E_s(a, M) = \emptyset$$

and $b < a$. Since the condition (B) holds, we obtain that there exists an infinite convex part of $E_s(a, M')$ satisfying the formula $\phi^{\emptyset}_{2s}(x)$, whence $\phi^{\emptyset}_{2s}(M')$ is the union of an infinite number of disjoint convex sets, which contradicts the condition (C).

Finally, it can be proved by standard methods that $Th(M')$ admits quantifier elimination up to atomic formulas and formulas $\theta_1(x), \theta_2(x), \ldots, \theta_n(x)$, $\phi^{\varepsilon 1}_{11}(x)$, …, $\phi^{\varepsilon n^{\varepsilon 1}_{1,n-1}}_{1,n-1}(x); \ldots;$ $\phi^{\varepsilon 1}_{51}(x), \ldots, \phi^{\varepsilon n^{\varepsilon}_{5,n-1}}_{5,n-1}(x)$, from which we obtain $Th(M')$ is a countably categorical weakly o-minimal theory. □

Corollary 2. *Let M be an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank n, and let $E_1(x,y), E_2(x,y), \ldots, E_{n-1}(x,y)$ be $\emptyset$-definable equivalence relations partitioning M into infinitely many infinite convex classes, such that for any $a \in M$*

$$E_1(a, M) \subset E_2(a, M) \subset \ldots \subset E_{n-1}(a, M).$$

Let M' be an expansion of the model M by a new equivalence relation $E^*(x,y)$ partitioning M' into infinitely many infinite convex classes. Then M' is an 1-indiscernible countably categorical weakly o-minimal structure if and only if the following conditions hold:

(A) *each E^*-class does not have endpoints in M';*

(B) *the induced order on E^*-classes is a dense linear order without endpoints;*

(C) for any $a \in M'$ exactly one of the following $2n-1$ items holds:

$(1)_s$ there is $1 \leq s \leq n-1$ such that $E^*(a,M) = E_s(a,M)$;

$(2)_1$ $E^*(a,M) \subset E_1(a,M)$, $\sup E^*(a,M) < \sup E_1(a,M)$ and $\inf\ E_1(a,M) < \inf E^*(a,M)$;

$(2)_{j+1}$ there is $1 \leq j \leq n-2$ such that $E_j(a,M) \subset E^*(a,M) \subset E_{j+1}(a,M)$, $\sup E_j(a,M) < \sup\ E^*(a,M)$, $\inf\ E_j(a,M) < \inf E^*(a,M)$, $\sup E^*(a,M) < \sup E_{j+1}(a,M)$ and $\inf E_{j+1}(a,M) < \inf E^*(a,M)$;

$(2)_n$ $E_{n-1}(a,M) \subset E^*(a,M)$, $\sup E_{n-1}(a,M) < \sup E^*(a,M)$ and $\inf E^*(a,M) < \inf E_{n-1}(a,M)$.

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