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DYNAMIC CONTACT ALGEBRAS AND QUANTIFIER-FREE LOGICS FOR SPACE AND TIME

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Abstract. The paper is in the field of Region Based Theory of Space (RBTS), sometimes called mereotopology. RBTS is a kind of point-free theory of space based on the notion of region. Its origin goes back to some ideas of Whitehead, De Laguna and Tarski to build the theory of space without the use of the notion of point. Contact algebras present an algebraic formulation of RBTS and in fact give axiomatizations of the Boolean algebras of regular closed sets of some classes of topological spaces with an additional relation of contact. Dynamic contact algebras (DCA), are generalizations of contact algebras studying regions changing in time and present formal explications of Whitehead's ideas of integrated point-free theory of space and time. In the present paper we propose several new types of dynamic contact algebras and quantifier-free logics based on them. The logics are finitary, based on Modus Ponens and some nonstandard inference rules which replace the non-universal axioms of the corresponding DCA. In fact these logics can be considered as axiomatizations of the universal fragment of the first-order theory of the corresponding DCA. They can be treated as kinds of non-standard temporal logics for spatial regions changing in time. The difference with standard temporal logic is that we do not use temporal operators but temporal predicates. We present also snapshot semantics and Kripke style relational semantics for some of these logics, based on relational models for the corresponding DCA. The relational semantics helps to use some techniques adapted from modal logic to study some metalogical properties of the studied systems.

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Introduction

The paper is in the field of Region Based Theory of Space (RBTS), sometimes called mereotopology. RBTS is a kind of point-free theory of space based on the notion of region and some mereological and mereotopological relations between regions. Its origin goes back to some ideas of Whitehead, De Laguna and Tarski to build the theory of space based on mereology without the use of the notion of point. More information on RBTS, mereotopology and its applications in computer science can be found, for instance in [1, 2, 3]. Contact algebras present an algebraic formulation of RBTS and in fact give axiomatizations of the Boolean algebras of regular closed sets of some classes of topological spaces with an additional relation of contact between regions. An exhaustive study of this theory is given in [4]. Dynamic contact algebra (DCA), introduced by the second author in [5], is a generalization of contact algebra studying regions changing in time and presents a formal explication of Whitehead's ideas of integrated point-free theory of space and time. More information and motivations about this theory and relevant references can be found in [5]. The axiomatic system of DCA is extracted from a concrete model called snapshot model describing the changing regions by a series of snapshots. Namely this model disclose the intuition formalized in the abstract definition of DCA.

In the present paper we propose three new versions of DCA (to be introduced in the main text): "strong dynamic contact algebra" (strong dca) - by adding two new axioms to DCA which are true in the snapshot model, "weak dynamic contact algebra" (weak dca) - by removing the so called Efremovich axiom from DCA, and "basic dynamic contact algebra (basic dca)", which is based on universal first-order axioms and is intended as the universal fragment of the mentioned three versions (note that the abbreviation DCA is used only for the algebras introduced in [5] and for the general notion of dynamic contact algebra we use the abbreviation dca). The motivation to introduce strong dca is that it has a simple relational model, called "relational dynamic space". This model appeared to be characteristic also for the weak dca in a sense that weak dca-s are representable in this kind of relational models. This shows that both, weak dca-s and DCA-s can be considered as certain subalgebras of strong dca-s. We introduce quantifier-free logics in the style of [6] corresponding to all mentioned notions of dca-s. Let us note that these logics are finitary, based on propositional logic, Modus Ponens and some nonstandard inference rules which replace the non-universal axioms of the corresponding dca. In fact these logics are axiomatizations of the universal fragment of the first-order theory of the mentioned dca-s. We prove strong and weak completeness theorems for these logics and show that all mentioned logics have equal and decidable sets of theorems. This shows that if we consider only weak completeness, all logics coincide with the logic corresponding to the basic dca, which do not possess non-standard rules. As a side product, this also shows that in proving theorems in these logics non-standard rules can be eliminated. All mentioned above quantifier free logics can be considered as a non-standard kind of temporal logic for spatial regions changing in time. The difference with standard temporal logic is that we do not use temporal operators but temporal predicates, which are certain spatio-temporal relations between changing (dynamic) regions. We present three kinds of semantics for these logics: algebraic semantics based on the above three kinds of dca, relational (Kripke style) semantics, based on relational models of dca-s, and snapshot semantics based on snapshot models. The relational semantics helps also to use some techniques adapted from modal logic in the study of some metalogical properties of the mentioned logical systems.

The structure of the paper is the following. Section 1 presents some preliminary information about contact and precontact relations in Boolean algebras and surveys the basic facts about the version of DCA introduced in [5]. In some sense we consider the present paper as a continuation of [5], so [5] will be considered also as a source of facts needed for the present exposition. In order to make the text more understandable and easy to follow, all notions, constructions and intuitive explanations from [5] will be given with details. Section 2 is devoted to strong and weak dca-s and Section 3 - for the relational representation theory of weak dca-s. Section 4 includes the theory of basic dca-s and their relationships to weak and strong dca-s. Section 5 is devoted to the quantifier-free logics for space and time. In Section 6 we mention some open problems and plans for future research.

We consider [7] and [8] correspondingly as standard books for Boolean algebras and Topology.

1. Preliminaries

1.1. **Contact and precontact relations in Boolean algebras.**

Definition 1.1. Contact relations [4]. *Let* $(B, 0, 1, \leq, +, ., *)$ *be a non-degenerate Boolean algebra and* C *be a binary relation in* B. C *is called a* ***contact*** *relation in* B *if the following axioms are satisfied:*

(C1) If aCb *then* $a \neq 0$ *and* $b \neq 0$,

(C2) If aCb *and* $a \leq a'$ *and* $b \leq b'$ *then* $a'Cb'$,

(C3') If $aC(b + c)$ *then* aCb *or* aCc,

(C3") If $(a + b)Cc$ *then* aCc *or* bCc,

(C4) If aCb *then* bCa,

(C5) If $a.b \neq 0$ *then* aCb.

We write $\overline{C}$ *for the complement of* C. *If* C *is a contact relation in* B, *then the pair* (B, C) *is called a* ***contact algebra***.

Let us mention that on the base of (C4) only one of the axioms (C3') and (C3") is needed. Note also that (C5) is equivalent to the following more simple axiom

(C5') If $a \neq 0$ then aCa.

The elements of the Boolean algebra are called regions and are considered as abstractions of spatial bodies and Boolean operations are considered as operations for constructing new regions from given ones. In this way Boolean algebra is considered as a model of mereology, a philosophical discipline of parts and wholes [9].

In the present context we treat the Boolean part of the contact algebra as its *mereological component* and the contact relation - as its *mereotopological component*. In our treating of mereology we consider the zero element 0 as a *non-existing region* and this can be used to define the ontological predicate of existence in the following way: *a ontologically exists* iff $a \neq 0$. For simplicity, instead of "ontologically exists" we will say simply "exists" and from the context it will be clear that this is not the existential quantifier. The predicate of existence will be very important in the theory of dynamic contact algebras.

The definitions of the main mereological relations "part-of" and "overlap" are the following:

• a is part of b iff $a \leq b$, i.e. part-of is just the Boolean ordering,

• a overlaps b (in symbols $a\mathrm{O}b$) iff there exists a region $c \neq 0$ such that $c \leq a$ and $c \leq b$ iff $a.b \neq 0$.

The main example of contact algebra is the Boolean algebra of regular-closed sets of topological space, called regions and two regions are in a contact if they have a common point [4]. It is proved in [4] that every contact algebra is representable as a contact algebra of topological kind.

Below we will give a non-topological (relational) example of of Boolean algebra with contact relation.

Definition 1.2. Precontact relations. *Let* $(B, 0, 1, \leq, +, ., *)$ *be a non-degenerate Boolean algebra and* C *be a binary relation in* B. C *is called a* ***precontact*** *relation in* B *if it satisfies the axioms (C1), (C2), (C3') and (C3"). If* C *is a* precontact relation *in* B *then the pair* (B, C) *is called a* precontact algebra. *Precontact relations will be used later on in the formalization of some temporal relations between changing regions in dynamic contact algebras.*

Precontact algebras were considered under another name in [10]. We will be interested later on contact and precontact algebras satisfying the following additional axiom for C_R:

(CE) If $a\overline{C}b$ then $(\exists c)(a\overline{C}c$ and $c^*\overline{C}b)$.

This axiom is called sometimes *Efremovich axiom*, because it is used in the definition of Efremovich proximity spaces.

The following construction from [10] gives examples of Boolean algebras with precontact relations. Let (W, R) be a relational system with W a non-empty set and R be a binary relation in W (such pairs are called in [10] adjacency spaces). For subsets a, b of W define aC_Rb iff there exist points $x \in a$ and $y \in b$ such that xRy. Then C_R is a precontact relation. Every precontact algebra is representable as a contact algebra of such a kind [10]. The following fact is proved in [10]:

Lemma 1.3. *(i)* C_R *satisfies the axiom C4 iff* R *is a symmetric relation in* W.

(ii) C_R *satisfies the axiom C5 iff* R *is a reflexive relation in* W.

(iii) C_R *satisfies the Efremovich axiom (CE) iff* R *is a transitive relation in* W.

If (W, R) is a relational system with reflexive and symmetric relation R, then by Lemma 1.3 (i) and (ii) the precontact relation C_R is a contact relation in the Boolean algebra of all subsets of W. This non-topological example of contact algebra is typical in a sense that every contact algebra is representable as a contact algebra of this form (see [10]).

Let (W, R, S) be a relational system with two relations. We consider the following two first-order conditions for R and S:

$(R \circ S \subseteq S)$ If xRy and ySz, then xSz (The composition of R with S is included in S).

$(S \circ R \subseteq S)$ If xSy and yRz, then xSz (The composition of S with R is included in S).

We consider also the following two conditions for precontact relations C_R and C_S similar to the Efremowich axiom (CE):

(C_RC_S) If $a\overline{C}_Sb$, then there exists $c \subseteq W$ such that $a\overline{C}_Rc$ and $c^*\overline{C}_Sb$, and

(C_SC_R) If $a\overline{C}_Sb$, then there exists $c \subseteq W$ such that $a\overline{C}_Sc$ and $c^*\overline{C}_Rb$.

We call the conditions (C_RC_S) and (C_SC_R) **compositional axioms** for C_R and C_S

The proof of the following lemma is similar to the proof of Lemma 1.3 (iii):

Lemma 1.4. *(i) The condition (C_RC_S) is fulfilled between precontact relations C_R and C_S iff the condition $(R \circ S \subseteq S)$ is satisfied,*

(ii) The condition (C_SC_R) is fulfilled between precontacts relations C_R and C_S iff the condition $(S \circ R \subseteq S)$ is satisfied.

1.2. **Dynamic contact algebras, abstract definition.** We adopt from [5] Part III the following definition of dynamic contact algebra:

Definition 1.5. *By a **dynamic contact algebra** (DCA for short) we mean any system $\underline{B} = (B, 0, 1, ., +, ^*, C^s, C^t, \mathcal{B}, TR, UTR, NOW)$ where $(B, 0, 1, ., +, ^*)$ is a non-degenerate Boolean algebra, and the following conditions are satisfied:*
*(i) C^s is a contact relation on B called **space contact**,*
*(ii) C^t is a contact relation on B, called **time contact** satisfying the following additional axioms:*
$(C^s \to C^t)$ $aC^sb \to aC^tb$,
(C^tE) *If $a\overline{C}^tb$, then $(\exists c)(a\overline{C}^tc$ and $c^*\overline{C}^tb)$, the Efremovich axiom for C^t (see 1.1),*
*(iii) $\mathcal{B}$ is a precontact relation in B called **precedence relation**.*
*(iv) TR - **time representatives**, and UTR - **universal time representatives**, are subsets of B satisfying the following axioms:*
(TR1) $c \in TR$ iff $c \neq 0$ and $(\forall a, b)(aC^tc$ and $bC^tc \to aC^tb)$.
(TR2) $c \in UTR$ iff $c \in TR$ and $c\overline{C}^tc^*$.
(TRC^t) If aC^tb, then $(\exists c \in UTR)(aC^tc$ and $bC^tc)$.
(TRC^s) If aC^sb then $(\exists c \in UTR)((a.c)C^sb)$.
($TR\mathcal{B}1$) If $c \in TR$, $c\mathcal{B}b$ and aC^tc, then $a\mathcal{B}b$.
($TR\mathcal{B}2$) If $d \in TR$, $a\mathcal{B}d$ and bC^td, then $a\mathcal{B}b$.
($TR\mathcal{B}3$) If $a\mathcal{B}b$, then $\exists c \in UTR$ such that $c\mathcal{B}b$ and aC^tc.
($TR\mathcal{B}4$) If $a\mathcal{B}b$, then $\exists d \in UTR$ such that $a\mathcal{B}d$ and bC^td.
In the next axioms $c(i)$ and $c(j)$ are arbitrary elements of UTR.
($UTR\mathcal{B}11$) $(\forall p \in B)(p\mathcal{B}c(i)$ or $p^*\mathcal{B}c(j))$ iff $(\exists c(k) \in UTR)(c(k)\mathcal{B}c(i)$ and $c(k)\mathcal{B}c(j))$.
($UTR\mathcal{B}12$) $(\forall p \in B)(p\mathcal{B}c(i)$ or $c(j)\mathcal{B}p^*)$ iff $(\exists c(k) \in UTR)(c(k)\mathcal{B}c(i)$ and $c(j)\mathcal{B}c(k))$.
($UTR\mathcal{B}21$) $(\forall p \in B)(c(i)\mathcal{B}p$ or $p^*\mathcal{B}c(j))$ iff $(\exists c(k) \in UTR)(c(i)\mathcal{B}c(k)$ and $c(k)\mathcal{B}c(j))$.
($UTR\mathcal{B}22$) $(\forall p \in B)(c(i)\mathcal{B}p$ or $c(j)\mathcal{B}p^*)$ iff $(\exists c(k) \in UTR)(c(i)\mathcal{B}c(k)$ and $c(j)\mathcal{B}c(k))$.
$(UTRNOW)$ $NOW \in UTR$.

Since DCA-s are algebraic systems, we adopt for them the standard definitions of subalgebra, homomorphism, isomorphism, isomorphic embedding, etc. Note that axioms (TR1) and (TR2) show that the sets TR and UTR are definable with first-order formulas of the relation C^t. We however include these sets in the signature of DCA, because we want they to be preserved in the representation theory of DCA-s.

Remark 1.6. *(1) The implications from right to left in the axioms $UTR\mathcal{B}11$, $UTR\mathcal{B}12$, $UTR\mathcal{B}121$ and $UTR\mathcal{B}22$ are provable by some (universal) axioms of DCA and hence are superfluous.*

As an example we present the proof of the following formula which implies the implication from the right to the left part in axiom $UTR\mathcal{B}21$:

If $c \in UTR$, $a\mathcal{B}c$, and $c\mathcal{B}b$, then $a\mathcal{B}p$ or $p^*\mathcal{B}b$.

Suppose that this implication is not true. Then we have: (1) $c \in UTR$, (2) $a\mathcal{B}c$, (3) $c\mathcal{B}b$, (4) $a\overline{\mathcal{B}}p$ and (5) $p^*\overline{\mathcal{B}}b$. From (2) and (4) we get (by axioms (TR2) and (TR$\mathcal{B}$3)) (6) $c\overline{C}^t p$. By the same reasons from (3) and (5) we get (7) $c\overline{C}^t p^*$. By the contact axioms of C^t we obtain from (6) and (7) $c\overline{C}^t(p + p^*)$ and $c\overline{C}^t 1$, which implies $c = 0$. But (1) implies $c \neq 0$ - a contradiction.

We will consider axiomatic extensions of DCA with some special axioms called **time axioms** to be introduced later on (see Remark 1.11).

The intuitive meaning of all relations in DCA can be obtained from the standard example of DCA which is given in the next section.

1.3. **Snapshot standard example of DCA.** In this section, following [5] Part II we will introduce a series of definitions and constructions needed to give a standard example of DCA, called in this paper "snapshot example". The snapshot example is important, because it gives the main intuition of the abstract DCA. It is called "standard" because the main representation theorem for DCA (see [5]) states that every DCA is isomorphic with some standard DCA. First we present the notion of time structure given in the next section.

1.3.1. *Time structures.* Classical physics describes changing objects by presenting their main features as functions of time. So it presupposes that the time is given by its sets of time points identifying them with real or rational numbers with their specific arithmetic structure. This structure of the set of time points is not obligatory for all situations where we have to describe change. Very often time structures have the form of an abstract relational system of the form $(T, \prec)$, where T is a non-empty set of time points and $\prec$ is a binary relation on T such that $m \prec n$ means that *m is before n.* This intuition motivates to call $\prec$ *before-after* relation or *time order*. Time structures $(T, \prec)$ of such a kind are studied in temporal logic. For instance if T is the set of real numbers the time order coincides with one of the standard ordering relations $<$ or $\leq$ of strict or partial order of numbers. In the general time structures the relation $\prec$ may satisfy various abstract properties. In the following list we describe some of them with their specific names and notations which will be used in this paper.

- **(RS)** *Right seriality* $(\forall m)(\exists n)(m \prec n)$,
- **(LS)***Left seriality* $(\forall m)(\exists n)(n \prec m)$,
- **(Up Dir)** *Updirectedness* $(\forall i, j)(\exists k)(i \prec k$ and $j \prec k)$,
- **(Down Dir)** *Downdirectedness* $(\forall i, j)(\exists k)(k \prec i$ and $k \prec j)$,
- **(Dens)** *Density* $i \prec j \rightarrow (\exists k)(i \prec k$ and $k \prec j)$,
- **(Ref)** *Reflexivity* $(\forall m)(m \prec m)$,
- **(Irr)** *Irreflexivity* $(\forall m)($ not $m \prec m)$,
- **(Lin)** *Linearity* $(\forall m, n)(m \prec n$ or $n \prec m)$,
- **(Tri)** *Trichotomy* $(\forall m, n)(m = n$ or $m \prec n$ or $n \prec m)$,
- **(Tr)** *Transitivity* $(\forall ijk)(i \prec j$ and $j \prec k \rightarrow i \prec k)$.

We call the set of formulas (RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri), (Tr) *time conditions.* "Before-after" relation satisfying the condition (Irr) will be called "strict". If $\prec$ satisfies (Ref) the reading of $m \prec n$ should be more precise: "m is equal or before n".

Note that the above listed conditions for time ordering are not independent. Taking some meaningful subsets of them we obtain various notions of time order.

1.3.2. *Snapshot construction. Definitions of space contact, time contact and precedence.*
Snapshot construction. We present a specific dynamic model of space based on a given time structure $(T, \prec)$ by means of the so called **snapshot construction**. The intuition based on this construction is the following. Suppose that we want to describe a dynamic environment consisting of a regions changing in time. First we suppose that we are given a time structure $\underline{T} = (T, \prec)$ and in order to know what is the spatial configuration of regions at each moment of time $m \in T$ we make a series of pictures corresponding to each moment m of time from T. We assume that for each $m \in T$ the spatial configuration of the regions forms a contact algebra $(\underline{B}_m, C_m) = (B_m, 0_m, 1_m, \leq_m, +_m, \cdot_m, *_m, C_m)$, called a **coordinate contact algebra** corresponding to m. In other words $(\underline{B}_m, C_m)$ is a static "snapshot" of this configuration. We identify a given changing region a with the series $< a_m >_{m\in T}$ of snapshots and call such a series a dynamic region. In a sense this series can be considered also as a "trajectory" or "time history" of a. We denote by $B(\underline{T})$ the set of all dynamic regions. If $a =< a_m >_{m\in T}$ is a given dynamic region then a_m can be considered as "a at the time point m". The "static" region a_m will be called also the "m-th coordinate of a". For instance the expression $a_m \neq 0_m$ means that "a exists at the time point m", and the expression $a_m C_m b_m$ means that "a and b are in a contact at the moment m". Thus $(\underline{B}_m, C_m)$ contains all m-th coordinates of the changing regions. We assume that the set $B(\underline{T})$ is a Boolean algebra, i.e. a mereology with Boolean constants and operations defined as follows: $1 =< 1_m >_{m\in T}$, $0 =< 0_m >_{m\in T}$, Boolean ordering $a \leq b$ iff $(\forall m \in T)(a_m \leq_m b_m)$ and Boolean operations are defined "coordinate-wise": $a + b =_{def}< a_m(+_m)b_m >_{m\in T}$, $a.b =_{def}< a_m(\cdot_m)b_m >_{m\in T}$, $a^* =_{def}< a^*_m >_{m\in T}$. Thus defined Boolean algebra is called **dynamic model of space over the time structure** $(T, \prec)$. Note that the Boolean algebra $B(\underline{T})$ is a subalgebra of the Cartesian product $\prod_{m\in T} B_m$ of the contact algebras $(\underline{B}_m, C_m)$, $m \in T$. A model which coincides with the Cartesian product is called a **full model**. $B(\underline{T})$ is called a **rich model** if it contains all dynamic regions a such that for all $m \in T$ we have $a_m = 0_m$ or $a_m = 1_m$. Obviously full models are rich.

Dynamic model of space is a quite rich spatio-temporal structure in which one can give explicit definitions of various spatio-temporal relations between dynamic regions. First we will study the following three basic spatio-temporal relations between dynamic regions mentioned in the abstract definition: space contact,time contact and precedence relation. Let a, b are dynamic regions.

- **Space contact** $aC^s b$ iff $(\exists m \in T)(a_m C_m b_m)$.

Intuitively space contact between a and b means that there is a time point in which a and b are in a contact.

- **Time contact** $aC^t b$ iff $(\exists m \in T)(a_m \neq 0_m$ and $b_m \neq 0_m)$.

Intuitively time contact between a and b means that there exists a time point in which a and b exist simultaneously. Note that $a_m \neq 0_m$ and $b_m \neq 0_m$ means just that a and b exist at the time point m. This relation can be considered also as a kind of **simultaneity relation** or **contemporaneity relation** studied in Whitehead's works. This suggests to call a and b contemporaries if $aC^t b$.

- **Precedence** $a\mathcal{B}b$ iff $(\exists m, n \in T)(m \prec n$ and $a_m \neq 0_m$ and $b_n \neq 0_n)$.

Intuitively a is in a precedence relation with b (in words *a precedes b*) means that there is a time point in which a exists which is before a time point in which b exists, which motivates the name of $\mathcal{B}$ as a precedence relation.

The following lemma from [5] Part II (Lemma 1.4) verifies part of the axioms od DCA.

Lemma 1.7. *(i) C^s is a contact relation,*

(ii) C^t is a contact relation satisfying the following additional condition

$(C^s \to C^t)$ $aC^sb \to aC^tb$.

If the dynamic model of space $B(\underline{T})$ is a rich one, then C^t satisfies the Efremovich axiom,

(C^tE) *If $a\overline{C}^tb$, then there exists c such that $a\overline{C}^tc$ and $c^*\overline{C}^tb$.*

(iii) $\mathcal{B}$ is a precontact relation.

The following lemma is not from [5] and is a new one. It gives the possibility to add two new axioms to the abstract definition of DCA which are true in the standard model.

Lemma 1.8. *Suppose that the dynamic model of space $B(\underline{T})$ is rich. Then the following two compositional axioms (see 1.4) for C^t and $\mathcal{B}$ are true:*

(i) $(C^t\mathcal{B})$ If $a\overline{\mathcal{B}}b$, then there exists c such that $a\overline{C}^tc$ and $c^*\overline{\mathcal{B}}b$.

(ii) $(\mathcal{B}C^t)$ If $a\overline{\mathcal{B}}b$, then there exists c such that $a\overline{\mathcal{B}}c$ and $c^*\overline{C}^tb$.

Proof. (i) Suppose $a\overline{\mathcal{B}}b$ and define c coordinate wise:

$$c_k = \begin{cases} 0_k, & \text{if } a_k \neq 0_k \\ 1_k, & \text{if } a_k = 0_k. \end{cases}$$

Since the algebra is rich then c exists. The verification of the conclusion $a\overline{\mathcal{B}}c$ and $c^*\overline{C}^tb$ is straightforward.

(ii) The proof is similar to the above one - use the following definition of c:

$$c_l = \begin{cases} 0_l, & \text{if } b_l = 0_l \\ 1_l, & \text{if } b_l \neq 0_l. \end{cases}$$

□

1.3.3. *A characterization of abstract properties of time structures with some time axioms.* We do not presuppose in the formal definition of dynamic model of space that the time structure $(T, \prec)$ satisfies some abstract properties of the precedence relation. In this section we shall see that all abstract properties of the precedence relation mentioned in Section 1.3.1, are in an exact correlation with some special conditions of time contact C^t and precedence relation $\mathcal{B}$ called **time axioms**. The correlation is given in the next table:

(RS) *Right seriality* $(\forall m)(\exists n)(m \prec n) \Longleftrightarrow$

(rs) $a \neq 0 \to a\mathcal{B}1$,

(LS) *Left seriality* $(\forall m)(\exists n)(n \prec m) \Longleftrightarrow$

(ls) $a \neq 0 \to 1\mathcal{B}a$,

(Up Dir) *Updirectedness* $(\forall i, j)(\exists k)(i \prec k \text{ and } j \prec k) \Longleftrightarrow$

(up dir) $a \neq 0 \wedge b \neq 0 \to a\mathcal{B}p \vee b\mathcal{B}p^*$,

(Down Dir) *Downdirectedness* $(\forall i, j)(\exists k)(k \prec i \text{ and } k \prec j) \Longleftrightarrow$

(down dir) $a \neq 0 \wedge b \neq 0 \rightarrow p\mathcal{B}a \vee p^*\mathcal{B}b$,
(Dens) *Density* $i \prec j \rightarrow (\exists k)(i \prec k \wedge k \prec j) \Longleftrightarrow$
(dens) $a\mathcal{B}b \rightarrow a\mathcal{B}p$ or $p^*\mathcal{B}b$,
(Ref) *Reflexivity* $(\forall m)(m \prec m) \Longleftrightarrow$
(ref) $aC^tb \rightarrow a\mathcal{B}b$,
(Irr) *Irreflexivity* $(\forall m)($ not $m \prec m) \Longleftrightarrow$
(irr) $a\mathcal{B}b \rightarrow (\exists c, d)(c\mathcal{B}d$ and aC^tc and bC^td and $c\overline{C}^td)$,
(Lin) *Linearity* $(\forall m, n)(m \prec n \vee n \prec m) \Longleftrightarrow$
(lin) $a \neq 0 \wedge b \neq 0 \rightarrow a\mathcal{B}b \vee b\mathcal{B}a$,
(Tri) *Trichotomy* $(\forall m, n)(m = n$ or $m \prec n$ or $n \prec m) \Longleftrightarrow$
(tri) $(aC^tc$ and bC^td and $c\overline{C}^td) \rightarrow (a\mathcal{B}b$ or $b\mathcal{B}a)$,
(Tr) *Transitivity* $i \prec j$ and $j \prec k \rightarrow i \prec k \Longleftrightarrow$
(tr) $a\overline{\mathcal{B}}b \rightarrow (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$.

The following lemma corresponds to Lemma 2.1 from [5] Part II.

Lemma 1.9. Correspondence Lemma. *Let $B(\underline{T})$ be a rich model of space over the time structure $(T, \prec)$. Then all the correspondences in the above table are true in the following sense: the left site of a given equivalence is true in $(T, \prec)$ iff the right site is true in $B(\underline{T})$.*

Definition 1.10. *Time axioms. The formulas **(rs)**, **(ls)**, **(up dir)**, **(down dir)**, **(dens)**, **(ref)**, **(irr)**, **(lin)**, **(tri)**, **(tr)**, included in the above table are called "time axioms" and will be considered as additional axioms for DCA-s and also as additional axioms for some other versions of DCA.*

Remark 1.11. *The above lemma is very important because it states that the abstract properties of the time structure of a given rich model of space are determined by the time axioms which contain only variables for dynamic regions and time points are not mention. This correlation suggests to consider (abstract) DCA-s satisfying some of the time axioms.*

Remark 1.12. *Let us observe that all time axioms contain only the predicate $\mathcal{B}$ except the axioms (tri) and (irr) which contain the predicates $\mathcal{B}$ and C^t. Let us note that Lemma 1.9 remains true if we replace correspondingly (tri) and (irr) by simpler formulas defining the same time conditions in rich models:*

(tri) If $a \neq 0$, $b \neq 0$, then aC^tb or $a\mathcal{B}b$ or $b\mathcal{B}a$,

(irr) If $a\mathcal{B}b$, then $(\exists c \neq 0)(\exists d \neq 0)(c \leq a$ and $d \leq b$ and $c\overline{C}^td)$.

We preserves the old names and later on in this paper by (tri) and (irr) we will understand only the above two formulas.

As an example we show the proof of Lemma 1.9 for the case (tri) $\Leftrightarrow$ (Tri) taking the new formula for (tri).

Proof. (tri) $\Rightarrow$ (Tri). Suppose (tri) and for the sake of contadiction let (Tri) be not true, i.e. for some i and j we have $i \neq j$, $i \not\prec j$ and $j \not\prec i$. Define a and b coordinate wise as follows:

$$a_k = \begin{cases} 1_k, & \text{if } i = k \\ 0_k, & \text{if } i \neq k. \end{cases}, \quad b_k = \begin{cases} 1_k, & \text{if } j = k \\ 0_k, & \text{if } j \neq k. \end{cases}$$

Since $B(\underline{T})$ is a rich model of space, then the definition of a and b is correct. It is easy to see that $a \neq 0$, $b \neq 0$, $a\overline{C}^tb$, $a\overline{\mathcal{B}}b$ and $b\overline{\mathcal{B}}a$ which contradicts (tri). Thus (Tri) is true.

(tri) $\Leftarrow$ (Tri). Suppose (Tri). In order to prove (tri) suppose $a \neq 0$, $b \neq 0$. Then $\exists i, i \neq 0_i$ and $\exists j, j \neq 0_i$. By (Tri) we have $i = j$ or $i \prec j$ or $j \prec i$. This implies $aC^t b$ or $a\mathcal{B}b$ or $b\mathcal{B}a$ which completes the proof. □

1.3.4. *Time representatives and NOW.* In this section, following [5] Part II, we define in dynamic model of space the remaining notions from the signature of abstract DCA: time representatives TR, universal time representatives UTR and a special universal time representative NOW.

First about the intuitions behind these notions. Consider the phrases: "the epoch of Leonardo", "the epoch of Renaissance", "the geological age of the dinosaurs", "the time of the First World War". All these phrases indicate a concrete unit of time named by something which happened or existed at that time and not in some other moment of time. These examples suggest to introduce in the dynamic model of space a special set of dynamic regions called *time representatives*, which are regions existing at a unique time point. The formal definition is the following:

Definition 1.13. *A region c in a dynamic model of space is called a* ***time representative*** *if there exists a time point $i \in T$ such that $c_i \neq 0_i$ and for all $j \neq i$, $c_j = 0_j$. We say also that c is a representative of the time point i and indicate this by writing $c = c(i)$. In the case when $c_i = 1_i$, c is called* ***universal time representative****. We denote by TR (UTR) the set of (universal) time representatives in a given dynamic model of space.*

The following lemma corresponds to Lemma 3.2 from [5] Part II.

Lemma 1.14. *Let $B(\underline{T})$ be a rich dynamic model of space over the time structure $(T, \prec)$. Then for each time point $i \in T$ there exist an universal time representative $c(i)$ of i. If a is a region such that $a_i \neq 0_i$ and $a_i \neq 1_i$ then $c(i).a$ is a time representative which is not universal.*

The universal time representative NOW. The above lemma shows that in rich models there exist time representatives corresponding to each moment of time. It suggests also to enrich the time structure $(T, \prec)$ with a special moment of time denoted by **now**, corresponding to the "present moment of time". We denote by **NOW** the universal time representative corresponding to **now**.

Now we are ready to introduce the exact definition of standard DCA.

Definition 1.15. *By a standard dynamic contact algebra we mean any rich dynamic model of space with time structure $(T, \prec, now)$ with explicit definitions of the relations C^s, C^t, $\mathcal{B}$, time representatives TR, universal time representatives UTR and the universal time representative NOW.*

Because for the purposes of the present paper we will introduce later on a new standard model for DCA, the present one will be called ***"snapshot model of DCA"****.*

The results of [5] Part II (Lemma 1.4, Lemma 3.3 and Lemma 3.7) show that standard DCA satisfies axioms of the abstract definition of DCA. Also the meaning of time axioms is established as statements determining the properties of the time structure of the model. This model of DCA is called *standard* because it is shown in [5] Part III that every DCA with a number of additional time axioms is representable as a standard DCA over a time structure satisfying the time conditions determined by the corresponding time axioms.

It was shown also in [5] Part II that time representatives, universal time representatives and NOW increase considerably the expressive power of DCA (see Section 3.1 of [5] Part II) allowing to express different temporal statements for dynamic regions including different forms of Past, Present and Future. For instance the sentence a **exists sometimes in the future** can be expressed by the formula $\mathbf{NOW}\mathcal{B}a$.

1.3.5. *Translation Lemma.* Some properties of universal time representatives suggest a translation τ from the first-order language of time structures into the language of DCA-s defined as follows. If i is a variable for time points let $c(i)$ denote a variable denoting universal time representative and let for different i and j $c(i)$ and $c(j)$ be also different. Then replace all atomic formulas of the form $i = j$ and $i \prec j$ with the formulas $c(i)C^t c(j)$ (or $c(i) = c(j)$) and $c(i)\mathcal{B}c(j)$ respectively. Example: $A = (\forall i)(\exists j)(i \prec j)$, $\tau(A) = (\forall c(i))(\exists c(j))(c(i)\mathcal{B}c(j))$.

The following lemma corresponds to Lemma 3.5 from [5] Part II.

Lemma 1.16. Translation Lemma. *Let $B(\underline{T})$ be a rich standard DCA with time structure $(T, \prec)$. Then for any first-order formula A in this language: A is universally true in $(T, \prec)$ iff $\tau(A)$ is universally true in DCA. In particular, for all formulas A from the set* {(Rs), (Ls),(Up Dir), (Down Dir), (Dens), (Ref), (Irr), (Lin), (Tri) *and* (Tr)} *we have A is true in $(T, \prec)$ iff $\tau(A)$ is true $B(\underline{T})$.*

2. Strong dynamic contact algebras and weak dynamic contact algebras

Lemma 1.8 shows that the snapshot model verifies two conditions for C^t and $\mathcal{B}$ which are not listed in the axioms of DCA - the following two compositional axioms:

$(C^t\mathcal{B})$ If $a\overline{\mathcal{B}}b$, then there exists c such that $a\overline{C}^t c$ and $c^*\overline{\mathcal{B}}b$.

$(\mathcal{B}C^t)$ If $a\overline{\mathcal{B}}b$, then there exists c such that $a\overline{\mathcal{B}}c$ and $c^*\overline{C}^t b$.

Adding these axioms to the abstract definition of DCA we obtain the notion of **"strong dynamic contact algebra"** (strong dca). It follows from Lemma 1.8 that the snapshot model of space is a model also for strong dca.

If we delete the Efremovich axiom (C^tE) from the definition of the abstract DCA we obtain the definition of **"weak dynamic contact algebra"** (weak dca) . A common name for algebras of the just mentioned three classes is "dynamic contact algebra" - "dca" (with small letters). The notation DCA (with capital letters) will be used only for the algebras introduced in [5] and Section 1.5.

We denote by Σ_{DCA}, the class of all DCA-s, by Σ_{weak} - the class of all weak dca-s, and by Σ_{strong} - the class of all strong dca-s.

Obviously snapshot models are also models for weak dca.

Remark 2.1. *Inspecting the proof of the representation theorem for DCA in* [5] *Part III, it can be shown that the proof can be modified to hold without the use of Efremovich axiom and hence to hold for weak dca. So representation theorem based on snapshot models is also possible for weak dca-s and obviously for strong dca-s.*

Further in this paper we will present a representation theorem for weak dca based on a kind of relational models, which will be used also for Kripke style semantics for logical systems based on (weak, strong) dca-s.

2.1. **Relational dynamic spaces.** We will introduce in this section relational systems (called relational dynamic spaces) from which one may construct models for strong dca-s. These models, of course, will be also models for DCA-s and weak dca-s. They will be used later on for a Kripke style semantics for quantifier-free logics based on the language of DCA.

Definition 2.2. *By a* ***relational dynamic space*** *we mean a relational system* $\underline{W} = (W, R^t, R^s, \prec, \textbf{\textit{now}})$ *such that* $\textbf{\textit{now}} \in W$*,* R^t *is an equivalence relation,* R^s *is a reflexive and symmetric relation included in* R^t *and* $\prec$ *is a binary relation satisfying the following "compositional axioms" (see Lemma 1.4)*

($R^t \circ \prec$) If xR^ty *and* $y \prec z$*, then* $x \prec z$*,*

($\prec \circ R^t$) If $x \prec y$ *and* yR^tz*, then* $x \prec z$*.*

We will associate some intuition for the abstract definition of relational space. Intuitively, the elements of W are considered as space points existing at some moment of time. The relation xR^ty means that the moment of existence of x is the same as the moment of existence of y, i.e. x and y exist simultaneously at some moment of time. The relation $x \prec y$ means that the moment of existence of x precedes (is before) the moment of existence of y. So R^t and $\prec$ are typical time relations between space points. The point **now** is a space point existing now, at the present time. Having in mind this intuition we can see that the two axioms ($R^t \circ \prec$) and ($\prec \circ R^t$) are quite natural. This suggests also to consider the substructure $(W, R^t, \prec, \textbf{now})$ (in which the relation R^s is deleted) as the **time substructure** of $\underline{W}$. The relation xR^sy means that the space points x and y exist simultaneously at some moment of time (i.e. we have xR^ty) and also at that moment they are **near spatially**. So R^s is a spatial relation between space points. Subsets of W can be considered as spatial regions containing, in general, points existing at different times. Each equivalence class with respect to R^t, in particular, have to be considered as the region of all points simultaneously existing at a given moment of time and not in another moment of time.

Starting from arbitrary relational dynamic space $\underline{W}$ we construct a strong dca $B(W)$ over W as follows.

Applying the construction of relational model of precontact relation (see the text before Lemma 1.3) define for subsets $a, b \subseteq W$:

• **time contact** aC^tb iff there exist $x \in a$ and $y \in b$ such that xR^ty.

• **space contact** aC^sb iff there exist $x \in a$ and $y \in b$ such that xR^sy.

• **precedence** $a\mathcal{B}b$ iff there exist $x \in a$ and $y \in b$ such that $x \prec y$.

• By $UTR(W)$ we denote the set of equivalence classes of W under R^t and call them **"universal time representatives"** .

• By a **"time representative"** we mean any nonempty subset of a equivalence class under R^t and denote its set by $TR(W)$.

• **NOW** (denoted also by **NOW**(W)) is a the equivalence class determined by **now** and hence belongs to $UTR(W)$.

The intuitions behind the above definitions follows from the intuition associated to the relations R^t, R^s, $\mathcal{B}$ and the subsets of W as regions. For instance we call the equivalence classes with respect to R^t universal time representatives because each equivalence class contains all points existing in a fixed moment of time and not in other moments of time. The same is the motivation for the name " time representative"

It remains to show that the Boolean algebra of all subsets of W with the above defined relations satisfies the axioms of strong dca.

By Lemma 1.3 it follows that C^t is a contact relation satisfying the Efremovich axiom (CE), that C^s is a contact relation included in C^t and that $\mathcal{B}$ is a a precontact. It follows by Lemma 1.4 that the two new axioms for strong dca - the compositional axioms $(C^t\mathcal{B})$ and $(\mathcal{B}C^t)$ are also satisfied.

Let us show that the axioms for TR and UTR are also true. The group of axioms TR1, TR2, TRC^s, TR$\mathcal{B}$1,..., TR$\mathcal{B}$4 present no difficulties and follow by direct calculations. More involved are the axioms from the group UTR$\mathcal{B}$11, UTR$\mathcal{B}$12, UTR$\mathcal{B}$121 and UTR$\mathcal{B}$22. Let us consider the implication from left to right in UTR$\mathcal{B}$22.

In a simpler notations we have to prove the following implication:
(#) $(\forall p \subseteq W)(c\mathcal{B}p$ or $d\mathcal{B}p^*) \Rightarrow (\exists e \in UTR(W))(c\mathcal{B}e$ and $d\mathcal{B}e)$,
where $c, d \in UTR(W)$.

We will use some tricks from modal logic. Let R be a binary relation in W and for $a \subseteq W$ denote $\langle R\rangle a =_{def} \{x \in W : (\exists y \in a)(xRy)\}$ (this is the possibility modality of a in the modal algebra over (W, R)). Then for the precontact relation C_R we have the following:

aC_Rb iff $a \cap \langle R\rangle b \neq \varnothing$ iff $\langle R^{-1}\rangle a \cap b \neq \varnothing$.

For the right part of the implication (#) one can easily prove the following equivalence (c and d are in UTR(W)):
(1) $(\exists e \in UTR(W))(c\mathcal{B}e$ and $d\mathcal{B}e)$ iff $\langle\succ\rangle c \cap \langle\succ\rangle d \neq \varnothing$, where $\succ$ is the converse relation for $\prec$.

Suppose now that (#) does not hold. Then we have
(2) $(\forall p \subseteq W)(c\mathcal{B}p$ or $d\mathcal{B}p^*)$, and
(3) not$(\exists e \in UTR(W))(c\mathcal{B}e$ and $d\mathcal{B}e)$

On the base of (1) (3) is equivalent to
(4) $\langle\succ\rangle c \cap \langle\succ\rangle d = \varnothing$.

But for $p = \langle\succ\rangle d$ and using (4) one can show that (2) is falsified.

Thus, we have just proved the following lemma:

Lemma 2.3. *$B(W)$ is a strong DCA.*

Remark 2.4. *We want to prove a lemma which is analogous to Lemma 1.9 and establish a correspondence between time formulas and time conditions. We consider the same time axioms listed in Definition 1.10 (see also Remark 1.11). However, since we are in a new model of dynamic space some of time conditions have to be replaced by new ones, and these are the conditions (Irr) of irreflexivity and (Tri) of trichotomy:*
(Irr) Irreflexivity $(\forall m)($ *not* $m \prec m)$,
(Tri) Trichotomy $(\forall m, n)(m = n$ *or* $m \prec n$ *or* $n \prec m)$.

These two conditions have to be replaced correspondingly by the following two new conditions:
(Irr') *If $x \prec y$, then $x\overline{R}^t y$,*
(Tri') *$xR^t y$ or $x \prec y$ or $y \prec x$.*

Now we are ready to formulate the new correspondence lemma.

Lemma 2.5. ***Relational correspondence for time axioms.*** *Let α be any formula from the list of* time axioms *(rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr) and A be its corresponding formula from the list of* time conditions *(RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr'), (Lin), (Tri'), (Tr). Then A is true in $\underline{W}$ iff α is true in $B(W)$.*

Proof. This lemma is analogous to similar correspondence lemmas considered in [6]: Lemma 2.1 and Lemma 2.2 and can be proved in a similar way.

Let us note that the correspondence **(Tr)⇔(tr)** follows from Lemma 1.3 (iii). To illustrate the proof we will consider the cases: **(Dens)⇔ (dens)**, **(Irr')⇔(irr)**.
(Dens)⇔ (dens). The implication from left to right is easy. For the proof of converse implication take $a = \{x\}$ and $b = \{y\}$. Substituting this in (dens) we obtain: $\{x\}\mathcal{B}\{y\} \rightarrow (\forall p \subseteq W)(\{x\}\mathcal{B}p \vee p^*\mathcal{B}\{y\}$ which is equivalent to $x \prec y \rightarrow (\forall p \subseteq W)(\{x\}\mathcal{B}p \vee p^*\mathcal{B}\{y\}$. Reasoning as in the proof of Lemma 2.3 one can show that $(\forall p \subseteq W)(\{x\}\mathcal{B}p \vee p^*\mathcal{B}\{y\}$ implies $(\exists z)(x \prec z \prec y)$. Note that using some notations used in the proof of Lemma 2.3 we see that $(\exists z)(x \prec z \prec y)$ is equivalent to $\langle>\rangle\{x\} \cap \langle<\rangle\{y\} \neq \varnothing$. So we have to prove the implication: $(\forall p \subseteq W)(\{x\}\mathcal{B}p \vee p^*\mathcal{B}\{y\}$ implies $\langle>\rangle\{x\} \cap \langle<\rangle\{y\} \neq \varnothing$. Suppose that this implication is not true. Then we have
(1) $(\forall p \subseteq W)(\{x\}\mathcal{B}p \vee p^*\mathcal{B}\{y\}$ and
(2) $\langle>\rangle\{x\} \cap \langle<\rangle\{y\} = \varnothing$.
Let us note that (1) is equivalent to
(3) $(\forall p \subseteq W)(\langle>\rangle\{x\} \cap p \neq \varnothing$ or $p^* \cap \langle<\rangle\{y\} \neq \varnothing)$.
Putting $p = \langle<\rangle\{y\}$ in (3) we obtain a result which contradicts (2), which finishes the proof.
(Irr')⇔(irr). ⇒. Suppose (Irr'), i.e. $(\forall x, y)(x \prec y \rightarrow x\overline{C}^t y)$. To prove (Irr) suppose $a\mathcal{B}b$. Then there exist $x \in a$ and $y \in b$ such that $x \prec y$. Define $c = \{x\}$ and $d = \{y\}$. Then obviously $c \neq 0$, $d \neq 0$, $c \leq a$, $d \leq b$ and $c\overline{C}^t d$.

⇐. Suppose (Irr'), i.e. $a\mathcal{B}b \rightarrow (\exists c, d \neq 0)(c \leq a,\ c \leq b$, and $c\overline{C}^t d$. Put $a = \{x\}$, $b = \{y\}$. The there exist $c, d \neq 0$ such that $c \subseteq \{x\}$, $d \subseteq \{y\}$ and $c\overline{C}^t d$. We obtain from here that $c = \{x\}$, $d = \{y\}$ and $\{x\}\overline{C}^t\{y\}$. Thus $\{x\}\mathcal{B}\{y\}$ implies $\{x\}\overline{C}^t\{y\}$ which is equivalent to (Irr'). □

Another thing which has to be done is to extend the translation τ mentioned in Section 1.3.5 in order to hold for the time substructure $(W, R^t, \prec)$, considered as a time substructure of the relational dynamic space. The modification is the following: for first-order variable x, we let $\tau(x)$ to be an universal time representative and for different variables x and y, $\tau(x)$ and $\tau(y)$ to be different variables ranging on universal time representatives, $\tau(xR^ty) =_{def} \tau(x)C^t\tau(y) =_{def} \tau(x) = \tau(y)$, $\tau(x \prec y)) =_{def} \tau(x)\mathcal{B}\tau(y)$. Now the analog of Lemma 1.16 can easily be proved.

Remark 2.6. *Each relational dynamic space $\underline{W} = (W, R^t, R^s, \prec, \boldsymbol{now})$ can be transformed into a snapshot dynamic model of space as follows. Let W_x be the equivalence class of x under R^t and let R^s_x be the restriction of R^s to W_x . Then the pair $\underline{W_x} = (W_x, R^s_x)$ is a relational structure with reflexive and symmetric relation R^s_x, the precontact algebra B_x over $\underline{W_x}$ is a contact algebra, called the local contact algebra corresponding to the time point x. Consider the subsystem $(W, R^t, \prec, \boldsymbol{now})$ as a time structure and to each $x \in W$ associate the contact algebra over the structure (W_x, R^s_x) considered as the coordinate contact algebra. Then define the Cartesian* *product $\prod_{x\in W} B_x$ and proceed as in the snapshot construction described in Section 1.3.2. There is a simple correspondence between subsets of W considered as regions and dynamic regions from snapshot construction: the intersection $a \cap W_x$ is just "a at the moment x", and the series $\langle a \cap W_x\rangle_{x\in W}$ is the corresponding dynamic region from the snapshot model. Also there is a simple correspondence between the relations $R^t, R^s, \prec$ from $\underline{W}$ and their definable analogs from the snapshot model.*

Conversely, from each snapshot model of space over a time structure $(T, \prec, \boldsymbol{now})$ we can extract a relational dynamic model of space. Each coordinate contact algebra (B_t, C_t) can be considered as a contact algebra over an adjacency space (see Definition 1.2) (W_t, R_t) and for different t we may assume that the corresponding spaces are disjoined. Take W to be the union of all W_t and the subsets W_t consider as the equivalence classes of a relation R^t. The relation R^s is just the union of the relations R_t. The definition of the relation $\prec$ can be defined as follows: for $x, y \in W$ we define $x \prec y$ iff there exist equivalence classes W_t and W_s such that $x \in W_t$, $y \in W_s$ and $t \prec s$ in the time structure $(T, \prec, \boldsymbol{now})$. In this way we define the structure $\underline{W} = (W, R^t, R^s, \prec)$. For the definition of the element $\boldsymbol{now}$ in this structure, let it be any fixed element from $W_{\boldsymbol{now}}$. It is not difficult to see that $\underline{W}$ satisfies the axioms of relational dynamic space. The informal meaning associated to the abstract definition of relational dynamic space is taken just from this concrete construction.

3. Relational representation theorem for weak dca-s

In this section we will prove a representation theorem for weak dca-s over relational dynamic space. It will follow from this representation theorem that every weak dca can be isomorphically embedded into a strong dca. The idea of the representation construction is a modification of the relational representation theory for precontact algebras developed in [10] (in [10] precontact algebras are called proximity algebras): first to each weak dca $\underline{B}$ we associate a relational dynamic space $W(\underline{B})$, called canonical dynamic space, and then we prove that $\underline{B}$ can be embedded into the strong dca over $W(\underline{B})$. The difference with the representation construction from [10] is that in [10] the canonical relational system was constructed over the set of all ultrafilters of $\underline{B}$, while in the present construction we take a class of special ultrafilters, called UTR-ultrafilters. To realize this program we first need some facts about weak dca-s and their ultrafilters.

3.1. Basic facts for weak dca-s.

We assume in this section that $\underline{B}$ is a fixed weak dca.

Lemma 3.1. *(i) If $a \neq 0$, then there exists $c \in UTR$ such that $a.c \neq 0$.*

(ii) If $c \in UTR$, then (aC^tc iff $a.c \neq 0$).

(iii) If $c \in TR$ and $d \in UTR$, then ($c.d \neq 0$ iff $c \leq d$).

(iv) If $c \in UTR$, $d \in TR$ and $c \leq d$, then $c = d$.

(v) Let $c, d \in UTR$. Then the following conditions are equivalent:

(v1) cC^td,

(v2) $c.d \neq 0$,

(v3) $c = d$.

(vi) If $c \in TR$, then there exists unique $d \in UTR$ such that $c \leq d$.

(vii) If $c \neq 0$, $d \in UTR$ and $c \leq d$, then $c \in TR$.

(viii) $c \in TR$ iff $c \neq 0$ and there exists $d \in UTR$ such that $c \leq d$.

(ix) $c \in TR$ *iff* $c \neq 0$ *and* $(\forall d \in UTR)(c.d \neq 0 \to c \leq d)$.
(x) If $c \in TR$ *and* $(\forall d \in TR)(c \leq d \to c = d)$, *then* $c \in UTR$.
(xi) $c \in UTR$ *iff* $c \in TR$ *and* $(\forall d \in TR)(c \leq d \to c = d)$.
(xii) $aC^s b$ *iff* $(\exists c \in UTR)((a.c)C^s(b.c))$.
(xiii) $aC^t b$ *iff* $(\exists c \in UTR)((a.c)C^t(b.c))$.
(xiv) $a\mathcal{B}b$ *iff* $(\exists c, d \in UTR)((a.c)\mathcal{B}(b.d))$.

Proof. **(i)** Let $c \neq 0$. Then $cC^s c$ and by the axiom (TRC^s) there exists $c \in UTR$ such that $(a.c)C^s a$ which implies by the contact axioms for C^s that $a.c \neq 0$.

(ii) Let $c \in UTR$.

(⇒) Suppose $aC^t c$ and for the sake of contradiction that $a.c = 0$. Then $a \leq c^*$ and by $aC^t c$ we get $c^* C^t c$. By axiom (TR2) this contradicts $c \in UTR$.

(⇐) Suppose $a.c \neq 0$. Then by the contact axioms for C^t we get $aC^t c$.

(iii) Let $c \in TR$ and $d \in UTR$.

(⇒) Suppose $c.d \neq 0$ and for the sake of contradiction that $c \not\leq d$. From here we get: $cC^t d$, $c.d^* \neq 0$ and $cC^t d^*$. Since $c \in TR$, then by axiom (TR1) we get from $cC^t d$ and $cC^t d^*$ that $dC^t d^*$, which contradicts $d \in UTR$.

(⇐) Suppose $c \leq d$. Then $c.d = c \neq 0$ (c is in TR).

(iv) Suppose $c \in UTR$, $d \in TR$ and $c \leq d$. Then $c.d = c \neq 0$ and applying (iii) we get $d \leq c$ and consequently $c = d$.

(v) This condition follows from (ii) and (iii).

(vi) Suppose $c \in TR$. Then by axiom (TR1) $c \neq 0$ and by (i) there exists $d \in UTR$ such that $c.d \neq 0$. Then by (iii) we get $c \leq d$. For the uniqueness of d suppose that for $d_1, d_2 \in UTR$ we have $c \leq d_1$ and $c \leq d_2$. Then $c \leq d_1.d_2$ and since $c \neq 0$, then $d_1.d_2 \neq 0$. Then by (v) we get $d_1 = d_2$.

(vii) Suppose $c \neq 0$, $d \in UTR$ and $c \leq d$ and for the sake of contradiction that $c \notin TR$. Then by axiom (TR2) $d \in TR$ and by (TR1) there are a, b such that $aC^t c$, $bC^t c$ and $a\overline{C}^t b$. From here and $c \leq d$ we get $aC^t d$, $bC^t d$ which, together with $d \in TR$ implies $aC^t b$ - a contradiction.

(viii) This condition follows from (vi) and (vii).

(ix) (⇒) This implication follows by (iii).

(⇐) Suppose (1) $c \neq 0$ and (2) $(\forall d \in UTR)(c.d \neq 0 \to c \leq d)$. From (1) we get by (i) that $c.d \neq 0$ for some $d \in UTR$ and by (2) we obtain that $c \leq d$. Then by (viii) we obtain that $c \in TR$.

(x) Suppose $c \in TR$ and $(\forall d \in TR)(c \leq d \to c = d)$ and for the sake of contradiction that $c \notin UTR$. Then by axiom (TR2) we get $cC^t c^*$. From $c \in TR$ by (vi) there exists $d \in UTR$ (and hence in TR) such that $c \leq d$. Then by the assumption we get c=d and substituting this in $cC^t c^*$ we obtain $dC^t d^*$ which contradicts $d \in UTR$.

(xi) This condition follows from (iv) and (x).

(xii), (xiii) and (xiv) have the same proofs as in the proof of Lemma 1.4 from [5] Part III. □

Definition 3.2. *Lemma 3.1 (vi) suggests to introduce a function denoted by* $\mathcal{U}tr(c)$ *which is defined only for the elements of TR and takes value in UTR such that for all* $c \in TR$, $\mathcal{U}tr(c) \in UTR$ *with the property* $c \leq \mathcal{U}tr(c)$.

The next lemma lists some properties of the function $\mathcal{U}tr$ easily proved by using some facts from Lemma 3.1.

Lemma 3.3. *(i) If* $c \in UTR$, *then* $\mathcal{U}tr(c) = c$.

(ii) If $c \in TR$, then $\mathcal{U}tr(\mathcal{U}tr(c)) = \mathcal{U}tr(c)$.
(iii) If $c, d \in TR$ and $cC^t d$, or $c \leq d$, or $c + d \in TR$, then $\mathcal{U}tr(c) = \mathcal{U}tr(d)$.

The following lemma contains two universal first-order statements which follow from some non-universal axioms of weak dca. We will use these statements as axioms of the notion of **basic dca** to be introduced in Section 4.

Lemma 3.4. *(i) If $d \in TR$, $c \neq 0$ and $c \leq d$, then $c \in TR$.*
(ii) If $c, d \in TR$ and $cC^t d$, then $(c + d) \in TR$.

Proof. The proof of (i) is an easy consequence of axiom (TR1).
(ii) Let $c, d \in TR$ and $cC^t d$, then obviously $c + d \neq 0$. To prove that $c + d \in TR$ suppose $aC^t(c+d)$, $bC^t(c+d)$ and proceed to show that $aC^t b$. Then this will imply by (TR1) that $c + d \in TR$. By the axioms of contact we obtain the following two disjunctions:
(1) $aC^t c$ or (2) $aC^t d$,
(1') $bC^t c$ or (2') $bC^t d$.
We have to consider four cases. Case (1)(1'): axiom (TR1) implies $aC^t b$ (because $c \in TR$). Similarly, for case (2),(2') (by the assumption that $d \in TR$). Case (1)(2'): $aC^t c$ and the assumption $cC^t d$ imply $aC^t c$ (because $d \in TR$). Then $bC^t d$ and $aC^t c$ imply again $aC^t d$. In a similar way we reason in the case (2)(1'). □

We mentioned in Section 1.3.5 a translation τ introduced and studied (for DCA-s) in [5] Part II and Part III, which translates formulas from the language of time structures into formulas of the language of DCA containing only variables for universal time representatives. In the present context the notion of time structure is different (see Section 2.1) and contains also the relation R^t, the translation for this relation is the following: $\tau(xR^t y) =_{def} \tau(x) = \tau(y)$. The following lemma expresses the relation between time axioms and the corresponding time condition by means of this translation. The lemma has the same proof as Lemma 1.5 from [5] Part III.

Lemma 3.5. ***Translation Lemma.*** *Let $\underline{B}$ be a weak dca and let A be any formula from the list of* time conditions *(RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr'), (Lin), (Tri'), (Tr) and let α be the corresponding formula from the list of* time axioms *(rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr). Then $\tau(A)$ is true in $\underline{B}$ iff α is true in $\underline{B}$.*

3.2. **Ultrafilters and UTR-ultrafilters.** We assume in this section that $\underline{B}$ is a fixed weak DCA. We denote by Ult(B) the set of ultrafilters in $\underline{B}$. Like in the representation theory for precontact algebras in [10] we define the canonical relations between ultrafilters corresponding to the precontact relations in $\underline{B}$ as follows.

Definition 3.6. *Suppose that C is any of the relations C^t, C^s and $\mathcal{B}$. The canonical relation R corresponding to C is defined by*
URV iff for all $a \in U$ and for all $b \in V$ we have aCb, $U, V \in Ult(B)$.
The canonical relation for C^t, C^s and $\mathcal{B}$ are denoted correspondingly by by R^t, R^s and $\prec$.

Lemma 3.7. *(i) R^t and R^s are reflexive and symmetric relations* [10].
(ii) R^s is included in R^t.

Proof. The condition (ii) follows from the fact that C^s is included in C^t. □

Definition 3.8. *Let $U \in Ult(B)$. U is called an UTR-ultrafilter if there exists $c \in UTR$ such that $c \in U$. Sometimes this will be denoted by $U(c)$ and this notation always assumes that $c \in UTR$. We denote by UTRult(B) the set of all UTR-ultrafilters in $\underline{B}$.*

Lemma 3.9. *Let $U, V \in Ult(B)$. Then:*

(i) If $c, d \in TR$, $cC^t d$, $c \in U$ and $d \in V$, then $UR^t V$.

(ii) If $c, d \in UTR$, $c \in U$ and $d \in V$, then $UR^t V$.

(iii) If $c, d \in TR$, $c \in U$ and $d \in V$, then ($UR^t V$ iff $cC^t d$).

(iv) If $c, d \in UTR$, $c \in U$ and $d \in V$, then ($UR^t V$ iff $c = d$).

In a more economic notation this can be written as follows:

$U(c)R^t V(d)$ iff $c = d$.

(v) If $U(c)R^s V(d)$, then $c = d$.

(vi) If $c, d \in TR$, $c\mathcal{B}d$, $c \in U$ and $d \in V$, then $U \prec V$.

(vii) If $c, d \in TR$ or $c, d \in UTR$, $c \in U$ and $d \in V$, then $U \prec V$ iff $c\mathcal{B}d$.

For the case $c, d \in UTR$ this can be written as follows:

$U(c) \prec V(d)$ iff $c\mathcal{B}d$.

Proof. **(i)** Suppose $c, d \in TR$, $cC^t d$, $c \in U$ and $d \in V$ and let $a \in U$ and $b \in V$. Then $a.c \neq 0$ (hence $aC^t c$), $b.d \neq 0$ (hence $bC^t d$). Conditions $d \in TR$, $cC^t d$ and $bC^t d$ imply by axiom (TR1) $bC^t c$, which with $aC^t c$ and $c \in TR$ imply $aC^t b$. This proves $UR^t V$.

(ii) follows from (i).

(iii) (⇒) This implication follows from the definition of the relation R^t.

(⇐) This implication follows from (i).

(iv) This follows from (iii) and Lemma 3.1 (v).

(v) Suppose $U(c)R^s V(d)$. This implies $U(c)R^t V(d)$ which by (iv) gives $c = d$.

(vi) The proof is similar to that of (i).

(vii) The proof follows from (vi) and the definition of $\mathcal{B}$.

□

Lemma 3.10. ***UTR-ultrafilter characterizations of the relations** $\leq$, C^t, C^s, TR and UTR.*

(i) $a \leq b$ iff for every UTR-ultrafilter U: if $a \in U$, then $b \in U$.

(ii) $aC^t b$ iff there exist UTR-ultrafilters U, V such that $UR^t V$, $a \in U$ and $b \in V$.

(iii) $aC^s b$ iff there exist UTR-ultrafilters U, V such that $UR^s V$, $a \in U$ and $b \in V$.

(iv) $a\mathcal{B}b$ iff there exist UTR-ultrafilters U, V such that $U \prec V$, $a \in U$ and $b \in V$.

(v) $c \in TR$ iff there exists an UTR-ultrafilter U such that $c \in U$ and for all UTR-ultrafilters V: $c \in V$ implies $UR^t V$.

(vi) $c \in UTR$ iff there exists an UTR-ultrafilter U such that $c \in U$ and for all UTR-ultrafilters V: $c \in V$ iff $UR^t V$.

Proof. **(i)** (⇒) This implication follows from the definition of ultrafilter.

(⇐). Suppose that $a \not\leq b$. Then $a.b^* \neq 0$ and by Lemma 3.1 (i) there exists $c \in UTR$ such that $a.b^*.c \neq 0$. So, there exists an ultrafilter U such that $a.b^*.c \in U$. Consequently $c \in U$ (and hence $U \in UTRult(B)$), $a \in U$ and $b \notin U$.

(ii) (⇒) Suppose $aC^t b$. Then by Lemma 3.1 (xiii) there exists $c \in UTR$ such that $(a.c)C^t(b.c)$. By the representation theory of precontact algebras developed in [10] there exist ultrafilters U, V such that $a.c \in U$, $b.c \in V$ and $UR^t V$. From here we get that $U, V \in UTRult(B)$, $a \in U$ and $b \in V$.

($\Leftarrow$) This implication trivially follows from the definition of R^t.

(iii) and (iv) can be proved in a similar way.

(v) ($\Rightarrow$) Suppose $c \in TR$. Then by Lemma 3.1 (ix) we get (1) $c \neq 0$, i.e. $c \not\leq 0$, and (2) $(\forall d \in UTR)(c.d \neq 0 \rightarrow c \leq d)$. Applying (i) to (1) we get a $U(d)$ such that $c \in U(d)$, so, $c.d \neq 0$ and by (2) we get $c \leq d$. It remains to show the following: for every $V(e)$: if $c \in V(e)$, then $U(d)R^tV(e)$. Suppose $c \in V(e)$. Since $c \leq d$ we get $d \in V(e)$ and hence $d.e \neq 0$. Then by Lemma 3.1 (v) we obtain $d = e$ and by Lemma 3.9 (iv) we obtain $U(d)R^tV(e)$.

($\Leftarrow$) Suppose: (1) there exists an UTR-ultrafilter $U(d)$ such that $c \in U(d)$, and (2): $(\forall V(l) \in UTRult(B))(c \in V(l) \rightarrow V(l)R^tU(d))$. This is equivalent to (2') $(\forall V(l) \in UTRult(B))(c \in V(l) \rightarrow l = d)$. Let us note that it can be shown that $l = d$ iff $d \in V(l)$. Substituting this in (2') we get: (2") $(\forall V(l) \in UTRult(B))(c \in V(l) \rightarrow d \in V(l)$, which by (i) implies $c \leq d$. By (1) we also get that $c \neq 0$. Thus we have: $c \neq 0$, $c \leq d$ and $d \in UTR$. Then by Lemma 3.1 (vii) we get $c \in TR$.

(vi) can be proved in a similar way.

□

3.3. **Canonical construction and the representation theorem.** We assume in this section that $\underline{B}$ is a fixed weak DCA. We associate to $\underline{B}$ a canonical system $W(B) = (W, R^t, R^s, \prec, \mathbf{now})$ as follows:

W is the set UTRult(B) of all UTR-ultrafilters in $\underline{B}$, R^t, R^s and $\prec$ are the canonical relations corresponding to C^t, C^s and $\mathcal{B}$ over the set W and **now** is the fixed UTR-ultrafilter containing **NOW**.

Lemma 3.11. *The system $W(B)$ is a relational dynamic space (see Definition 2.2). called* ***canonical relational dynamic space.***

Proof. First we will verify the compositional axioms

($R^t\circ \prec$) If xR^ty and $y \prec z$, then $x \prec z$, and

($\prec \circ R^t$) If $x \prec y$ and yR^tz, then $x \prec z$.

On the base of Lemma 3.9 (iv) and (v) we have the following calculations:

($R^t\circ \prec$) Suppose $X(c)R^tY(d)$ and $Y(d) \prec Z(e)$. This is equivalent to $c = d$ and $d\mathcal{B}e$ which implies $c\mathcal{B}e$ and consequently $X(c)R^tZ(d)$.

In a similar way one can easily verify ($\prec \circ R^t$) and the remaining axioms for relational dynamic space. □

We construct a strong dca $B(W(B))$ over the canonical relational dynamic space $W(B)$ as in Section 2.1. This strong dca will be called the **canonical strong dca associated to $\underline{B}$.**

We define h to be the Stone mapping defined by

$h(a) = \{U \in UTRult(B) : a \in U\}$.

Lemma 3.12. Canonical correspondence Lemma. *Let A be any formula from the list of* time conditions *(RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr'), (Lin), (Tri'), (Tr) and α be the corresponding formula from the list of* time axioms *(rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr). Then α is true in $\underline{B}$ iff A is true in $W(B)$.*

Proof. By Lemma 3.5 α is true in $\underline{B}$ iff $\tau(A)$ is true in $\underline{B}$. It is easy to see by Lemma 3.9 (v) and (vii) that $\tau(A)$ is true in $\underline{B}$ iff A is true in $W(B)$. Let us demonstrate this by an example: A=(Tri') = $(\forall\Gamma, \Delta)(\Gamma R^t\Delta$ or $\Gamma \prec \Delta$ or $\Delta \prec \Gamma)$, $\tau(A) = (\forall a, b \in UTR)(a = b$ or $a\mathcal{B}b$ or $b\mathcal{B}a)$

($\Rightarrow$). Suppose $\tau(A)$ is true in $\underline{B}$ and let $\Gamma = \Gamma(a)$ and $\Delta = \Delta(b)$ be UTR-filters. Then by the assumption for $\tau(A)$ we have: $a = b$ or $a\mathcal{B}b$ or $b\mathcal{B}a$. Then by Lemma 3.9 (v) and (vii) we obtain $\Gamma R^t \Delta$ or $\Gamma \prec \Delta$ or $\Delta \prec \Gamma$.

($\Leftarrow$) Suppose A holds in $W(B)$ and let $a, b \in UTR$. By axiom (TR2) $a \neq 0$. Then there exists an ultrafilter Γ containing a. Analogously b is contained in an UTR-filter $\Delta = \Delta(b)$. By the assumption on A we have: $\Gamma(a)R^t\Delta(b)$ or $\Delta(a) \prec \Delta(b)$ or $\Delta(b) \prec \Gamma(a)$. Then by Lemma 3.9 we get $a = b$ or $a\mathcal{B}b$ or $b\mathcal{B}a$. □

Lemma 3.13. ***Embedding Lemma.*** *(i) The mapping h preserves the Boolean operations of B.*

(ii) $a \leq b$ iff $h(a) \subseteq h(b)$ in $B(W(B))$.

(iii) aC^tb iff $h(a)C^th(b)$ in $B(W(B))$.

(iv) aC^sb iff $h(a)C^sh(b)$ in $B(W(B))$.

(v) $a\mathcal{B}b$ iff $h(a)\mathcal{B}h(b)$ in $B(W(B))$.

(vi) $c \in TR$ iff $h(c) \in TR(W)$.

(vii) $c \in UTR$ iff $h(c) \in UTR(W)$.

(viii) $h(\boldsymbol{NOW})$ is $\boldsymbol{NOW}(W)$

Proof. **(i)** Because the set $W = UTRult(B)$ is a set of ultrafilters, h preserves Boolean operations.

(ii) Applying Lemma 3.10 (i) we obtain: $a \leq b$ iff $(\forall U \in UTRult(B))$ $(a \in U \rightarrow b \in U)$ iff $(\forall U \in UTRult(B))(U \in h(a) \rightarrow U \in h(b))$ iff $h(a) \subseteq h(b)$ in $B(W(B))$.

(iii) Applying Lemma 3.10 (ii) we obtain: aC^tb iff $(\exists U, V \in UTRult(B))$ $(a \in U, b \in V$ and $UR^tV)$ iff $(\exists U, V \in UTRult(B))(U \in h(a), V \in h(b)$ and $UR^tV)$ iff $h(a)C^th(b)$ in $B(W(B))$.

The conditions (iv)-(viii) can be proved in a similar way applying the corresponding clauses of Lemma 3.10. □

Theorem 3.14. ***Relational representation theorem for weak dca.*** *Let $\underline{B}$ be a weak dca. Then there exists a relational dynamic space $\underline{W}$ and an embedding h of $\underline{B}$ into the strong dca $B(W)$ over $\underline{W}$.*

In addition we have the following. Let A be any formula from the list of time conditions *(RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr'), (Lin), (Tri'), (Tr) and α be the corresponding formula from the list of* time axioms *(rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr). Then α is true in $\underline{B}$ iff A is true in $\underline{W}$.*

Proof. Take $\underline{W}$ to be the canonical dynamic space $W(B)$ over $\underline{B}$. Then the theorem follows from Lemma 3.13 and Lemma 3.12 □

Corollary 3.15. *Every weak dca $\underline{B}$ can be embedded into a strong dca $\underline{B}'$. If in addition $\underline{B}$ satisfies some additional time axioms then $\underline{B}'$ can be chosen to satisfy the same time axioms. In particular the same is true for DCA.*

Proof. The proof follows from Theorem 3.14 and the fact that the dca over a dynamic model of space is always a strong dca. □

4. Basic dynamic contact algebras

In this section we will introduce another version of dca which is a generalization of weak dca having as axioms only universal first-order formulas which are true in weak dca-s. The language will contain also the definable in weak dca-s function $\mathcal{U}tr$

introduced by Definition 3.2. As axioms we take all first-order axioms of weak dca plus several firs-order sentences which follow from the remaining axioms of weak dca. Our aim is to show that the universal first-order theory for basic dca coincides with the universal theory for weak dca, DCA and strong dca. One problem is that the function $\mathcal{U}tr$ is partial and $\mathcal{U}tr(a)$ is defined only for $a \in TR$. The problem is that first-order languages does not accept partial functions. However there is no problem to extend the definition of $\mathcal{U}tr$ putting $\mathcal{U}tr(a) = 0$ for $a \notin TR$. This extension of the definition of $\mathcal{U}tr$ is convenient for two purposes: first it makes this function defined for all values of its argument which is important in first-order axiomatizations, and second, $\mathcal{U}tr(a) = 0$ will indicate that $a \notin TR$ and $a \notin UTR$. This treatment of $\mathcal{U}tr$ will be used in the rest of the paper.

4.1. **The abstract definition.**

Definition 4.1. *A system $\underline{B} = (B, C^t, C^s, \mathcal{B}, TR, UTR, NOW, \mathcal{U}tr)$ is called basic dca if $\mathcal{B}$ is a precontact relation, C^t and C^s are contact relations satisfying the axiom $(C^s \to C^t)$, TR, UTR and* ***NOW*** *satisfy the following list of axioms from weak dca:*
(TR1') If $c \in TR$, then $c \neq 0$,
(TR1") If $c \in TR$ and aC^tc and bC^t, then aC^tb,
(TR2) $c \in UTR$ iff $c \in TR$ and $c\overline{C}^t c^*$.
(TR$\mathcal{B}$1) If $c \in TR$, $c\mathcal{B}b$ and aC^tc, then $a\mathcal{B}b$.
(TR$\mathcal{B}$2) If $d \in TR$, $a\mathcal{B}d$ and bC^td, then $a\mathcal{B}b$.
We assume also the following additional axioms for TR and UTR:
(TR$\leq$) If $c \in TR$, $d \leq c$ and $d \neq 0$ then $d \in TR$,
(TR $\cup$) If $c, d \in TR$ and cC^td, then $(c + d) \in TR$,
(TR$\mathcal{U}tr$ 1) If $c \in TR$, then $\mathcal{U}tr(c) \in UTR$ and $c \leq \mathcal{U}tr(c)$.
(TR$\mathcal{U}tr$ 2) If $c \notin TR$, then $\mathcal{U}tr(c) = 0$.

We denote by Σ_{basic} the class of all basic dca-s. Let Θ be a set of the so called 'time axioms (Section 1.3.3). Then Σ^{Θ}_{basic} denote the class of dca-s satisfying the axioms from Θ.

Let us note that the axiom (TR$\mathcal{U}tr$ 2) is motivated by the discussion above.

Sometimes we will use more informative notations: $C^t = C^t_B$, $C^s = C^s_B$, $\mathcal{B} = \mathcal{B}_B$, $TR = TR(B)$, $UTR = UTR_B$ and $\mathcal{U}tr = \mathcal{U}tr_B$. This notation will be convenient when we consider basic subalgebras of $\underline{B}$. Note that (TR1') and (TR1") are the universal part of (TR1), axioms (TR$\leq$) and (TR $\cup$) are taken from Lemma 3.4 and (TR$\mathcal{U}tr$) is from Definition 3.2. Also the function $\mathcal{U}tr$ is defined only for members from TR and its value is a member from UTR. If $a \in TR$, then the value $\mathcal{U}tr(a) = u$ is the unique element $u \in UTR$ such that $a \leq u$. The element u will be called the UTR-witness of a.

Lemma 4.2. *(i) If $a \in UTR$, then: aC^tb iff $a.b \neq 0$.*
(ii) If $a, b \in TR$ and $b \in UTR$ then: $a.b \neq 0$ iff $a \leq b$.
(iii) If $a, b \in UTR$ then the following conditions are equivalent: (a) aC^tb, (b) $a.b \neq 0$, (c) $a = b$.

Proof. The proof is the same as the proof of Lemma 3.1 (ii), (iii) and (v). □

Lemma 4.3. *(i) The relation C^t is an equivalence relation in the set TR.*
(ii) If $c \in UTR$, then $\mathcal{U}tr(c) = c$.

(iii) If $c \in TR$*, then* $\mathcal{U}tr(\mathcal{U}tr(c)) = \mathcal{U}tr(c)$.

(iv) If $c, d \in TR$ *and* $cC^t d$*, or* $c \leq d$*, or* $c + d \in TR$*, then* $\mathcal{U}tr(c) = \mathcal{U}tr(d)$.

(v) If $\{c_1, \cdots, c_n\} \subseteq TR$ *and for all* $i, j \in \{1, \cdots, n\}$ *we have* $c_i C^t c_j$*, then* $c_1 + \cdots + c_n \in TR$.

(vi) If $d = c_1 + \cdots + c_n \in TR$*, and for all* $i \in \{1, \cdots, n\}$ *we have* $c_i \neq 0$*, then* $\{c_1, \cdots, c_n\} \subseteq TR$ *and for all* $i, j \in \{1, \cdots, n\}$ *we have* $c_i C^t c_j$*. More over* $\mathcal{U}tr(d) = \mathcal{U}tr(c_1) = \ldots = \mathcal{U}tr(c_n)$.

(vii) If $c, d \in UTR$ *and* $c \neq d$*, then* $c.d = 0$.

(viii) If $a.b \in TR$*, then* $a.\mathcal{U}tr(a.b) \in TR$ *and* $\mathcal{U}tr(a.\mathcal{U}tr(a.b)) = \mathcal{U}tr(a.b)$.

(ix) If $a_1 ... a_n \in TR$ *and* $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$*, then* $a_{i_1} ... a_{i_n}.\mathcal{U}tr(a_1 ... a_n) \in TR$ *and* $\mathcal{U}tr(a_{i_1} ... a_{i_n}.\mathcal{U}tr(a_1 ... a_n)) = \mathcal{U}tr(a_1 ... a_n)$.

(x) If $c_1, \ldots, c_n, d \in UTR$*,* $c_1^* ... c_n^*.a \neq 0$ *and* $c_1^* ... c_n^*.a \leq d$*, then* $d \notin \{c_1, \ldots, c_n\}$.

(xi) If $c_1, \ldots, c_n \in UTR$ *and* $c_1 + \ldots + c_n = 1$*, then* $UTR = \{c_1, \ldots, c_n\}$.

Proof. The statements (i)-(vii) are easy consequences from the axioms of basic dca.

(viii). By axiom (TR$\mathcal{U}tr$) we have $\mathcal{U}tr(ab) \in UTR$ (and also $\mathcal{U}tr(ab) \in TR$) and $a.b \leq \mathcal{U}tr(ab)$. Since $a.b \leq a$ we get $a.b \leq (\mathcal{U}tr(ab)).a$ and because $a.b \neq 0$ we get $(\mathcal{U}tr(a.b)).a \neq 0$. From this we obtain $(\mathcal{U}tr(a.b).a \leq \mathcal{U}tr(a.b)$. But $\mathcal{U}tr(a.b) \in TR$, $(\mathcal{U}tr(a.b)).a \neq 0$ and $(\mathcal{U}tr(a.b)).a \leq \mathcal{U}tr(a.b)$ imply by axiom (TR$\leq$) that $(\mathcal{U}tr(a.b)).a \in TR$. The proof of the second part of (viii) follows by using similar reasoning.

(ix) The proof is analogous to that of (viii).

(x) Suppose $d = c_i$, $1 \leq i \leq n$. Then $c_1^* ... c_n^*.a \leq c_i$ and multiplying both sides of the inequality with c_i^* we get $c_1^* ... c_n^*.a \leq 0$ - a contradiction.

(xi) Let $d \in UTR$, then $d = d.1 = d.(c_1 + \ldots + c_n) \neq 0$. So there exists $1 \leq i \leq n$ such that $d.c_i \neq 0$ which by (vii) implies that $d = c_i$ and that $UTR = \{c_1, \ldots, c_n\}$. □

Lemma 4.3 (ix) suggests to introduce the following definition.

Definition 4.4. ***UTR-finite basic dca.*** *Let* $\underline{B}$ *be a basic dca.* $\underline{B}$ *is called UTR-finite basic dca if there is a finite subset* $\{u_1, \ldots, u_n\} \subseteq UTR(B)$ *such that* $u_1 + \ldots + u_n = 1$.

Lemma 4.5. *Every UTR-finite basic dca is weak dca.*

Proof. Let $\underline{B}$ be a UTR-finite basic dca. Then there exists a finite subset $\{u_1, \ldots, u_n\}$ of different elements of $UTR(B)$ such that $u_1 + \ldots + u_n = 1$. By Lemma 4.3 (xi) $UTR(B) = \{u_1, \ldots, u_n\}$, so each UTR member is one of u_i, $i = 1, \ldots, n$. We shall verify that $\underline{B}$ satisfies the axioms of weak dca. This will be done by a series of claims.

Claim 1. If $c \neq 0$, then there exists $u \in UTR$ such that $c.u \neq 0$

Proof. Let $c \neq 0$. We also have $c = c.1 = c.(u_1 + \ldots + u_n) = c.u_1 + \ldots + c.u_n$, so there exists $i = 1, ...n$ such that $c.u_i \neq 0$ (otherwise $c = 0$). □

Claim 2. Let $c \neq 0$ and for all $a, b \in B$: $aC^t c$ and $bC^t c$ implies $aC^t b$. Then $c \in TR$.

Proof. Suppose $c \neq 0$ and for all $a, b \in B$: $aC^t c$ and $bC^t c$ implies $aC^t b$. We will show that there exists $u_i \in UTR$ such that $c \leq u_i$. Suppose the contrary, i.e. for all $u_i \in UTR$ we have $c \nleq u_i$ and hence $c.u_i^* \neq 0$ and $u_i^* C^t c$. By Claim 1 we obtain that there exists $u_j \in UTR$ such that $c.u_j \neq 0$ and hence $u_j C^t c$. Taking i=j we get $u_j^* C^t c$. Take $a = u_j$ and $b = u_j^*$. Then by the assumption of the claim we obtain $u_j C^t u_j^*$ which contradicts the fact that $u_j \in UTR$. □

Claim 2 together with axioms (TR1') and (TR1") shows that axiom (TR1) of weak dca is fulfilled.

Claim 3. If $aC^t b$, then there exists $u \in UTR$ such that $aC^t u$ and $bC^t u$.

Proof. $aC^t b$ iff $a.1C^t b.1$ iff $a.(u_1 + \ldots + a_n)C^t(b.(u_1 + \ldots + u_n)$ iff $(a.u_1 + \ldots + a.u_n)C^t(b.u_1 + \ldots + b.u_n)$ iff there exists i, j, $1 \leq i \leq j \leq n$ such that $a.u_i C^t b.u_j$. The last formula implies $u_i C^t u_j$ and by Lemma 4.2 we get $u_i = u_j$ and $i = j$. Thus, there exists $i : 1 \leq i \leq n$ such that $a.u_i C^t b.u_i$. This last formula implies (by the axioms of contact) that $aC^t u_i$ and $bC^t u_i$. □

Claim 3 verifies axiom (TRC^t). In a similar way one can verify the axioms (TRC^s), (TR$\mathcal{B}$3) and (TR$\mathcal{B}$4). This finishes the proof of Lemma 4.5.

Lemma 4.6. ***Finite generation Lemma.***

*Let $\underline{B} = (B, C^t, C^s, \mathcal{B}, TR, UTR, NOW, \mathcal{U}tr)$ be a basic dca and let $A = \{a_1, \ldots, a_n\}$ be a finite subset of B containing **NOW**. Then there exists a finite subalgebra B_0 of $\underline{B}$ containing A.*

Proof. In order to make the proof easy to follow we will prove the statement for the following special representative case: $A = \{u, v, c, d\}$ where u, v be two different elements of UTR one of which is **NOW** and c, d be two different elements of B which are different from 0 and 1 and are not from UTR. Since $u \neq v$ it can be easily derived that $u.v^* \neq 0$ and $u^*.v \neq 0$. The case $u^*.v^* = 0$ implies that $u+v = 1$ which by Lemma 4.3 (ix) shows that the only UTR elements of $\underline{B}$ are u and v. In this case take the Boolean subalgebra B_0 generated by the set A and consider it with the same contacts C^t, C^s and the precontact $\mathcal{B}$. Define $UTR_{B_0} = \{u, v\} = UTR_B$, $TR_{B_0} = \{a \in B_0 : a \in UTR_B\}$ and $\mathcal{U}tr_{B_0}$ to be the restriction of $\mathcal{U}tr_B$ to B_0. Obviously $\mathcal{U}tr_{B_0}$ is defined for the elements of TR_{B_0} and takes values in UTR_{B_0}, so $\underline{B}_0$ is a basic dca-subalgebra of $\underline{B}$. Let us now consider the case $u^*.v^* \neq 0$.

Consider the following 16 elements of B grouped in the following 4 groups:

(I) $u.v.c.d$, $u.v.c.d^*$, $u.v.c^*.d$, $u.v.c^*.d^*$,

(II) $u.v^*.c.d$, $u.v^*.c.d^*$, $u.v^*.c^*.d$, $u.v^*.c^*.d^*$,

(III) $u^*.v.c.d$, $u^*.v.c.d^*$, $u^*.v.c^*.d$, $u^*.v.c^*.d^*$,

(IV) $u^*.v^*.c.d$, $u^*.v^*.c.d^*$, $u^*.v^*.c^*.d$, $u^*.v^*.c^*.d^*$.

Let us note that all elements from the group (I) are 0 because $u.v = 0$ (u and v are two different elements of UTR, see Lemma 4.3). We claim that it is not possible that all elements from group (II) to be equal to 0. Suppose that this is so, then we get the following: $0 = u.v^*.c.d + u.v^*.c.d^* + u.v^*.c^*.d + u.v^*.c^*.d^* = u.v^*.(c.d + c.d^* + c^*.d + c^*.d^*) = u.v^*.1$, hence $u.v^* = 0$ which is not true. In a similar way we show that not all members from the groups (III) and (IV) are equal to 0 (for (III) we use the fact that $u^*.v \neq 0$ and for (IV) that $u^*.v^* \neq 0$).

Now consider all possible sums of the members of the above groups. In particular, some of these sums are equal to the elements u, v, c, d and the sum of the members of all groups gives the element 1 (all these are elementary facts from the theory of Boolean algebras). They form a Boolean subalgebra of $\underline{B}$ which may not be closed with respect to the operation $\mathcal{U}tr$, which is different from 0 only on members which are from the set TR. We claim that all non-zero elements from groups (II) and (III) (and such exist) are members of TR. Suppose for instance the first member of (II) $u.v^*.c.d \neq 0$. But we have also that $u.v^*.c.d \leq u \in UTR$ which implies by axioms (TR$\leq$) and (TR2) that $u.v^*.c.d \in TR$. For this member we have $\mathcal{U}tr(u.v^*.c.d) = u$ and similarly for the other members of groups (II) and (III).

Other candidates for TR from the above groups are the non-zero members of group (IV) (such elements exist) and we look in the algebra $\underline{B}$ if this is so. Let for simplicity all members from group (IV) are members of TR. Applying to them the function $\mathcal{U}tr$ we find four elements w_1, w_2, w_3, w_4 from UTR such that the following holds:

(#) $u^*.v^*.c.d \leq w_1$, $u^*.v^*.c.d^* \leq w_2$, $u^*.v^*.c^*.d \leq w_3$, $u^*.v^*.c^*.d^* \leq w_4$.

We claim that w_1, w_2, w_3, w_4 are different from u and v. Suppose for instance that $w_1 = u$ then we have $u^*.v^*.c.d \leq u$.. Multiplying both sides of this inequality with u^* we obtain $u^*.v^*.c.d = 0$ which is impossible, because $u^*.v^*.c.d \in TR$. The same result will be obtained if we proceed with $w_1 = v$. So w_1, w_2, w_3, w_4 are new members which we should include in the subalgebra we are looking for. For that purpose we consider the group (V) of the following elements:

(V) (1) $w_1.w_2^*.w_3^*.w_4^*$, (2) $w_1^*.w_2.w_3^*.w_4^*$, (3) $w_1^*.w_2^*.w_3.w_4^*$, (4) $w_1^*.w_2^*.w_3^*.w_4$, (5) $w_1^*.w_2^*.w_3^*.w_4^*$.

Now we multiply each element from the groups (I) - (IV) by each element from the set (V) and then consider all possible sums. They generate a new finite Boolean subalgebra of $\underline{B}$, denoted by B_0, containing the elements u, v, c, d and w_1, w_2, w_3, w_4. We are interested if this subalgebra is closed under the operation $\mathcal{U}tr$ applied to members of B_0 which are members of TR. The possible values of this function should be the elements $u, v, w_1, w_2, w_3, w_4 \in UTR$. For that purpose let us look which are the members of TR in this subalgebra and if their UTR-witnesses are in the set u, v, w_1, w_2, w_3, w_4. Note that all multiplications of the members from the groups (II) and (III) with the elements (1), (2), (3) and (4) from the group (V) are equal to 0, because they contain two different elements from UTR. So the only possible non-zero multiplications from these groups are with the element (5) $w_1^*.w_2^*.w_3^*.w_4^*$. For instance for the first member of (II) the result is $u.v^*.c.d.w_1^*.w_2^*.w_3^*.w_4^* \leq u$. If it is non-zero then it is a member of TR with UTR-witness u. The other possible members of TR from these multiplications will have as UTR-witness either u or v. Now let us consider possible multiplications of the members from the group (IV) with the elements from the group (V). The member $u^*.v^*.c.d$ can have possible non-zero multiplication only with the element (1) $w_1.w_2^*.w_3^*.w_4^*$ and for the result we have $u^*.v^*.c.d.w_1.w_2^*.w_3^*.w_4^* \leq w_1$. If it is non-zero then it is a member of TR with UTR-witness w_1. Why the other combinations give zero multiplication? Consider for instance the multiplication of $u^*.v^*.c.d$ with (2): $u^*.v^*.c.d.w_1^*.w_2.w_3^*.w_4^*$. This element is $\leq w_1$ and $\leq w_1^*$ which implies that it is equal to 0. We obtain the same result of multiplying of $u^*.v^*.c.d$ with the elements (3), (4) and (5). So, the possible TR-members from the multiplications of the group (IV) and (V) have UTR-witnesses the elements from the set w_1, w_2, w_3, w_4. Are there candidates for new TR-members from the sums and which are their UTR-witnesses? If a sum $d = a_1 + \ldots + a_k$ of non-zero members of the above considered groups is a member of TR, then by Lemma 4.3 (vi) and (v) we may conclude that all a_i are also members of TR and the UTR-witness of the sum d and all a_i are equal. So it is possible to have new TR members but their UTR-witnesses are from the set u, v, w_1, w_2, w_3, w_4 which are contained in the finite Boolean subalgebra generated by u, v, c, d plus w_1, w_2, w_3, w_4. This shows that this finite Boolean subalgebra is also closed with respect to the function $\mathcal{U}tr$ applied for the members of B_0 which are members of TR.

We consider $C^t_{B_0}$, $C^s_{B_0}$, $\mathcal{B}_{B_0}$, TR_{B_0} and UTR_{B_0} to be the restrictions of the corresponding relations from B in the set B_0, then this makes B_0 a finite basic DCA which is a subalgebra of $\underline{B}$. Let us note that the proof of the general case can go in the same way. □

Let us note that we may consider basic dca-s satisfying some of the time axioms (rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr). However we can not state here that the Translation Lemma which is true for weak DCA-s (and also for DCA and strong dca, see Lemma 3.5) is true for basic dca. The proof of this lemma for weak dca, DCA and strong dca essentially uses the non-universal axioms which are excluded from the definition of basic dca. Note that all these axioms are universal statements except (irr) and (tr). Since universal statements are preserved under subalgebras, we can obtain the following version of Lemma 4.6 as a simple corollary.

Corollary 4.7. *Let $\underline{B} = (B, C^t, C^s, \mathcal{B}, TR, UTR, NOW, \mathcal{U}tr)$ be a basic dca and let $A = \{a_1, \ldots, a_n\}$ be a finite subset of B containing **NOW**. Suppose in addition that $\underline{B}$ satisfies a set Θ of universal time axioms. Then there exists a finite subalgebra B_0 of $\underline{B}$ containing A and satisfying the axioms from Θ.*

4.2. **Relational models for basic dca-s.** In this section we introduce a generalization of relational dynamic space introduced by Definition 2.2.

Definition 4.8. *By a **basic relational dynamic space** we mean the relational system $\underline{W} = (W, W^0, R^t, R^s, \prec, \boldsymbol{now})$ such that $W \neq \varnothing$, W^0 is a subset of W containing **now** and the following additional conditions are satisfied:*

- *R^t is a symmetric and reflexive relation in W which is transitive in W^0 (hence is an equivalence relation in W^0), and the following additional conditions are satisfied:*

  (♣) If $x \in W^0$ and xR^ty, then $y \in W^0$.
- *R^s is a reflexive and symmetric relation included in R^t,*
- *The relation $\prec$ satisfies the following generalized "compositional axioms" (see Lemma 1.4)*
- *$(R^t\circ \prec, W^0)$ If xR^ty, $y \in W^0$ and $y \prec z$, then $x \prec z$,*
- *$(\prec \circ R^t, W^0)$ If $x \prec y$, $y \in W^0$ and yR^tz, then $x \prec z$.*

The subsystem $(W, W^0, R^t, \prec, \boldsymbol{now})$ is called the time substructure of the relational dynamic space.c

The class of all basic relational dynamic spaces is denoted by Δ_{basic}. Let Ω be a subset of the so called time conditions (special conditions on the relation $\prec$, see Section 1.3.3). Then Δ^{Ω}_{basic} denote the class of all relational basic dynamic spaces satisfying the conditions from Ω.

Obviously W^0 with the restriction of all relations to W^0 is a relational dynamic space, so if we add the additional condition that $W^0 = W$ then the system coincides with the system of relational dynamic space. This shows that indeed Definition 4.8 is more general than Definition 2.2 and that all relational dynamic spaces are basic relational dynamic spaces, so $\Delta_{rel} \subseteq \Delta_{basic}$.

Let $\underline{W}$ be a basic relational dynamic space. We associate a basic dca to $\underline{W}$ via the following constructions. The elements of the algebra are all subsets of W. Define the contact relations C^t, C^s and the precontact $\mathcal{B}$ as this is done for the relational dynamic space in Section 2. Define $UTR(W)$ to be the set of equivalence classes of W^0 under the relation R^t and $TR(W)$ to be the set of nonempty subsets of the equivalence classes in W^0. Define **NOW** to be the equivalence class containing **now**. Finally, for $a \in TR(W)$ define $\mathcal{U}tr(a)$ to be the unique equivalence class containing a. It is a routine verification that all axioms of basic dca are fulfilled. We will verify only that if c is an equivalence class then it satisfies the condition $c\overline{C}c^*$. Suppose the contrary, i.e. that cC^tc^*. Then there exist $x \in c$ and $y \notin c$ such that xR^ty, Since $c \subseteq W_0$ then $x \in W^0$. Then xR^ty implies by (♣) that $y \in W^0$ and since c is equivalence class, that $y \in c$ - a contradiction.

So we have the following lemma.

Lemma 4.9. *The above constructed algebra over the basic relational dynamic space is a basic dca.*

The analog of Lemma 2.5 has the same formulation and the same proof. Let us introduce it.

Lemma 4.10. *Let α be any formula from the list of* time axioms *(rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr) and A be its corresponding formula from the list of* time conditions *(RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr'), (Lin), (Tri'), (Tr). Then A is true in $\underline{W}$ iff α is true in $B(W)$.*

4.3. **P-morphisms between basic relational dynamic spaces.** In this section we will study some mappings between basic relational dynamic spaces called p-morphisms. The definition for the particular systems in the form (W, R) was given in [6] and is motivated by a similar (but different) notion from modal logic. This notion is important because of the following fact: if f is a p-morphism from a system $\underline{W}_1$ onto $\underline{W}_2$ then f^{-1} is an isomorphic embedding of the algebra over $\underline{W}_2$ into the algebra over $\underline{W}_1$. Using p-morphisms we will prove that every basic dca over basic relational dynamic space can be embedded into a strong dca. We first repeat the definition of p-morphism from [6] and then we adapt it for the case of relational dynamic spaces.

Definition 4.11. *Let (W_i, R_i), i=1,2 be two relational systems with binary relations R_i and let $f : W_1 \to W_2$ be a surjective mapping from W_1 onto W_2. f is called a p-morphism if it satisfies the following two conditions:*

(PR1) For all $x_1, y_1 \in W_1$: if $x_1R_1y_1$, then $f(x_1)R_2f(y_1)$,

(PR2) For all $x_2, y_2 \in W_2$: if $x_2R_2y_2$, then there exist $x_1, y_1 \in W_1$ such that $x_1R_1y_1$ and $f(x_1) = x_2$ and $f(y_1) = y_2$.

Define for $a_2 \subseteq W_2$, $h_f(a_2) = f^{-1}(a_2) = \{x_1 \in W_1 : f(x_1) \in a_2\}$.

The system $\underline{W}_2$ is called a p-morphic image of $\underline{W}_1$ and $\underline{W}_1$ is called a p-morphic preimage of $\underline{W}_2$.

The following lemma is an algebraic analog of Lemma 2.4 from [6].

Lemma 4.12. *Let $(B(W_i), C_{R_i})$, i=1,2 be the precontact algebra over the relational system (W_i, R_i) (see Section 1.1). Then h_f is an isomorphic embedding of $(B(W_2), C_{R_2})$ into $(B(W_1), C_{R_1})$.*

Obviously the composition of p-morphisms is a p-morphism.

The modification of the above definition for basic relational dynamic spaces is the following.

Definition 4.13. *Let* $\underline{W_i} = (W_i, W_i^0, R_i^t, R_i^s, \prec_i, \boldsymbol{now}_i)$*,* $i = 1, 2$ *be two basic relational dynamic spaces and let* $f : W_1 \to W_2$ *be a surjective mapping from* W_1 *onto* W_2*.* f *is called a p-morphism if it satisfies the following conditions:*

($\bullet$1) *The conditions (PR1) and (PR2) from Definition 4.11 for the three relations* $R = R^t, R^s, \prec$*.*

($\bullet$2) *Let* $\underline{W_i^0}$ *be the restriction of the system* $\underline{W_i}$ *to the set* W_i^0*,* $i = 1, 2$*. Then* f *is an isomorphism from* $\underline{W_1^0}$ *onto* $\underline{W_2^0}$*. In particular we have* $f(\boldsymbol{now}_1)=\boldsymbol{now}_2$*.*

Lemma 4.14. ***P-morphism Lemma.*** *Let* $\underline{W_i} = (W_i, W_i^0, R_i^t, R_i^s, \prec_i, \boldsymbol{now}_i)$*,* $i = 1, 2$ *be two basic relational dynamic spaces and let* $f : W_1 \to W_2$ *be a p-morphism from* W_1 *onto* W_2*. Let* $\underline{B}(W_i)$ *be the basic dca over the space* $\underline{W_i}$*,* $i = 1, 2$*. Then* h_f *is an isomorphic embedding from* $\underline{B}(W_2)$ *into* $\underline{B}(W_1)$*.*

Proof. The proof is the same as the proof of Lemma 4.12. Condition ($\bullet$1) is used as in Lemma 4.12 to show that h_f preserves Boolean and precontact relations and condition ($\bullet$2) is used to show that h_f preserves TR and UTR sets, **NOW** and the function $\mathcal{U}tr$. □

In Section 4.3.1 we will show that each basic relational dynamic space is a p-morphic preimage of a basic relational space with R^t being equivalence relation. In Section 4.3.2 we will show that each basic relational space with R^t an equivalence relation is a p-morphic image of relational dynamic space. This will imply what we need, namely that every basic dca over basic relational dynamic space can be embedded into a strong dca.

4.3.1. *The first p-morphism.*

Lemma 4.15. *Let* $\underline{W_1} = (W_1, W_1^0, R_1^t, R_1^s, \prec_1, \boldsymbol{now}_1)$ *be a basic relational dynamic space. Then there exist a basic relational dynamic space* $\underline{W_2} = (W_2, W_2^0, R_2^t, R_2^s, \prec_2, \boldsymbol{now}_2)$ *with* R_2^t *being an equivalence relation and a p-morphism* f_1 *from* $\underline{W_2}$ *onto* $\underline{W_1}$*.*

Proof. Let $W_2^0 = W_1^0$ and $W_2 = W_1^0 \cup \{(x, \alpha) : x \in \alpha \text{ and } \alpha = \{u, v\}, uR^tv, \alpha \cap W_1^0 = \varnothing\}$.

We define R_2^t in W_2 by cases as follows:
(1) $x, y \in W_1^0$: xR_2^ty iff xR_1^ty,
(2) $(x, \alpha)R_2^t(y, \beta)$ iff $\alpha = \beta$,
(3) $x \in W_1^0$: $x\overline{R}_2^t(y, \beta)$, $(y, \beta)\overline{R}_2^tx$.

Definition of R_2^s:
(1) $x, y \in W_1^0$: xR_2^sy iff xR_1^sy,
(2) $(x, \alpha)R_2^s(y, \beta)$ iff xR_1^sy and $\alpha = \beta$,
(3) $x \in W_1^0$: $x\overline{R}_2^s(y, \beta)$, $(y, \beta)\overline{R}_2^sx$.

Definition of $\prec_2$:
(1) $x, y \in W_1^0$: $x \prec_2 y$ iff $x \prec_1 y$,
(2) $(x, \alpha) \prec_2 (y, \beta)$ iff $x \prec_1 y$,
(3) $x \in W_1^0$: $x \prec_2 (y, \beta)$ iff $x \prec_1 y$, $(y, \beta) \prec_1 x$ iff $y \prec_1 x$.

$\mathbf{now}_2 =_{def} \mathbf{now}_1$.

Definition of the first p-morphism denoted by f_1. (1) For $x \in W^0$: $f_1(x) = x$. (2) $f_1((x, \alpha)) = x$.

It is a routine task to verify that the above defined relational system $\underline{W_2}$ is a basic relational dynamic space and that R_2^t is an equivalence relation. We will verify the conditions of p-morphism.

For ($\bullet$1) we consider three cases. For (PR^t1) suppose xR_1^ty and show $f_1(x)R_2^tf_1(y)$. There are 3 cases:

Case 1: $x, y \in W_1^0$. Here $f_1(x) = x$ and $f_1(y) = y$ and obviously $f_1(x)R_2^tf_1(y)$.

Case 2: $x, y \notin W_1^0$: put $\alpha = \{x, y\}$. Then obviously $(x, \alpha)R_2^t(x, \alpha)$.

Case 3: $x \in W_1^0$ and $y \notin W_1^0$, or $x \notin W_1^0$ and $y \in W_1^0$. This case is impossible because if xR_1^ty and $x \in W_1^0$, then $y \in W_1^0$ and similarly for the second case.

The verification of (PR^s1) is similar using the fact that R_1^s is included in R_1^t.

The verification of $(P \prec 1)$. Suppose $x \prec_1 y$ and consider the three cases for x and y as above. The case 1 is obvious and for the case 2 take $\alpha = \{x\}$ and $\beta = \{y\}$ (by reflexivity we have xR_1^tx, so (x, α) is correctly defined and similarly for (y, β)). Then obviously $(x, \alpha) \prec_2 (y, \beta)$ and $f_1((x, \alpha)) = x$ and $f_1((y, \beta)) = y$. We reason in a similar for the case 3.

The verification of the condition (PR^t2). For the case 1 of the definition of R_2^t the condition is obvious. For the case 2: from $x \in \alpha$, $y \in \beta$ and $\alpha = \beta$ it can easily be derived that xR_1^ty.

The verification of (PR^s2) and $(P \prec 2)$ follow directly from the corresponding definitions.

The condition ($\bullet$2) is obvious because f_1 here is an identity and $W_2^0 = W_1^0$. □

4.3.2. *The second p-morphism.*

Lemma 4.16. *Let $\underline{W_1} = (W_1, W_1^0, R_1^t, R_1^s, \prec_1, \boldsymbol{now}_1)$ be a basic relational dynamic space such that R_1^t is an equivalence relation. Then there exist a relational dynamic space $\underline{W_2} = (W_2, W_2^0, R_2^t, R_2^s, \prec_2, \boldsymbol{now}_2)$ and a p-morphism f_2 from $\underline{W_2}$ onto $\underline{W_1}$.*

Proof. Let $W_2^0 = W_1^0$ and $W_2 = W_1^0 \cup \{(x, i) : x \notin W_1^0 \text{ and } i \in \{1, 2\}\}$.

We define R_2^t in W_2 by cases as follows:

(1) $x, y \in W_1^0$: xR_2^ty iff xR_1^ty,

(2) $(x, i)R_2^t(y, j)$ iff xR_1^ty and $(i = j = 1$ or $i = j = 2$ and $x = y)$,

(3) $x \in W_1^0$: $x\overline{R}_2^t(y, j)$, $(y, j)\overline{R}_2^tx$.

Definition of R_2^s:

(1) $x, y \in W_1^0$: xR_2^sy iff xR_1^sy,

(2) $(x, i)R_2^s(y, j)$ iff xR_1^sy and $(i = j = 1$ or $i = j = 2$ and $x = y)$,

(3) $x \in W_1^0$: $x\overline{R}_2^s(y, j)$, $(y, j)\overline{R}_2^sx$.

Definition of $\prec_2$:

(1) $x, y \in W_1^0$: $x \prec_2 y$ iff $x \prec_1 y$,

(2) $(x, i) \prec_2 (y, j)$ iff $x \prec_1 y$ and $i = j = 2$,

(3) $x \in W_1^0$: $x \prec_2 (y, j)$ iff $x \prec_1 y$ and $j = 2$, $(y, j) \prec_1 x$ iff $y \prec_1 x$ and $j = 2$.

$\mathbf{now}_2 =_{def} \mathbf{now}_1$.

Definition of the second p-morphism denoted by f_2. (1) For $x \in W^0$: $f_2(x) = x$. (2) $f_2((x, \alpha)) = x$.

It is a routine task to verify that the above defined relational system $\underline{W_2}$ is a relational dynamic space. The verification of conditions for p-morphism is the same as for f_1. □

As a consequence of Lemma 4.15 and 4.16 we obtain the following corollary.

Corollary 4.17. *Every basic relational dynamic space is a p-morphic image of a relational dynamic space.*

Proof. Let $\underline{W_1}$ be a basic relational dynamic space. By Lemma 4.15 there exists a basic relational dynamic space $\underline{W_2}$ in which the relation R^t_2 is an equivalence relation and a p-morphism f_1 from $\underline{W_2}$ onto $\underline{W_1}$. By Lemma 4.16 there exist a relational dynamic space $\underline{W_3}$ and a p-morphism f_2 from $\underline{W_3}$ onto $\underline{W_2}$. Then the composition $f = f_2 \circ f_1$ of the two p-morphisms is a p-morphism from $\underline{W_3}$ onto $\underline{W_1}$. □

Proposition 4.18. *Let $\underline{W}$ be a basic relational space and let $\underline{B}(\underline{W})$ be the basic dca over $\underline{W}$. Then there exists a strong dca $\underline{B}$ and an isomorphic embedding of $\underline{B}(\underline{W})$ into $\underline{B}$.*

Proof. Let $\underline{W}$ be a basic relational dynamic space and let $\underline{B}(\underline{W})$ be the basic dca over $\underline{W}$. By Corollary 4.17 there exists a relational dynamic space $\underline{W'}$ and a p-morphism f from $\underline{W'}$ onto $\underline{W}$. Let $\underline{B}(\underline{W'})$ be the strong dca over $\underline{W'}$ (see Lemma 2.3) Then the mapping h_f (see Definition 4.11 and Lemma 4.14) is an embedding from the basic dca $\underline{B}(\underline{W})$ into the strong dca $\underline{B}(\underline{W'})$. □

Definition 4.19. *Let $\underline{W_1}$ and $\underline{W_2}$ be basic relational spaces, f be a p-morphism from $\underline{W_1}$ onto $\underline{W_2}$ and A be a time condition from the list (RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr'), (Lin), (Tri'), (Tr). We say that f preserves A if the following holds: $\underline{W_2}$ satisfies A whenever $\underline{W_1}$ satisfies A.*

Lemma 4.20. *The first and second p-morphisms from Lemma 4.15 and Lemma 4.16 preserve the time conditions **(Dens)**, **(Irr')** and **(Tr)**.*

Proof. The proof is by a routine verification. □

Corollary 4.21. *Let $\underline{W}$ be a basic relational space satisfying some (or all) of the time conditions **(Dens)**, **(Irr')** and **(Tr)** and let $\underline{B}(\underline{W})$ be the basic dca over $\underline{W}$. Then there exists a strong dca $\underline{B}$ satisfying the corresponding time axioms and an isomorphic embedding of $\underline{B}(\underline{W})$ into $\underline{B}$.*

Proof. The proof follows by a modification of the proof of Proposition 4.18 using Lemma 4.20 and Lemma 2.5. □

4.4. **Relational representation theory for finite basic dca-s.** In this section we will prove a representation theorem for finite basic dca-s showing that every finite basic dca is isomorphic with a basic dca over a finite basic relational space. We do not know if such a representation theorem holds for arbitrary basic dca.

4.4.1. *The canonical basic relational space of finite basic dca.* Let $\underline{B}$ be a finite basic dca. Since $\underline{B}$ is a finite Boolean algebra then it is atomic and is isomorphic with the Boolean set algebra over the set of its atoms. Let $At(B)$ be the set of atoms of $\underline{B}$. We will define a relational system $\underline{W}(\underline{B}) = (W, W^0, R^t, R^s, \prec, \mathbf{now})$ associated with $\underline{B}$ and called **canonical basic relational dynamic space** as follows. Put $W = At(B)$. Define $W^0 = \{a \in At(B) : a \in TR(B)\}$. For $a, b \in At(B)$ define: aR^tb iff aC^tb, aR^sb iff aC^sb and $a \prec b$ iff $a\mathcal{B}b$. To define **now** consider the region **NOW**. Since **NOW**$\neq \varnothing$ it contains at least one atom and let **now** is one of them. By axiom (TR$\leq$) **now**$\in TR(B)$ and hence **now**$\in W^0$.

Lemma 4.22. *The above defined system* $\underline{W}(\underline{B}) = (W, W^0, R^t, R^s, \prec, \textbf{\textit{now}})$ *is a basic relational dynamic space.*

Proof. The only nontrivial part of the proof is to verify the condition (♣) if $a \in W^0$, $b \in W$ and aR^tb, then $b \in W^0$. The proof goes as follows. From $a \in W^0$ we get $a \in TR(B)$. Let $c = \mathcal{U}tr(a)$ so $c \in UTR$ and $a \leq c$. By definition aR^tb means aC^tb and by $a \leq c$ we obtain cC^tb. By Lemma 4.2 (i) we get $c.b \neq 0$. Then there exists an atom d such that $d \leq (c.b)$. From here we get $d \leq b$ and since d and b are atoms, then $d = b$, hence $b \leq (c.b) \leq c$. But $c \in UTR$, so $c \in TR$ and $b \leq c$. Since b is an atom, then $b \neq 0$ which together with $b \leq c$ imply (by axiom TR $\leq$) that $b \in TR(B)$, hence $b \in W^0$. □

The following lemma is an analog of Lemma 2.5 but with different proof.

Lemma 4.23. *Let* $\underline{B}$ *be a finite basic dca and let* $\underline{W}(\underline{B}) = (W, W^0, R^t, R^s, \prec, \textbf{\textit{now}})$ *be its canonical basic relational dynamic space. Let* α *be any formula from the list of* time axioms *(rs), (ls), (up dir), (down dir), (dens), (ref), (irr), (lin), (tri), (tr) and A be its corresponding formula from the list of* time conditions *(RS), (LS), (Up Dir), (Down Dir), (Dens), (Ref), (Irr'), (Lin), (Tri'), (Tr). Then A is true in* $\underline{W}(\underline{B})$ *iff* α *is true in* $\underline{B}$.

Proof. By the canonical construction we have that W is the set $At(B)$ of the atoms of B and let $At(B) = \{a_1, \dots, a_n\}$. We will illustrate the proof by considering the case (Dens)⇔(dens). All other cases can be proved in a similar way working with atoms.
(Dens)⇔(dens). ⇒ . Suppose that (Dens) is true. In order to prove (dens) suppose $a\mathcal{B}b$. We have to show that for all p we have $a\mathcal{B}p$ or $p^*\mathcal{B}b$. Let us assume that $a = a_{i_1} + \ldots + a_{i_k}$ and $b = a_{j_1} + \ldots + a_{j_l}$. Then by the distribution axioms of precontact relation we obtain from $a\mathcal{B}b$ that $a_{i_s}\mathcal{B}a_{j_t}$ for some $s \leq k$ and $t \leq l$ (obviously $a_{i_s} \leq a$ and $b_{j_t} \leq b$). This shows that $a_{i_s} \prec a_{j_t}$. By (Dens) there exists an atom a_m such that $a_{i_s} \prec a_m \prec a_{j_t}$ i.e. $a_{i_s}\mathcal{B}a_m$ and $a_m\mathcal{B}a_{j_t}$. Let p be an arbitrary element of B. There are two cases for a_m: $a_m \leq p$ or $a_m \prec p^*$ (this is an obvious property of any atom).

Case 1: $a_m \leq p$. Then from $a_{i_s} \leq a$, $a_{i_s}\mathcal{B}a_m$ we get $a\mathcal{B}p$.

Case 2: $a_m \prec p^*$. In a similar way we obtain $p^*\mathcal{B}b$.

⇐. Suppose that (dens) is true. Let a_k and a_l be two atoms and suppose $a_k \prec a_l$ (i.e. $a_k\mathcal{B}a_l$). We have to show that there exists an atom a_m such that $a_k \prec a_m \prec a_l$ i.e. $a_k\mathcal{B}a_m$ and $a_m\mathcal{B} \prec a_l$. Suppose the contrary, namely

(♯) for all a_m: either $a_k\overline{\mathcal{B}}a_m$ or $a_m\overline{\mathcal{B}}a_l$.

Since $a_k \neq 0$ and $a_l \neq 0$, then by (dens) we have that the following holds:

(♮) For all $p \in B$: either $a_k\mathcal{B}p$ or $p^*\mathcal{B}a_l$.
Let P be the set of all atoms a_m such that $a_k\overline{\mathcal{B}}\underline{a_m}$ and let p be their sum. Then by the distributivity axioms of precontact we get $a_k\overline{\mathcal{B}}p$ and by (♮) we get that $p^*\mathcal{B}a)_l$. Obviously p^* will be the sum of all elements from the complement of P for which we have: $a_k\mathcal{B}a_m$. But by (♯) we obtain that for these elements we have $a_m\underline{\mathcal{B}}a_l$ and for their sum p^* that $p^*\overline{\mathcal{B}}a_l$ which contradicts $p^*\mathcal{B}a_l$. □

4.4.2. *The isomorphism Theorem for finite basic dca.*

Proposition 4.24. ***Relational representation of finite basic dca-s***. *Let $\underline{B}$ be a finite basic dca and let $\underline{W} = \underline{W}(\underline{B})$ be its canonical basic relational space. Denote by $\underline{B}(\underline{W})$ the basic dca over $\underline{W}$. Then:*

(i) $\underline{B}$ is isomorphic with $\underline{B}(\underline{W})$.

(ii) If $\underline{B}$ satisfies some of the time axioms then $\underline{B}(\underline{W})$ satisfies the same axioms.

Proof. (i) Because $\underline{W}(\underline{B})$ is a finite Boolean algebra then there is a Boolean isomorphism h of B with the Boolean algebra of subsets of W, namely $h(a) = \{c \in At(B) : c \leq a\}$. Let $h(a) = \{c_1, \cdots, c_k\}$. It is easy to see that $a = c_1 + \cdots + c_k$. Using this observation it can be easily shown that h preserves also the relations $C^t, C^s, \mathcal{B}$, the sets $TR(B)$ and $UTR(B)$, and that $h(\mathbf{NOW}(B)) = \mathbf{NOW}(B(W))$. As an example we will verify only the equivalence:

$a \in UTR(B)$ iff $h(a) \in UTR(B(W))$.

For the direction from left to the right suppose that $a \in UTR(B)$ and let $a = c_1 + \cdots + c_k$ where $h(a) = \{c_1, \cdots, c_k\}$. Then we have $c_1 + \cdots + c_k \leq a$ and also $a \in TR$. By Lemma 4.3 (vi) we have $c_i C^t c_j$ (hence $c_i R^t c_j$) for all $i, j \leq k$ and $c_i \in TR(B)$ for all $i \leq k$. We will show that the set $\{c_1, \cdots, c_k\}$ is an R^t-equivalence class. Suppose that $b \in W$ and that $c_1 R^t b$. Then $c_1 C^t b$. Since $c_1 \leq a$ we have $aC^t b$. Since $c_1 \in W^0$ and $c_1 R^t b$ then $b \in W^0$, so $b \in TR(B)$. We will show that $b \leq a$. Suppose not, i.e. $b \not\leq a$. Then $a^*.b \neq 0$ and $a^* C^t b$. Then from $a^* C^t b$, $aC^t b$ and $b \in TR(B)$ we get that $aC^t a^*$ which contradicts that $a \in UTR$. So we have that $b \leq a = c_1 + \cdots + c_k$. This implies that there exists $i \leq k$ such that $b = c_i$ which completes the proof that $h(a)$ is an equivalence class with respect to R^t and that $h(a) \in UTR(B(W))$.

For the implication from the right to the left, suppose that $h(a) \in UTR(B(W))$ i.e. that (by definition) $h(a) = \{c_1, \cdots, c_k\}$ is an equivalence class with respect to R^t. First we show that $a \in TR$. We have that for all $i, j \leq k$ $c_i R^t c_j$. Then by Lemma 4.3 (v) $a = c_1 + \cdots + c_k \in TR$. It remains to show that $a\overline{C}^t a^*$. Suppose for the sake of contradiction that $aC^t a^*$. So $a^* \neq 0$. Let $h(a^*) = \{d_1, \ldots, d_l\}$. Then $h(a) \cap h(a^*) = \varnothing$ (h is a Boolean isomorphism) and consequently no $d_j \in \{c_1, \ldots, c_k\}$, $j \leq l$. However, $aC^t a^*$ implies that for some $i, j : i \leq k$ and $j \leq l$ we have that $c_i C^t d_j$, i.e. $c_i R^t d_j$ and since $h(a)$ is an equivalence class, then $d_j \in h(a)$ - a contradiction.

(ii) Let $\underline{B}$ satisfies some of the time axioms. Then by Lemma 4.23 the canonical space $\underline{W}(\underline{B})$ satisfies the corresponding time conditions. Applying Lemma 4.10 we get that $\underline{B}(\underline{W})$ satisfies the considered time axioms. □

Proposition 4.25. *(i) Every finite basic dca $\underline{B}$ can be isomorphically embedded into a strong dca $\underline{B'}$*

(ii) If in addition $\underline{B}$ satisfies some (or all) of the axioms (dens), (irr) and (tr), then $\underline{B'}$ can be chosen to satisfy the same axioms.

Proof. The proposition directly follows from Proposition 4.24 and Proposition 4.18. □

5. Quantifier-free logics for space and time

In this paper we considered several classes of dca-s: DCA introduced in [5] (see also Definition 1.5), weak and strong dca introduced in Section 2 and basic dca, introduced in Section 4.

All these algebras are based on one and the same first-order language except that the language of basic dca contains a function $\mathcal{U}tr$ which in the other kinds of dca-s is definable (see comments for this function at the beginning of Section 4). Consequently we may assume that all four types of dca-s are based on one and the same first-order language. In this section we will present four minimal quantifier free logics based on this language corresponding to the four types dca-s with various kinds of semantics: algebraic semantics, based on the four kinds dca, relational semantics (Kripke style), based on some relational models of the algebras, and snapshot semantics, based on snapshot models for the DCA introduced in [5] and described also in Section 1.2 (see also Remark 2.6). We denote these four minimal logics correspondingly by $\mathbf{L}^{min}_{basic}$, $\mathbf{L}^{min}_{weak}$, $\mathbf{L}^{min}_{DCA}$ and $\mathbf{L}^{min}_{strong}$.

The quantifier-free axiomatization of the corresponding minimal logic can be obtained as follows. We consider classical propositional logic as a base logic plus all universal axioms and several universal consequences of the axioms from the corresponding dca. The rules of inference are Modus Ponens and some non-standard rules corresponding to non-universal axioms of the corresponding dca. Then, we may extend the corresponding minimal logic with some number of axioms or rules corresponding to the so called "time axioms" (see Section 1.3.3).

5.1. **The language and syntax.** We consider a first-order language $\mathbb{L}$ without quantifiers containing the following symbols:

(I) propositional operations of negation $\neg$, conjunction $\wedge$, disjunction $\vee$, implication $\Rightarrow$ and equivalence $\Leftrightarrow$,

(II) a denumerable set of term variables called Boolean variables,

(III) three constant symbols: $0, 1$ called Boolean constants and **NOW**,

(IV) Boolean operations $*$ complement, . product and + sum and the one place function symbol $\mathcal{U}lt$,

(V) unary predicate symbols TR and UTR, and binary predicate symbols: $\leq$ Boolean ordering, C^t, C^s and $\mathcal{B}$.

(VI) brackets (and).

Terms are defined in the standard way from the term variables and constants by using Boolean operations and $\mathcal{U}lt$. Atomic formulas are the following (a, b are terms): $TR(a)$, $UTR(a)$, $a = b$, $a \leq b$, aC^tb, aC^sb, $a\mathcal{B}b$. Non-atomic formulas are defined from atomic formulas in a standard way using propositional operations.

Notation: Instead of $TR(a)$ and $UTR(a)$ we will use the more common notation $a \in TR$ and $a \in UTR$.

Abbreviations: $a = b =_{def} (a \leq b) \wedge (b \leq a)$, $a \neq b =_{def} \neg(a = b)$
$a \not\leq b =_{def} \neg(a \leq b)$, $a \notin TR =_{def} \neg(a \in TR)$, $a \notin UTR =_{def} \neg(a \in UTR)$,
If C is one of the symbols $C^t, C^s, \mathcal{B}$, then $a\overline{C}b =_{def} \neg(aCb)$, $\top =_{def} (a \leq a)$, $\bot =_{def} (a \not\leq a)$.

We adopt the standard rules in first-order logic for omissions of brackets.

5.2. **Algebraic semantics.** First we introduce algebraic semantics for the language $\mathbb{L}$. It can be interpreted in any of the four types of dca as follows. Let $\underline{B}$ be a dca of the given type and v be a valuation (function) of the Boolean variables in B. Then the pair $M = (B, v)$ is called a model. The extension of v for arbitrary terms is as follows (by induction on complexity of the term): $v(0) = 0$, $v(1) = 1$, $v(\mathbf{NOW}) = \mathbf{NOW}(B)$, $v(\mathcal{U}tr(a)) = \mathcal{U}tr(v(a))$, $v(a^*) = v(a)^*$, $v(a.b) = v(a).v(b)$, $v(a + b) = v(a) + v(b)$.

The truth of a formula A in a model (B, v), denoted by $(B, v) \models A$ is defined inductively by the complexity of A as follows. For atomic formulas:
$(B, v) \models a \in TR$ iff $v(a) \in TR(B)$, $(B, v) \models a \in UTR$ iff $v(a) \in UTR(B)$,
$(B, v) \models a \leq b$ iff $v(a) \leq v(b)$, $(B, v) \models aC^t b$ iff $v(a)C^t v(b)$,
$(B, v) \models aC^s b$ iff $v(a)C^s v(b)$, $(B, v) \models a\mathcal{B}b$ iff $v(a)\mathcal{B}v(b)$.

For nonatomic formulas: $(B, v) \models \neg A$ iff $(B, v) \not\models A$, $(B, v) \models A \wedge B$ iff $(B, v) \models A$ and $(B, v) \models B$ and similarly for disjunctions implications and equivalencies.

We say that the formula A is true in the algebra $\underline{B}$ (in symbols $\underline{B} \models A$) if for every valuation v in B we have that $(B, v) \models A$. If Σ is a class of algebras (of one of the given four types), then A is true in Σ (in symbols $\Sigma \models A$), if for every $\underline{B} \in \Sigma$ we have $\underline{B} \models A$. If Σ is a class of algebras we denote by $\mathcal{L}(\Sigma)$ the set of all formulas A which are true in Σ and call this set the logic of Σ.

If Σ is a class of algebras then Σ^{fin} denotes the class of finite members of Σ. Let us remind that Σ_{basic} is the class of all basic dca-s, Σ_{weak} is the class of all weak dca-s, Σ_{DCA} is the class of all DCA-s and Σ_{strong} is the class of all strong dca-s and that we have the following inclusions $\Sigma_{basic} \supseteq \Sigma_{weak} \supseteq \Sigma_{DCA} \supseteq \Sigma_{strong}$. Let Θ be a set of time axioms (see Section 1.3.3) we denote by Σ^{Θ} the class of all members of Σ satisfying the axioms of Θ. We have also the following inclusions: $\Sigma^{\Theta}_{basic} \supseteq \Sigma^{\Theta}_{weak} \supseteq \Sigma^{\Theta}_{DCA} \supseteq \Sigma^{\Theta}_{strong}$.

The following lemma is obvious:

Lemma 5.1. *Let Σ_1 and Σ_2 be two classes of dca-s and $\Sigma_1 \subseteq \Sigma_2$. Then $\mathcal{L}(\Sigma_2) \subseteq \mathcal{L}(\Sigma_1)$.*

Proposition 5.2. *Let Θ be a set of time axioms which are universal sentences (i.e. Θ does not contain **irr** and **tr**). Then $\mathcal{L}(\Sigma^{\Theta}_{basic}) = \mathcal{L}(\Sigma^{fin,\Theta}_{basic})$.*

Proof. The proof follows from Lemma 5.1 and Lemma 4.7. □

Proposition 5.3. $\mathcal{L}(\Sigma_{basic}) = \mathcal{L}(\Sigma_{strong})$

Proof. The proof follows by Lemma 5.1 and Proposition 5.2 and Proposition 4.25. □

As a corollary we obtain the following theorem.

Theorem 5.4. *The logics $\mathcal{L}(\Sigma_{basic})$, $\mathcal{L}(\Sigma_{weak})$, $\mathcal{L}(\Sigma_{DCA})$ and $\mathcal{L}(\Sigma_{strong})$ are equal.*

Proof. By Lemma 5.1 and Lemma 5.3 we have $\mathcal{L}(\Sigma_{basic}) \subseteq \mathcal{L}(\Sigma_{weak}) \subseteq \mathcal{L}(\Sigma_{DCA}) \subseteq \mathcal{L}(\Sigma_{strong}) = \mathcal{L}(\Sigma_{basic})$, which implies the required equality. □

Theorem 5.5. *Let Θ be a set of time axioms. Then the logics $\mathcal{L}(\Sigma^{\Theta}_{weak})$, $\mathcal{L}(\Sigma^{\Theta}_{DCA})$, $\mathcal{L}(\Sigma^{\Theta}_{strong})$ are equal.*

Proof. We have the obvious inclusions: $\Sigma^{\Theta}_{strong} \subseteq \Sigma^{\Theta}_{DCA} \subseteq \Sigma^{\Theta}_{weak}$. By Lemma 5.1 we obtain $\mathcal{L}(\Sigma^{\Theta}_{weak}) \subseteq \mathcal{L}(\Sigma^{\Theta}_{DCA}) \subseteq \mathcal{L}(\Sigma^{\Theta}_{strong})$. By Theorem 3.14 we obtain $\mathcal{L}(\Sigma^{\Theta}_{strong}) \subseteq \mathcal{L}(\Sigma^{\Theta}_{weak})$ which with the previous inclusions gets the equality of the three logics. □

A stronger form of Proposition 5.3 is the following.

Proposition 5.6. *let Θ be a set consisting of some (or of all) of the time axioms (dens), (irr) and (tr). Then $\mathcal{L}(\Sigma^{\Theta}_{basic}) = \mathcal{L}(\Sigma^{\Theta}_{strong})$.*

Proof. The proof follows from Lemma 5.1, Lemma 4.25 and Proposition 5.2. □

Corollary 5.7. *Let Θ be a set consisting of some (or of all) of the time axioms (dens), (irr) and (tr). Then the logics $\mathcal{L}(\Sigma^{\Theta}_{basic})$, $\mathcal{L}(\Sigma^{\Theta}_{weak})$, $\mathcal{L}(\Sigma^{\Theta}_{DCA})$, $\mathcal{L}(\Sigma^{\Theta}_{strong})$ are equal.*

5.3. **Relational semantics.** Let Δ_{basic} (respectively Δ^{fin}_{basic}) be the class of all basic (finite) relational dynamic spaces and Δ_{rel} (respectively Δ^{fin}_{rel}) be the class of all relational (finite) dynamic spaces and note that that $\Delta_{rel} \subseteq \Delta_{basic}$. Also if Ω is a set of "time conditions" (see Remark 2.4) then Δ^{Ω}_{basic} is the class of all spaces satisfying Ω and similarly for Δ^{Ω}_{rel} and we have $\Delta^{\Omega}_{rel} \subseteq \Delta^{\Omega}_{basic}$. If Δ is a class of spaces then Δ^{fin} denote the class of its finite members.

Relational (Kripke style) semantics for the language $\mathbb{L}$ can be defined as follows. Let $\underline{W}$ be a basic relational dynamic space and $\underline{B}(\underline{W})$ be the dca over $\underline{W}$. Let v be a function (valuation) associated to each variable a a subset $v(a) \subseteq W$ and the pair $(\underline{W}, v)$ is called a relational (Kripke) model. We say that a formula A is true in the model $(\underline{W}, v)$ if it is true in the algebraic model $(\underline{B}(\underline{W}), v)$, and similarly the notions "true in a space $\underline{W}$" and "true in a class of spaces". Let Δ be a class of basic relational dynamic spaces and let us denote by $\mathcal{L}(\Delta)$ the set of all formulas true in Δ and call this set the logic of Δ. If Δ is a class of basic relational dynamic spaces we denote by $\Sigma(\Delta)$ the class of all dca-s over the members of Δ. Obviously we have $\mathcal{L}(\Delta) = \mathcal{L}(\Sigma(\Delta))$. Thus all notions related to Kripke semantics can be reduced to corresponding notions related to algebraic semantics. It is easy to see that all interesting statements proved in the preceding section concerning logics of some classes of algebraic models can be easily transformed for statements about logics of some classes of basic relational dynamic spaces. In order to safe space we will skip this.

5.4. **Snapshot semantics.** As we have already seen, relational semantics of the language $\mathbb{L}$ is a special algebraic semantics when algebras are over some relational systems. The main representation theorem for DCA-s is for algebras arising from snapshot models of space (see [5] and Section 1.3.2. By Remark 2.1 representation theorem for weak dca-s (and hence for strong dca-s) with respect to snapshot models of space can also be proved. It is mentioned in Remark 2.6 how snapshot models can be extracted from relational models of space. In this text we will not go into details for snapshot semantics for the language $\mathbb{L}$.

5.5. **Axiomatization of $\mathbf{L}^{min}_{basic}$.** The axiomatic system for $\mathbf{L}^{min}_{basic}$, which corresponds to the class Σ_{basic} of all basic dca-s, contains all first-order axioms for Boolean algebra plus the corresponding axioms for equality (they are universal formulas), and all axioms for basic dca which are also universal first-order formulas plus the following replacement axiom for the operation $\mathcal{U}tr$:

(Replacement $\mathcal{U}tr$) $(a = b) \Rightarrow (\mathcal{U}tr(a) = \mathcal{U}tr(b))$.

We may consider extensions of $\mathbf{L}^{min}_{basic}$ by some universal time axioms. Note that from the ten time axioms considered in Section 1.3.3 only two are not universal:

(irr) $a\mathcal{B}b \rightarrow (\exists c, d)(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d)$,

(tr) $a\overline{\mathcal{B}}b \rightarrow (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{\mathcal{B}}b)$.

Note also that the above two formulas can be easily transformed into the following equivalent forms:

(irr') $(\forall c, d)\neg(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d) \Rightarrow a\overline{\mathcal{B}}b$,

(tr') $(\forall c)(a\mathcal{B}c \vee c^*\mathcal{B}b) \Rightarrow a\mathcal{B}b$.

It is important to note that **(irr')** and **(tr')** are in the following special non-universal form:

(✠) $(\forall b_1, \dots, b_m)A(a_1, \dots, a_n, b_1, \dots, b_m) \Rightarrow B(a_1, \dots, a_n)$,
where $a_1, \dots, a_n$ are terms, $b_1, \dots, b_m$ are term variables which are not included in the formula $B(a_1, \dots, a_n)$, and the terms $a_1, \dots, a_n$. Also the notation $A(a_1, \dots, a_n, b_1, \dots, b_m)$ means that $a_1, \dots, a_n, b_1, \dots, b_m$ are the only terms included in A (respectively the same for $B(a_1, \dots, a_n)$.

We transform the formula (✠) into the following quantifier-free rule of inference.

RULE (✠) $\frac{C \Rightarrow A(a_1,\dots,a_n,b_1,\dots,b_m)}{C \Rightarrow B(a_1,\dots,a_n)}$.

The RULE (✠) is subject to the following constraints: $a_1, \dots, a_n$ are terms, $b_1, \dots, b_m$ are term variables which are not included in the formulas C, $B(a_1, \dots, a_n)$, and consequently in the terms $a_1, \dots, a_n$. The formula $C \Rightarrow A(a_1, \dots, a_n, b_1, \dots, b_m)$ is called the premise of the rule and the formula $C \Rightarrow B(a_1, \dots, a_n)$ is called the conclusion of the rule.

Note that replacing C with arbitrary true formula not containing $b_1, \dots, b_m$ we obtain the following simplified version of the rule, which however can not replace **RULE** (✠) for obtaining completeness results.

Simplified **RULE** (✠) $\frac{A(a_1,\dots,a_n,b_1,\dots,b_m)}{B(a_1,\dots,a_n)}$.

Following this pattern of transformation we obtain the following two rules of inference corresponding to **(irr)** and **(tr)**:

RULE (irr) $\frac{C \Rightarrow \neg(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c \overline{C}^t d)}{C \Rightarrow a \overline{\overline{\mathcal{B}}} b}$, where c, d are variables not occurring in the terms a, b and the formula C.

RULE (tr) $\frac{C \Rightarrow a\mathcal{B}c \vee c^* \mathcal{B} b}{C \Rightarrow a\mathcal{B}b}$, where c is a variable not occurring in the terms a, b and the formula C.

The above two rules replace the time axioms **irr** and **tr** and consequently will be called "time rules". We may consider extensions of $\mathbf{L}^{min}_{basic}$ with some universal time axioms (i.e. time axioms different from **irr** and **tr**) and some of the rules **RULE(irr)** and **RULE (tr)**.

5.6. **Nonstandard rules of inference.** The rules in the type **RULE** (✠) are called nonstandard rules of inference. In the context of logics for region-based theories of space such rules were studied for the first time in [6]. Fortunately all non-universal axioms for weak dca-s, DCA-s and strong dca-s can be transformed equivalently in the form of (✠) and consequently they can be reformulated into additional nonstandard rules in the form **RULE** (✠) which will be used in the axiomatizations of the minimal logics $\mathbf{L}^{min}_{weak}$, $\mathbf{L}^{min}_{DCA}$ and $\mathbf{L}^{min}_{strong}$. Inspecting the definitions of weak dca, DCA and strong dca we can see that we have the following non-universal axioms written in the quantified extension of the language $\mathbb{L}$.

(I) Non-universal axioms for weak dca:

(TR1←) $c \neq 0 \wedge (\forall a)(\forall b)(aC^tc \wedge bC^tc \rightarrow aC^tb) \Rightarrow c \in TR$.

(TRC^t) $aC^tb \Rightarrow (\exists c)(c \in UTR \wedge aC^tc \wedge bC^tc)$.

(TRC^s) $aC^sb \Rightarrow (\exists c)(c \in UTR \wedge (a.c)C^sb)$.

(TR$\mathcal{B}$3) $a\mathcal{B}b \Rightarrow (\exists c)(c \in UTR \wedge c\mathcal{B}b \wedge aC^tc)$.

(TR$\mathcal{B}$4) $a\mathcal{B}b \Rightarrow (\exists d)(d \in UTR \wedge a\mathcal{B}d \wedge bC^td)$.

(UTR$\mathcal{B}$11→)

$(\forall p)(c \in UTR \wedge d \in UTR \wedge (p\mathcal{B}c \vee p^*\mathcal{B}d)) \Rightarrow (\exists e)(e \in UTR \wedge e\mathcal{B}c \wedge e\mathcal{B}d)$.

(UTR$\mathcal{B}$12→)

$(\forall p)(c \in UTR \wedge d \in UTR \wedge (p\mathcal{B}c \vee d\mathcal{B}p^*)) \Rightarrow (\exists e)(e \in UTR \wedge e\mathcal{B}c \wedge d\mathcal{B}e)$.

(UTR$\mathcal{B}$21→)
$(\forall p)(c \in UTR \wedge d \in UTR \wedge (c\mathcal{B}p \vee p^*\mathcal{B}d)) \Rightarrow (\exists e)(e \in UTR \wedge c\mathcal{B}e \wedge e\mathcal{B}d)$.
(UTR$\mathcal{B}$22→)
$(\forall p)(c \in UTR \wedge d \in UTR \wedge (c\mathcal{B}p \vee d\mathcal{B}p^*)) \Rightarrow (\exists e)(e \in UTR \wedge c\mathcal{B}e \wedge d\mathcal{B}e)$.

Some of the above axioms are parts of axioms in the original definition which are in the form $A \Leftrightarrow B$. In the corresponding name we add → or ← to indicate which part of the equivalence is non-universal.

(II) Non-universal axioms for DCA: the axioms from group (I) plus the Efremovich axiom
(CE) $a\overline{C}^t b \Rightarrow (\exists c)(a\overline{C}^t c \wedge c^*\overline{C}^t b)$

(III) Non-universal axioms for strong dca: the axioms from group (I) and (II) plus the following two compositional axioms:
($C^t\mathcal{B}$) $a\overline{\mathcal{B}}b \Rightarrow (\exists c)(a\overline{C}^t c \wedge c^*\overline{\mathcal{B}}b)$.
($\mathcal{B}C^t$) $a\overline{\mathcal{B}}b \Rightarrow (\exists c)(a\overline{\mathcal{B}}c \wedge c^*\overline{C}^t b)$.

All formulas from the above list can be equivalently transformed (by decidable transformations) into the shape of the formula (✠) and then to obtain the corresponding non-standard rule of inference in the form of **RULE** (✠). The names of the new rules are the names of the corresponding axioms preceded by **RULE**. As an example we will do this for (UTR$\mathcal{B}$11→). First using standard Boolean transformations and negation on quantifiers we obtain the formula
(1) $(\forall p)(p\mathcal{B}c \vee p^*\mathcal{B}d) \wedge (\forall e)(\neg(e \in UTR \wedge e\mathcal{B}c \wedge e\mathcal{B}d) \Rightarrow \neg(c \in UTR \wedge d \in UTR)$.
Then we move the quantifier $(\forall e)$ trough $(p\mathcal{B}c \vee p^*\mathcal{B}d)$ and obtain the formula which is in the shape of (✠):
(2) $(\forall p)(\forall e)(p\mathcal{B}c \vee p^*\mathcal{B}d) \wedge (\neg(e \in UTR \wedge e\mathcal{B}c \wedge e\mathcal{B}d) \Rightarrow \neg(c \in UTR \wedge d \in UTR)$.
Now the rule is the following:
RULE(UTR$\mathcal{B}$11→) $\frac{C \Rightarrow (p\mathcal{B}c \vee p^*\mathcal{B}d) \wedge \neg(e \in UTR \wedge e\mathcal{B}c \wedge e\mathcal{B}d)}{C \Rightarrow \neg(c \in UTR \wedge d \in UTR)}$, where p and e are variables not occurring in the terms c, d and the formula C.

Lemma 5.8. *Every non-standard rule of inference of the form* ***RULE*** *(✠) preserves the validity in any class of dca-s satisfying the non-universal axiom (✠) corresponding to the rule.*

Proof. Consider the non-standard rule in the form
RULE (✠) $\frac{C \Rightarrow A(a_1,\ldots,a_n,b_1,\ldots,b_m)}{C \Rightarrow B(a_1,\ldots,a_n)}$.
which corresponds to the formula
(✠) $(\forall b_1,\ldots,b_m)A(a_1,\ldots,a_n,b_1,\ldots,b_m) \Rightarrow B(a_1,\ldots,a_n)$,
and is subject to the following constraints: $a_1,\ldots,a_n$ are terms and $b_1,\ldots,b_m$ are term variables which are not included in the formulas C, $B(a_1,\ldots,a_n)$, and consequently in the terms $a_1,\ldots,a_n$.

Let Σ be a class of dca-s which satisfies the condition (✠). We have to show that whenever the premise $C \Rightarrow A(a_1,\ldots,a_n,b_1,\ldots,b_m)$ is true in Σ, then the conclusion $C \Rightarrow B(a_1,\ldots,a_n)$ is also true in Σ. Suppose that this is not so. Then the premise $C \Rightarrow A(a_1,\ldots,a_n,b_1,\ldots,b_m)$ is true in Σ and there is an algebra $B \in \Sigma$ and a model (B,v) such that $(B,v) \not\models C \Rightarrow B(a_1,\ldots,a_n)$. This means that $(B,v) \models C$ and $(B,v) \not\models B(a_1,\ldots,a_n)$, so $B(v(a_1),\ldots,v(a_n))$ is not true in B. But B satisfies the condition (✠). So there are $c_1\ldots c_m \in B$ such that $A(v(a_1),\ldots,v(a_n),c_1\ldots c_m)$ is not true in B. Define v' for the variables $b_1,\ldots,b_n$ as follows $v'(b_1) = c_1$,...,$v'(b_m) = c_m$. Define v' for the variables in C and in $a_1,\ldots,a_n$ as v. By the constraints on $b_1,\ldots,b_m$ we obtain that $v'(a_1) = v(a_1)$,...

$v'(a_n) = v(a_n)$ and $(B, v') \models C$. Hence $(B, v') \not\models C \Rightarrow B(a_1, \ldots, a_n)$. Substituting in A we get: $A(v'(a_1), \ldots, v'(a_n), v'(b_1), \ldots, v'(b_m))$ is not true in B. Since $(B, v') \models C$ we obtain that $(B, v') \not\models C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)$, contrary to the assumption that $C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)$ is true in Σ. □

5.7. **Axiomatizations of the logics $\mathbf{L}^{min}_{weak}$, $\mathbf{L}^{min}_{DCA}$ and $\mathbf{L}^{min}_{strong}$.** The minimal logic $\mathbf{L}^{min}_{weak}$ corresponds to the class of all weak dca-s. Its axiomatization is an extension of the axiomatization of the minimal logic $\mathbf{L}^{min}_{basic}$ (see Section 5.5) with the nonstandard rules which can be obtained from the formulas of the list (I) from Section 5.6.

The minimal logic $\mathbf{L}^{min}_{DCA}$ corresponds to the class of all DCA-s. Its axiomatization is an extension of the axiomatization of $\mathbf{L}^{min}_{weak}$ with the non-standard rule corresponding to the Efremovich axiom (CE).

The minimal logic $\mathbf{L}^{min}_{strong}$ corresponds to the class of strong dca-s. Its axiomatization is an extension of the axiomatization of $\mathbf{L}^{min}_{DCA}$ with the non-standard rules corresponding to the non-universal axioms ($C^t\mathcal{B}$) and ($\mathcal{B}C^t$).

Definition 5.9. ***Time axioms and time rules.*** *Let us note that in the context of logics, the term "time axiom" corresponds to the following list of formulas mentioned in Remark 1.12, which are written in the language $\mathbb{L}$:* ***(rs)****,* ***(ls)****,* ***(up dir)****,* ***(down dir)****,* ***(dens)****,* ***(ref)****,* ***(lin)****,* ***(tri)****. The non-universal formulas* ***irr*** *and* ***tr*** *are excluded. Instead of them we add to this list the two rules of inference* ***RULE irr*** *and* ***RULE tr*** *called "time rules". Let Θ be a set of time axioms. A class Σ^Θ of dca-s is corresponding to Θ if it satisfies all axioms from Θ and if Θ contains some of the rules* ***RULE irr*** *and* ***RULE tr****, then Σ satisfies* ***irr****, respectively* ***tr****.*

Theorem 5.10. ***Soundness Theorem****. Let $\mathbf{L}$ be any of the minimal logics $\mathbf{L}^{min}_{basic}$, $\mathbf{L}^{min}_{weak}$, $\mathbf{L}^{min}_{DCA}$ and $\mathbf{L}^{min}_{strong}$ possibly extended with some time axioms and rules. Then $\mathbf{L}$ is sound in its corresponding class of dca-s.*

Proof. The proof follows from the fact that the logic contains as axioms all axioms satisfied in the corresponding class of dca-s. By Lemma 5.8 all nonstandard rules preserve the validity in the corresponding class of algebras because it satisfies by assumption the axioms corresponding to the non-standard rules. Modus Ponens always preserves validity in any semantics. □

6. Completeness theorems

In this section we will prove completeness theorems for the minimal logics $\mathbf{L}^{min}_{basic}$, $\mathbf{L}^{min}_{weak}$, $\mathbf{L}^{min}_{DCA}$ and $\mathbf{L}^{min}_{strong}$ and their extensions with time axioms and rules considering only algebraic semantics. One of the reasons is just for brevity and the other reason is that the relational and snapshot semantics, considered in Sections 5.3 and 5.4 are reducible to kinds of algebraic semantics. The method is based on a version of canonical model construction which is a modification of Henkin's completeness proof for the classical first-order logic. In the context of logics for region-based theories of space this method was applied for the first time in [6] (for relational and topological models) and in [2] for algebraic semantics.

6.1. **Canonical models.** Let $\mathbf{L}$ be any of the minimal logics $\mathbf{L}^{min}_{basic}$, $\mathbf{L}^{min}_{weak}$, $\mathbf{L}^{min}_{DCA}$, $\mathbf{L}^{min}_{strong}$ possibly extended with some new time axioms and rules. We will describe a construction of canonical models for $\mathbf{L}$ following [6] and [2]. First we will introduce a special notion of a $\mathbf{L}$-theory.

A pair $T = (\Lambda, \Gamma)$ is called a **L**-theory (or simply a theory) if Λ is a set of variables and Γ is a set of formulas satisfying the following conditions:

(i) All theorems of **L** belong to Γ.

(ii) If A and $A \Rightarrow B$ belong to Γ then B belongs to Γ.

(iii) Let **RULE** (✠) be any of the nonstandard rules of inference of **L** and suppose that the premise $C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)$ belongs to Γ for some variables $b_1, \ldots, b_m$ not belonging to Λ and to the conclusion $C \Rightarrow B(a_1, \ldots, a_n)$. Then the conclusion $C \Rightarrow B(a_1, \ldots, a_n)$ also belongs to Γ (for the rule **RULE** (✠) see section 5.5). Note that if **L** does not contain non-standard rules of inference then (iii) is an empty clause and in this case some simplifications can be considered which we omit just for brevity.

The elements of Λ are called "the free variables of T" and the element of Γ are called "the formulas of T". Sometimes we will write $T = (T_1, T_2)$. We say that a formula A belongs to T (in symbols $A \in T$) if $A \in T_2$. We say that T is included in T' if $T_1 \subseteq T'_1$ and $T_2 \subseteq T'_2$. T is called a **consistent theory** if $\bot \notin T$, T is called a **good theory** if there are infinitely many variables out of T_1. T is called a **complete theory** if it is a consistent theory and for any formula A: $A \in T$ or $\neg A \in T$. T is called a **rich theory** if for any non-standard rule of the logic (say **RULE** (✠)) the following holds: if the conclusion $C \Rightarrow B(a_1, \ldots, a_n)$ does not belong to Γ, then the premise $C \Rightarrow A(a_1, \ldots, a_n, b_1, \ldots, b_m)$ does not belong to Γ for some variables $b_1, \ldots, b_m$ not included in $a_1, \ldots, a_n$. A set of formulas is consistent if it is contained in a consistent theory.

Proposition 6.1. *(i)* ***Conservativeness Lemma****. Every consistent theory can be extended into a good consistent theory in a possible extensions of the language with an infinite set of new variables.*

(ii) ***Lindenbaum Lemma****. Every consistent good theory can be extended into a complete and rich theory.*

(iii) If A is not a theorem of **L** *then there exists a complete and rich theory T not containing A such that T is constructed without adding new variables to the language $\mathbb{L}$.*

Proof. The proof is similar to the proof of Lemma 7.11 and 7.10 from [6] and hence is omitted. □

Let T be a complete and rich theory of **L**. We will construct a dynamic contact algebra $B(\mathbf{L})$ depending on **L** in a canonical way as follows. Define an equivalence relation in the set of terms as follows: $a \equiv b$ iff $a = b \in T$. It is easy to see that $\equiv$ is a congruence relation and let $|a|$ denotes the equivalence class determined by a and let $B = \{|a| : a \text{ is a term }\}$. Boolean operations and constant $0, 1$ are defined in a standard way and for the operation $\mathcal{U}tr$ we also have $\mathcal{U}tr(|a|) =_{def} |\mathcal{U}tr(a)|$. It can be easily proved, using the axioms for Boolean algebra (which are included in the axiomatization of **L**) that B is a Boolean algebra. For the canonical predicates in B and **NOW** the definitions are the following:

NOW$= |$**NOW**$|$, $|a| \in TR$ iff $(a \in TR) \in T$, $|a| \in UTR$ iff $(a \in UTR) \in T$, $|a|R|b|$ iff $aRb \in T$ for $R = C^t, C^s, \mathcal{B}$.

Using the axioms of basic dca, which are included in the set of axioms of **L** it can be proved that the above definitions are correct and that B is a basic dca, called the canonical dca determined by the theory T. To indicate this we will often write $B(T)$.

The canonical valuation v in B is defined as follows: for variable p we put $v(p) = |p|$. Then it is easy to see that for terms we also have $v(a) = |a|$. The pair (B, v) is called the canonical model for **L** determined by T.

Lemma 6.2. ***Canonical algebra Lemma.*** *If* **L** *contains a non-standard rule then* $B(T)$ *satisfies the non-universal axiom corresponding to the rule.*

Proof. Let us consider, for example, the rule
RULE (irr) $\frac{C \Rightarrow \neg(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d)}{C \Rightarrow a\overline{\mathcal{B}}b}$, where c, d are variables not occurring in the terms a, b and the formula C.
Note that the **RULE (irr)** is corresponding to the axiom **(irr)**:
(irr) $a\mathcal{B}b \rightarrow (\exists c, d)(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d)$,

Suppose $|a|\mathcal{B}|b|$, then $a\mathcal{B}b \in T$. Since T is a complete theory this is equivalent to $a\overline{\mathcal{B}}b \notin T$. Since T is a rich theory, then for some variables c, d we have the following:

$\neg(c \neq 0 \wedge d \neq 0 \wedge c \leq a \wedge d \leq b \wedge c\overline{C}^t d) \notin T$ (here we applied the rule in its simplified form where $C = \top$). Again by the completeness of T this is equivalent to the following:

$c\overline{\mathcal{B}}d \notin T$ and $a\overline{C}^t c \notin T$ and $b\overline{C}^t d \notin T$ and $cC^t d \notin T$. This, by the definitions of the canonical relations is equivalent to: there are $|c|, |d| \in B$ such that $|c|, |d| \neq |0|$, $|c| \leq |a||$, $|d| \leq |b||$ and $|c|\overline{C}^t|d|$ hold in B which shows that the axiom **(irr)** holds in B. □

Lemma 6.3. ***Truth Lemma.*** *Let* A *be a formula and* T *be a complete and rich theory of* **L**. *Then the following two conditions are equivalent:*

(i) A *is true in the canonical model determined by* T*,*

(ii) $A \in T$

Proof. The proof goes by induction on the complexity of A. □

Proposition 6.4. ***Canonical model Lemma.*** *The following two conditions are equivalent for any formula* A*:*

(i) A *is a theorem of* **L**,

(ii) A *is true in all canonical models of* **L** *constructed without adding new variables to the language* $\mathbb{L}$.

Proof. (i)→(ii) Suppose A is a theorem of **L** and let T be any complete and rich theory of **L**. Then by the truth Lemma A belongs to T and hence A is true in the canonical model determined by T.

(ii)→(i) We will reason by contraposition. Suppose that A is not a theorem of **L**. Then by Proposition 6.1 (iii) there exists a complete and rich theory T constructed without adding new variables to the language $\mathbb{L}$ such that $A \notin T$. Then by the Truth Lemma $A \notin T$. □

6.2. **Completeness theorems - weak form.** This section is devoted to the weak form of completeness theorems for the minimal logics $\mathbf{L}^{min}_{basic}$, $\mathbf{L}^{min}_{weak}$, $\mathbf{L}^{min}_{DCA}$ and $\mathbf{L}^{min}_{strong}$ and their extension with time axioms and rules. For that purpose we identify the logics with their sets of theorems. If Θ is a set of some time axioms and rules (see Definition 5.9) and **L** be any of the four minimal logics, then $\mathbf{L}^{\Theta}$ denotes the extension of **L** with the axioms and rules from Θ.

Theorem 6.5. ***Completeness theorem for minimal logics, weak form***. *Let* $\mathbf{L}$ *be any of the minimal logics and let* Σ *be the corresponding class of dca-s for* $\mathbf{L}$. *Then the following conditions are equivalent for any formula* A*:*

(i) A *is a theorem of* $\mathbf{L}$*,*

(ii) A *is true in the class of dca-s corresponding to* $\mathbf{L}$

(iii) A *is true in all canonical models of* $\mathbf{L}$ *which are constructed without adding new variables to the language* $\mathbb{L}$.

Proof. (i)→(ii) this is true by the Soundness theorem for $\mathbf{L}$ (see Theorem 5.10).

(ii)→(iii) Let for instance $\mathbf{L}$ be the system $\mathbf{L}^{min}_{weak}$ and let (B, v) be a canonical model of $\mathbf{L}$. Then the corresponding class for $\mathbf{L}$ is the class Σ_{weak} of all weak dca-s. The canonical algebra B is a basic dca and by Lemma 6.2 it satisfies also the axioms corresponding to the non-standard rules of $\mathbf{L}$, hence B is weak dca and belongs to the corresponding algebras of $\mathbf{L}$. Consequently A is true in the canonical model (B, v).

(iii)→(i) This follows directly by the Canonical model Lemma 6.4. □

Corollary 6.6. *(i) All four minimal logics have equal sets of theorems which coincide with the set of theorems of* $\mathbf{L}^{min}_{basic}$.

(ii) Theorems of the minimal logics do not depend on the non-standard rules of inference.

(iii) The set of theorems of minimal logics is decidable.

Proof. (i) If we identify the logics with their sets of theorems then by the Completeness theorem for minimal logics, Theorem 6.5, we have the following equalities:

$\mathbf{L}^{min}_{basic} = \mathcal{L}(\Sigma_{basic})$, $\mathbf{L}^{min}_{weak} = \mathcal{L}(\Sigma_{weak})$, $\mathbf{L}^{min}_{DCA} = \mathcal{L}(\Sigma_{DCA})$ and $\mathbf{L}^{min}_{strong} = \mathcal{L}(\Sigma_{strong})$. By Theorem 5.4 we have $\mathcal{L}(\Sigma_{basic}) = \mathcal{L}(\Sigma_{weak}) = \mathcal{L}(\Sigma_{DCA}) = \mathcal{L}(\Sigma_{strong})$ which proves (i).

(ii) Since $\mathbf{L}^{min}_{basic}$ does not have non-standard rules of inference the statement follows from (i).

(iii) By the completeness theorem we have $\mathbf{L}^{min}_{basic} = \mathcal{L}(\Sigma_{basic})$ and by Proposition 5.2 we have $\mathcal{L}(\Sigma^{fin}_{basic}) = \mathcal{L}(\Sigma_{basic})$, which implies that the set of theorems of $\mathbf{L}^{min}_{weak}$ (and hence for the other minimal logics) is decidable. □

Theorem 6.7. ***Completeness theorem for extensions of minimal logics with time axioms and rules.*** *Let* $\mathbf{L}^{\Theta}$ *be any of the minimal logics extended with a set* Θ *of additional time axioms and rules and let* Σ^{Θ} *be the corresponding class of dca-s for* $\mathbf{L}^{\Theta}$. *Then the following conditions are equivalent for any formula* A*:*

(i) A *is a theorem of* $\mathbf{L}^{\Theta}$*,*

(ii) A *is true in* Σ^{Θ}*,*

(iii) A *is true in all canonical models of* $\mathbf{L}^{\Theta}$ *which are constructed without adding new variables to the language* $\mathbb{L}$.

Proof. The proof is similar to the proof of Theorem 6.5. □

Corollary 6.8. *(i) Let* Θ *be a set of time axioms. Then the logic* $\mathbf{L}^{min,\Theta}_{basic}$ *is decidable.*

(ii) Let Θ *be a set of time axioms and rules. Then* $\mathbf{L}^{min,\Theta}_{weak} = \mathbf{L}^{min,\Theta}_{DCA} = \mathbf{L}^{min\Theta}_{strong}$.

(iii) Let Θ *be a set consisting of some (or al of) time axiom* ***(dens)*** *and time rules* ***RULE(irr)*** *and* ***RULE(tr)***. *Then the logics* $\mathbf{L}^{min,\Theta}_{basic}$, $\mathbf{L}^{min,\Theta}_{weak}$, $\mathbf{L}^{min,\Theta}_{DCA}$, $\mathbf{L}^{min,\Theta}_{strong}$ *have equal sets of theorems.*

Proof. (i) By Theorem 6.7 the set of theorems of $\mathbf{L}_{basic}^{min,\Theta}$ coincides with $\mathcal{L}(\Sigma_{basic}^{\Theta})$. By Proposition 5.2 $\mathcal{L}(\Sigma_{basic}^{\Theta}) = \mathcal{L}(\Sigma_{basic}^{fin,\Theta})$. This shows that $\mathbf{L}_{basic}^{min,\Theta} = \mathcal{L}(\Sigma_{basic}^{fin,\Theta})$, which implies the decidability of $\mathbf{L}_{basic}^{min,\Theta}$.

(ii) The statement follows from Theorem 6.7 and Theorem 5.5.

(iii) The proof follows from Theorem 6.7 and Corollary 5.7. □

6.3. **Completeness theorems - strong form.** Let Ψ be a set of formulas and Σ be a class of dca-s. We say that Ψ has a model in Σ if there is a model (B, v) such that $B \in \Sigma$ and for any $A \in \Psi$ we have $(B, v) \models A$. In such a case we write $(B, v) \models \Psi$.

Theorem 6.9. ***Completeness theorem for extensions of minimal logics with time axioms and rules - strong form.*** *Let $\mathbf{L}^{\Theta}$ be any of the minimal logics extended with a set Θ of additional time axioms and rules and let Σ^{Θ} be the corresponding class of dca-s for $\mathbf{L}^{\Theta}$. Then the following conditions are equivalent for any set of formulas Ψ:*

(i) Ψ is a consistent set of formulas,

(ii) Ψ has a model in Σ^{Θ}.

Proof. (i)⇒(ii). Let Ψ be a consistent set of formulas. Then by 6.1 Ψ can be extended into a complete and rich theory $T = (\Lambda, \Gamma)$ in a possible extension of the language with a new variables. Then the canonical model based on T is a model for Ψ.

(ii) Let Ψ has a model (B, v) in Σ^{Θ}. Let Γ be the set of all formulas A such that $(B, v) \models A$. Obviously Ψ is included in Γ and $T = (\varnothing, \Gamma)$ is a consistent theory, so Ψ is a consistent set of formulas. □

7. Concluding remarks and open problems

There are a lot of open problems which can be formulated but we will mention only some of them. First we would like to prove coincidence of the extensions of the minimal logics with arbitrary time axioms and rules or at least for some interesting cases, which means to obtain various extensions of Corollary 5.5. This corollary depends essentially on the properties of the p-morphisms developed in Section 4.3.1 and Section 4.3.2, and which of time conditions are preserved by these p-morphisms. So studying modifications of these p-morphisms which preserve important sets of time conditions is one of our future plans. Kripke semantics and snapshot semantics for the logics introduced in Section 6 are considered briefly just for saving space and because they are special kinds of algebraic models. Let us note that the most intuitive semantics for these logics is snapshot semantics which arise from the snapshot models for DCA and the corresponding representation theorem. We mentioned in Remark 2.1 that snapshot models and the corresponding representation theory for weak and strong dca-s is also possible. We do not know, however, if basic dca-s have meaningful snapshot construction and the corresponding representation theory, and hence we formulate this as an open problem. Finally, we'd like an extension of Corollary 4.7 to be true for extensions of basic dca satisfying the non-universal axioms **irr** and **tr**.

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