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COUNTING SPANNING TREES IN COBORDISM OF TWO CIRCULANT GRAPHS

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Abstract. We consider a family of graphs $H_n(s_1,\dots,s_k;t_1,\dots,t_\ell)$ that is a generalisation of the family of I-graphs, which, in turn, includes the generalized Petersen graphs. We present an explicit formula for the number $\tau(n)$ of spanning trees in these graphs in terms of the Chebyshev polynomials and find its asymptotics. Also, we show that the number of spanning trees can be represented in the form $\tau(n) = p\,n\,a(n)^2$, where $a(n)$ is an integer sequence and p is a prescribed integer depending on the number of even elements in the sequence $s_1,\dots,s_k,t_1,\dots,t_\ell$ and the parity of n.

Keywords: circulant graph, I-graph, Petersen graph, spanning tree, Chebyshev polynomial, Mahler measure.

1. Introduction

A *tree* is an undirected graph in which any two vertices are connected by exactly one path. In other words, a connected graph without cycles is a tree.

A *spanning tree* T in a graph G is a subgraph that is a tree which includes all of the vertices of G.

The number $\tau(G)$ of spanning trees in a connected graph G is a well studied invariant. In some simplest cases it can be calculated directly:

- if G is itself a tree, then $\tau(G) = 1$;
- if G is the cycle graph C_n with n vertices, then $\tau(G) = n$;

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- if G is a complete graph K_n with n vertices, then by Caley's formula $\tau(G) = n^{n-2}$ (see, e.g., [1], pp. 141–146);
- if G is the complete bipartite graph $K_{p,q}$, then $\tau(G) = p^{q-1}q^{p-1}$ (see, e.g., [7], p. 100);
- if G is the n-dimensional hypercube graph Q_n, then $\tau(G) = 2^{2^n-n-1} \cdot \prod_{k=2}^{n} k^{\binom{n}{k}}$ (see, e.g., [6]).

More complicated formulas for the number of spanning trees are known for some special graphs, such as the wheel, fan, ladder, Möbius ladder [3], grids [13], lattices [15], prism and anti-prism [16]. We mention that the number of spanning trees for circulant graphs is expressed is terms of the Chebyshev polynomials; it was found in [19], [18], [11] and [17]. Similar results are also true for the I-graph [11].

Let $s_1, s_2, \ldots, s_k$ be integers such that $1 \le s_1 < s_2 < \ldots < s_k \le \frac{n}{2}$. The graph $C_n(s_1, s_2, \ldots, s_k)$ with n vertices $0, 1, 2, \ldots, n-1$ is called *circulant graph* if the vertex $i, 0 \le i \le n-1$ is adjacent to the vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k \pmod{n}$. All vertices of the graph are of even degree $2k$. If n is even and $s_k = \frac{n}{2}$, then the vertices i and $i + s_k$ are connected by two edges.

Let $G = C_n(s_1, s_2, \ldots, s_k)$ and $G' = C_n(t_1, t_2, \ldots, t_\ell)$ be circulant graphs. A *cobordism* $H(G, G')$ of graphs G and G' is the graph with the following vertex set and edge set

$$V(H(G,G')) = \{u_i, v_i \mid i = 1, 2, \ldots, n\},$$

$$E(H(G,G')) = \{u_i u_{i+s_j}, u_i v_i, v_i v_{i+t_h} \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, k, h = 1, 2, \ldots, \ell\}$$

where all subscripts are given modulo n. An example of cobordism $H(G, G')$ of graphs $G = C_6(1)$ and $G' = C_6(1, 2)$ is shown in Fig. 1.

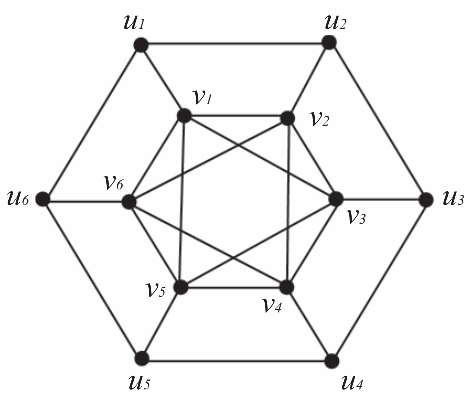

FIG. 1. Cobordism of graphs $C_6(1)$ and $C_6(1,2)$

To emphasize the dependence of $H(G, G')$ on the parameters, we also will write it in the form $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$. In the above definition, all vertices u_i are of valency $2k+1$, while all vertices v_i are of valency $2l+1$. In the case of even n when at least one of s_j or t_h is equal to $n/2$, the graph under consideration has multiple edges. Repeating the arguments from the papers [2], [8], [14] we conclude that the graph $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$ is connected if and only if greatest common divisor $\gcd(n, s_1, \ldots, s_k, t_1, \ldots, t_\ell) = 1$. If $\gcd(n, s_1, \ldots, s_k, t_1, \ldots, t_\ell) = m > 1$ then

$H_n(s_1,\ldots,s_k;t_1,\ldots,t_\ell)$ is a union of m copies of the graph $H_{n/m}(s_1/m,\ldots,s_k/m; t_1/m,\ldots,t_\ell/m)$. If $m=1$ and $\gcd(s_1,\ldots,s_k,t_1,\ldots,t_\ell)=d$ then the graphs $H_n(s_1,\ldots,s_k;t_1,\ldots,t_\ell)$ and $H_n(s_1/d,\ \ldots,s_k/d;t_1/d,\ldots,t_\ell/d)$ are isomorphic. Thus, without loss of generality we can assume that $\gcd(s_1,\ldots,s_k,t_1,\ldots,t_\ell)=1$. One can see that the graph $H_n(k;1)$ is isomorphic to the generalized Petersen graph $GP(n,k)$ and the graph $H_n(k;\ell)$ is isomorphic to I-graph $I(n,k,\ell)$. The number of spanning trees in the generalized Petersen graph and I-graph were investigated in [9] and [12] respectively. Even though the number of spanning trees in a given graph can be computed through eigenvalues of its Laplacian matrix, it is not easy to find the number of spanning trees for $H_n(s_1,\ldots,s_k;t_1,\ldots,t_\ell)$ using them.

In this paper, we obtain a closed formula for the number of spanning trees in the graph $H_n(s_1,\ldots,s_k;t_1,\ldots,t_\ell)$, investigate some arithmetical properties of this number and provide its asymptotic behavior.

2. Basic definitions and preliminary facts

We need the following basic properties of Chebyshev polynomials.

Let $T_n(z)=\cos(n\arccos z)$ and $U_{n-1}(z)=\dfrac{\sin(n\arccos z)}{\sin(\arccos z)}$ be the *Chebyshev polynomials* of the first and second kind respectively.

Then $T_n'(z)=nU_{n-1}(z), T_n(1)=1, U_{n-1}(1)=n$. For $z\neq 0$ we have the following identity $T_n(\frac{1}{2}(z+z^{-1}))=\frac{1}{2}(z^n+z^{-n})$.

Also, the polynomials $T_n(z)$ and $U_{n-1}(z)$ admit the following well-known presentation $T_n(z)=(q^n+q^{-n})/2$ and $U_{n-1}(z)=(q^n-q^{-n})/(q-q^{-1})$, where $q=z+\sqrt{z^2-1}$. See monograph [10] for other properties.

We denote the vertex and edge set of G by $V(G)$ and $E(G)$ respectively. Given $u,v\in V(G)$, we denote by a_{uv} the number of edges between vertices u and v. The matrix $A=A(G)=\{a_{uv}\}_{u,v\in V(G)}$, called *the adjacency matrix* of the graph G. The degree $d(v)$ of a vertex $v\in V(G)$ is defined by $d(v)=\sum_u a_{uv}$. Let $D=D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv}=d(v)$. Matrix $L=L(G)=D(G)-A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

In what follows, by I_n we denote the identity matrix of order n.

We refer to an $n\times n$ matrix to be *circulant,* and denote it by $circ(a_0,a_1,\ldots,a_{n-1})$ if it is of the form

$$circ(a_0,a_1,\ldots,a_{n-1})=\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1}\\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2}\\ \vdots & \vdots & & \ddots & \vdots\\ a_1 & a_2 & a_3 & \ldots & a_0\end{pmatrix}.$$

Recall [4] that the eigenvalues of matrix $C=circ(a_0,a_1,\ldots,a_{n-1})$ are given by the following simple formulas $\lambda_j=p(\varepsilon_n^j)$, $j=0,1,\ldots,n-1$, where $p(x)=a_0+a_1x+\ldots+a_{n-1}x^{n-1}$ and ε_n is the order n primitive root of the unity. Moreover, the circulant matrix $C=p(T_n)$, where $T_n=circ(\underbrace{0,1,0,\ldots,0}_{n})$ is the matrix shift operator $T_n:(x_0,x_1,\ldots,x_{n-2},x_{n-1})\to(x_1,x_2,\ldots,x_{n-1},x_0)$.

Denote by $L=L(H(G,G'))$ the Laplacian of $H(G,G')$, where $G=C_n(s_1,s_2,\ldots,s_k)$ and $G'=C_n(t_1,t_2,\ldots,t_\ell)$.

Then we have

$$L = \begin{pmatrix} (2k+1)I_n - \sum_{j=1}^{k}(T_n^{s_j} + T_n^{-s_j}) & -I_n \\ -I_n & (2l+1)I_n - \sum_{h=1}^{\ell}(T_n^{t_h} + T_n^{-t_h}) \end{pmatrix}.$$

3. Counting the number of spanning trees

The main result of this section is the following theorem.

Theorem 1. *The number of spanning trees in the graph $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$ is given by the formula*

$$\tau(n) = \frac{n}{q} \prod_{s=1}^{s_k+t_\ell-1} |2\,T_n(w_s) - 2|,$$

where $w_s, s = 1, 2, \ldots, s_k + t_\ell - 1$ are different from 1 roots of the algebraic equation

$$(2k + 1 - \sum_{j=1}^{k} 2\,T_{s_j}(w))(2\ell + 1 - \sum_{h=1}^{\ell} 2\,T_{t_h}(w)) = 1,$$

$q = \sum\limits_{j=1}^{k} s_j^2 + \sum\limits_{h=1}^{\ell} t_h^2$ *and $T_n(w)$ is the Chebyshev polynomial of the first kind.*

Proof. By the Kirchhoff theorem, the number of spanning trees $\tau(n)$ is equal to the product of nonzero eigenvalues of the Laplacian of graph $H(G, G')$ divided by the number of its vertices $2n$. To investigate the spectrum of Laplacian matrix

$$L = \begin{pmatrix} (2k+1)I_n - \sum\limits_{j=1}^{k}(T_n^{s_j} + T_n^{-s_j}) & -I_n \\ -I_n & (2l+1)I_n - \sum\limits_{j=1}^{l}(T_n^{t_j} + T_n^{-t_j}) \end{pmatrix},$$

we note that the eigenvalues of circulant matrix T_n are ε_n^j, where $\varepsilon_n = e^{\frac{2\pi i}{n}}$. Since all eigenvalues of T_n are distinct, the matrix T_n is conjugate to the diagonal matrix $\mathbb{T}_n = diag(1, \varepsilon_n, \ldots, \varepsilon_n^{n-1})$, where diagonal entries of $diag(1, \varepsilon_n, \ldots, \varepsilon_n^{n-1})$ are $1, \varepsilon_n, \ldots, \varepsilon_n^{n-1}$. To find spectrum of L, without loss of generality, one can assume that $T_n = \mathbb{T}_n$. In this case, all $n \times n$ blocks of L are diagonal matrices. If λ is eigenvalue of L and $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ is the corresponding eigenvector, we have the following system of equations

$$\begin{cases} A(T_n)x - y &= \lambda x \\ -x + B(T_n)y &= \lambda y \end{cases},$$

where $A(z) = 2k + 1 - \sum\limits_{j=1}^{k}(z^{s_j} + z^{-s_j})$ and $B(z) = 2\ell + 1 - \sum\limits_{h=1}^{\ell}(z^{t_h} + z^{-t_h})$.

From the first equation we have $y = A(T_n)x - \lambda x = (A(T_n) - \lambda)x$. Substituting y in the second equation we obtain $((A(T_n) - \lambda)(B(T_n) - \lambda) - 1)x = 0$.

Recall that the matrices under consideration are diagonal and the $(j+1, j+1)$-th entry of T_n is equal to ε_n^j. Hence, $((A(\varepsilon_n) - \lambda)(B(\varepsilon_n) - \lambda) - 1)x_{j+1} = 0$ and $y_{j+1} = (A(\varepsilon_n) - \lambda)x_{j+1}$.

As a result, for any $j = 0, \ldots, n-1$ the matrix L has exactly two eigenvalues $\lambda_{1,j}$ and $\lambda_{2,j}$ which are the roots of quadratic equation $(A(\varepsilon_n) - \lambda)(B(\varepsilon_n) - \lambda) - 1 = 0$.

The corresponding eigenvectors are (x, y), where $x = \mathbf{e}_{j+1} = (0, \dots, \underbrace{1}_{(j+1)-th}, \dots, 0)$ and $y = (A(\varepsilon_n) - \lambda)\mathbf{e}_{j+1}$. In particular, if $j = 0$ for $\lambda_{1,0}, \lambda_{2,0}$ we have $(1-\lambda)(1-\lambda) - 1 = \lambda(\lambda - 2) = 0$. That is, $\lambda_{1,0} = 0$ and $\lambda_{2,0} = 2$. Since $\lambda_{1,j}$ and $\lambda_{2,j}$ are roots of the same quadratic equation, we obtain $\lambda_{1,j}\lambda_{2,j} = P(\varepsilon_n^j)$, where

$$(1)\quad P(z) = A(z)B(z) - 1 = (2k+1-\sum_{j=1}^{k}(z^{s_j}+z^{-s_j}))(2\ell+1-\sum_{h=1}^{\ell}(z^{t_h}+z^{-t_h})) - 1.$$

Now we have

$$(2)\qquad \tau(n) = \frac{1}{2n}\lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = \frac{1}{n}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = \frac{1}{n}\prod_{j=1}^{n-1}P(\varepsilon_n^j).$$

To continue we need the following lemma.

Lemma 1. *The following identity holds $P(z) = Q(w)$, where*

$$Q(w) = (2k+1-\sum_{j=1}^{k}2\,T_{s_j}(w))(2\ell+1-\sum_{h=1}^{\ell}2\,T_{t_h}(w)) - 1,$$

$T_m(w)$ is the Chebyshev polynomial of the first kind and $w = \frac{1}{2}(z+z^{-1})$.

Moreover, if $\gcd(s_1, \dots, s_k, t_1, \dots, t_\ell) = 1$ then all the roots of the Laurent polynomial $P(z)$ counted with multiplicities are $1, 1, z_1, 1/z_1, \dots, z_{s_k+t_\ell-1}, 1/z_{s_k+t_\ell-1}$, where we have $|z_s| \neq 1, s = 1, 2, \dots, s_k + t_\ell - 1$. Polynomial $Q(w)$ has the roots $1, w_1, \dots, w_{s_k+t_\ell-1}$, where $w_s \neq 1$ for all $s = 1, 2, \dots, s_k + t_\ell - 1$.

Proof. Let us substitute $z = e^{i\varphi}$. It is easy to see that $w = \frac{1}{2}(z+z^{-1}) = \cos\varphi$. Then the first statement of the lemma follows from the identity $T_n(w) = \cos(n\arccos w) = \cos(n\varphi)$.

To prove the second statement of the lemma we suppose that the Laurent polynomial $P(z)$ has a root z_0 such that $|z_0| = 1$. Then $z_0 = e^{i\varphi_0}$, $\varphi_0 \in \mathbb{R}$. Now we have

$$(2k+1-\sum_{j=1}^{k}2\cos(s_j\varphi))(2\ell+1-\sum_{h=1}^{\ell}2\cos(t_h\varphi)) - 1 = 0.$$

Since $2k+1-\sum\limits_{j=1}^{k}2\cos(s_j\varphi_0) \geq 1$ and $2\ell+1-\sum\limits_{h=1}^{\ell}2\cos(t_h\varphi_0) \geq 1$ the equation holds if and only if $\cos(s_j\varphi_0) = 1$, $j = 1, \dots, k$ and $\cos(t_h\varphi) = 1$, $h = 1, \dots, \ell$. So $s_j\varphi_0 = 2\pi m_j$ and $\cos(t_h\varphi_0) = 2\pi n_h$ for some integer m_j and n_h. As $\gcd(s_1, \dots, s_k, t_1, \dots, t_h) = 1$ there exist two integer sequences p_j and q_h such that $s_1p_1 + \dots + s_kp_k + t_1q_1 + \dots + t_hq_h = 1$. Hence, $\varphi_0 = \varphi_0(s_1p_1 + \dots + s_kp_k + t_1q_1 + \dots + t_hq_h) = 2\pi(m_1p_1 + \dots + m_kp_k + n_1q_1 + \dots + n_hq_h) \in 2\pi\mathbb{Z}$. As a result, $z_0 = e^{i\varphi_0} = 1$. Now we have to show that the multiplicity of the root $z_0 = 1$ is 2. Indeed, $P(1) = P'(1) = 0$ and $P''(1) = -2(s_1^2 + \dots + s_k^2 + t_1^2 + \dots + t_h^2) \neq 0$. ☐

Set $H(z) = \prod\limits_{s=1}^{m}(z-z_s)(z-z_s^{-1})$, where $m = s_k + t_\ell - 1$ and z_s are roots of $P(z)$ different from 1. Then by Lemma 1, we have $P(z) = \frac{(z-1)^2}{z^{s_k+s_\ell}}H(z)$.

The following lemma has been proved in ([9], Lemma 5.3.)

Lemma 2. *Let* $H(z) = \prod\limits_{s=1}^{m}(z - z_s)(z - z_s^{-1})$ *and* $H(1) \neq 0$. *Then*

$$\prod_{j=1}^{n-1} H(\varepsilon_n^j) = \prod_{s=1}^{m} \frac{T_n(w_s) - 1}{w_s - 1},$$

where $w_s = \frac{1}{2}(z_s + z_s^{-1})$, $s = 1, \ldots, m$ *and* $T_n(x)$ *is the Chebyshev polynomial of the first kind.*

Proof. Taking into account Lemma 1 and Lemma 2, from (2) we get

$$\tau(n) = \frac{1}{n}\prod_{j=1}^{n-1} P(\varepsilon_n^j) = \frac{1}{n}\prod_{j=1}^{n-1} \frac{(\varepsilon_n^j - 1)^2}{(\varepsilon_n^j)^{s_k+t_\ell}} H(\varepsilon_n^j) = \frac{(-1)^{(n-1)(s_k+t_\ell)}n^2}{n}\prod_{j=1}^{n-1} H(\varepsilon_n^j)$$
$$(3) \qquad = (-1)^{(n-1)(s_k+t_\ell)} n \prod_{s=1}^{s_k+t_\ell-1} \frac{T_n(w_s) - 1}{w_s - 1}.$$

□

Since $\tau(n)$ the left hand side of equation (3) is a positive number, we obtain

$$(4) \qquad \tau(n) = n \prod_{s=1}^{s_k+t_\ell-1} \left| \frac{T_n(w_s) - 1}{w_s - 1} \right| = n \prod_{s=1}^{s_k+t_\ell-1} |T_n(w_s) - 1| / \prod_{s=1}^{s_k+t_\ell-1} |w_s - 1|.$$

Now we evaluate the product $\prod_{s=1}^{s_k+t_\ell-1} |w_s - 1|$. We note that $Q(w) = (2k + 1 - 2\sum\limits_{j=1}^{k} T_{s_j}(w))(2\ell + 1 - 2\sum\limits_{h=1}^{\ell} T_{t_h}(w)) - 1$ is an integer polynomial with the leading coefficient $a_0 = 2^{s_k+t_\ell}$. We have $Q(1) = 0$ and $Q'(1) = -2q$, where $q = \sum\limits_{j=1}^{k} s_j^2 + \sum\limits_{h=1}^{\ell} t_h^2$.

Herefrom we obtain

$$(5) \qquad \prod_{s=1}^{s_k+t_\ell-1} |w_s - 1| = |\frac{1}{a_0} Q'(1)| = \frac{2q}{2^{s_k+t_\ell}} = \frac{q}{2^{s_k+t_\ell-1}}.$$

Combining equations (4) and (5) we finish the proof of the theorem. □

Theorem 2. *Let* $\tau(n)$ *be the number of spanning trees of the graph* $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$. *Denote by* s *and* t *the number of odd numbers in the sequences* $s_1, \ldots, s_k$ *and* $t_1, \ldots, t_\ell$ *respectively. Let* p *is the square-free part of the number* $u = s + t + 4st$. *Then there exists an integer sequence* $a(n), n \in \mathbb{N}$ *such that*

1° $\tau(n) = n\, a(n)^2$, *if* n *is odd.*
2° $\tau(n) = p\, n\, a(n)^2$, *if* n *is even.*

Proof. Recall that all nonzero eigenvalues are $\{\lambda_{2,0}, \lambda_{1,j}, \lambda_{2,j} : j = 1, 2, \ldots, n-1\}$, where $\lambda_{2,0} = 2$. By the Kirchhoff theorem we have $2n\tau(n) = \lambda_{2,0}\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j} = 2\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}$. Hence $n\tau(n) = \prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}$. Note that $\lambda_{1,j}\lambda_{2,j} = P(\varepsilon_n^j) = P(\varepsilon_n^{n-j}) = \lambda_{1,n-j}\lambda_{2,n-j}$, where $P(z)$ is given by the formula (1). Therefore, we have $n\tau(n) = (\prod_{j=1}^{(n-1)/2}\lambda_{1,j}, \lambda_{2,j})^2$ if n is odd, and $n\tau(n) = \lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}(\prod_{j=1}^{n/2-1}\lambda_{1,j}, \lambda_{2,j})^2$ if n is even. Moreover, if n is even we get

$$\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}} = P(-1)$$
$$= \left(2k+1-\sum_{j=1}^{k}\left((-1)^{s_j}+(-1)^{-s_j}\right)\right)\left(2\ell+1-\sum_{h=1}^{\ell}\left((-1)^{t_h}+(-1)^{-t_h}\right)\right)-1$$
$$= \left(1+4\sum_{j=1}^{k}\frac{1-(-1)^{s_j}}{2}\right)\left(1+4\sum_{h=1}^{\ell}\frac{1-(-1)^{t_h}}{2}\right)-1$$
$$= (1+4s)(1+4t)-1 = 4(s+t+4st).$$

Let $u = s+t+4st$. We represent u in the form $u = p\,r^2$, where p is the square-free part of u.

Note that graph $H_n(s_1,\ldots,s_k;t_1,\ldots,t_\ell)$ admits a cyclic group of automorphisms isomorphic to $\mathbb{Z}_n$ acting freely on the set of spanning trees. Thus, $\frac{\tau(n)}{n}$ is an integer. Hence

1° $\frac{\tau(n)}{n} = \left(\frac{\prod_{j=1}^{(n-1)/2}\lambda_{1,j},\lambda_{2,j}}{n}\right)^2$ if n is odd,

2° $\frac{\tau(n)}{n} = p\left(\frac{2r\prod_{j=1}^{n/2-1}\lambda_{1,j},\lambda_{2,j}}{n}\right)^2$ if n is even.

Each algebraic number $\lambda_{i,j}$ comes into both products $\prod_{j=1}^{(n-1)/2}\lambda_{1,j},\lambda_{2,j}$ and $\prod_{j=1}^{n/2-1}\lambda_{1,j},\lambda_{2,j}$ with all of its Galois conjugate elements. Therefore, both products are integer numbers. From here we conclude that in equalities 1°, 2° the squared value a rational number. Because of $\frac{\tau(n)}{n}$ is integer and p is a square-free, all these rational numbers are integer. We set $a(n) = \frac{\prod_{j=1}^{(n-1)/2}\lambda_{1,j},\lambda_{2,j}}{n}$ if n is odd and $a(n) = \frac{2r\prod_{j=1}^{n/2-1}\lambda_{1,j},\lambda_{2,j}}{n}$ if n is even to finish the proof. □

4. Asymptotics for the number of spanning trees

The asymptotic formula for the number of spanning trees in the graph $H_n(s_1,\ldots,s_k;t_1,\ldots,t_\ell)$ is given in the following theorem.

Theorem 3. *Suppose that* $\gcd(s_1,\ldots,s_k,t_1,\ldots,t_\ell)=1$. *Then the number* $\tau(n)$ *of spanning trees of the graph* $H_n(s_1,\ldots,s_k;t_1,\ldots,t_\ell)$ *has the following asymptotics*

$$\tau(n) \sim \frac{n\,A^n}{\sum\limits_{j=1}^{k}s_j^2+\sum\limits_{h=1}^{\ell}t_h^2},\ n\to\infty,$$

where $A = \prod\limits_{P(z)=0,\,|z|>1}|z|$ *is the Mahler measure of the Laurent polynomial*

$$P(z) = (2k+1-\sum_{j=1}^{k}(z^{s_j}+z^{-s_j}))(2\ell+1-\sum_{h=1}^{\ell}(z^{t_h}+z^{-t_h}))-1.$$

Proof. By Theorem 1 we have

$$\tau(n) = \frac{n}{q}\prod_{s=1}^{s_k+t_\ell-1}|2T_n(w_s)-2|,$$

where $q = \sum\limits_{j=1}^{k} s_j^2 + \sum\limits_{h=1}^{\ell} t_h^2$ and $w_s, s = 1, 2, \ldots, s_k + t_\ell - 1$ are different from 1 roots of the polynomial

$$Q(w) = (2k + 1 - 2\sum_{j=1}^{k} T_{s_j}(w))(2\ell + 1 - 2\sum_{h=1}^{\ell} T_{t_h}(w)) - 1.$$

By Lemma 1, $T_n(w_s) = \frac{1}{2}(z_s^n + z_s^{-n})$, where the z_s and $1/z_s$ are roots of the polynomial $P(z)$ with the property $|z_s| \neq 1$, $s = 1, 2, \ldots, s_k + t_\ell - 1$. Replacing z_s by $1/z_s$, if it is necessary, we can assume that all $|z_s| > 1$ for all $s = 1, 2, \ldots, s_k + t_\ell - 1$. Then $T_n(w_s) \sim \frac{1}{2} z_s^n$ and $|2T_n(w_s) - 2| \sim |z_s|^n$ as $n \to \infty$. Hence

$$\frac{n}{q} \prod_{s=1}^{s_k+t_\ell-1} |2T_n(w_s) - 2| \sim \frac{n}{q} \prod_{s=1}^{s_k+t_\ell-1} |z_s|^n = \frac{n}{q} \prod_{P(z)=0,\, |z|>1} |z|^n = \frac{nA^n}{q}.$$

The theorem is proved. $\square$

Remark: It is known ([5], p. 67) that the Mahler measure $A = \prod\limits_{P(z)=0,\, |z|>1} |z|$ of the polynomial $P(z)$ can be calculated by the formula

$$A = \exp\left(\int_0^1 \log|P(e^{2\pi i t})| \mathrm{d}t\right).$$

5. Examples and Tables

Examples.

1° The graph $H_n(1, 2; 1)$. By Theorem 1 we have the following formula for the number of spanning trees

$$\tau(n) = \frac{n}{6}|(2\,T_n(-\sqrt{5/2}) - 2)(2\,T_n(\sqrt{5/2}) - 2))|.$$

By Theorem 2, there is an integer sequence $a(n)$ such that $\tau(n) = n\,a(n)^2$ if n odd and $\tau(n) = 6\,n\,a(n)^2$ if a is even. Also, by Theorem 3 we get $\tau(n) \sim \frac{n}{6}(4 + \sqrt{15})^n$, $n \to \infty$.

2° The graph $H_n(1, 2; 1, 2)$. By Theorem 1 we have the formula

$$\tau(n) = \frac{n}{10}(2\,T_n(-\frac{3}{2}) - 2)(2\,T_n(-\frac{1+\sqrt{33}}{4}) - 2))(2\,T_n(-\frac{1-\sqrt{33}}{4}) - 2)).$$

By Theorem 2, one can find an integer sequence $a(n)$ such that $\tau(n) = n\,a(n)^2$ if n odd and $\tau(n) = 6\,n\,a(n)^2$ if a is even. Also, by Theorem 3 we get $\tau(n) \sim \frac{n}{10}A^n$, $n \to \infty$, there $A = \frac{1}{4}(3 + \sqrt{5})(4 + \sqrt{3} + \sqrt{15 + 8\sqrt{3}}) \cong 14.53515$.

3° For the graph $H_n(1, 3; 1)$ there exists an integer sequence $a(n)$ such that $\tau(n) = n\,a(n)^2$ if n odd and $\tau(n) = 11\,n\,a(n)^2$ if a is even. By Theorem 1, the number of spanning trees is given by the formula

$$\tau(n) = \frac{n}{11}(2\,T_n(w_1) - 2)(2\,T_n(w_2) - 2)(2\,T_n(w_3) - 2)),$$

where $w_j, j = 1, 2, 3$ are the roots of the cubic equation $8w^3 - 4w^2 - 8w - 7 = 0$. We get $\tau(n) \sim \frac{n}{11} A^n$, $n \to \infty$, where $A \cong 8.62367$ is a suitable root of the algebraic equations $z^4 - 9z^3 + 3z^2 + 2z + 1 = 0$.

4° Graph $H_n(1, 3; 1, 3)$. We have $\tau(n) = n\, a(n)^2$, if n is odd and $\tau(n) = 5\, n\, a(n)^2$, if n is even, where $a(n)$ is an integer sequence.

Tables of complexity for graphs $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$.

TABLE 1. Graph $H_n(1, 2; 1)$

n	$\tau(n)$	$a(n)$
3	243	9
4	2400	10
5	25205	71
6	236196	81
7	2187367	559
8	19660800	640
9	174319209	4401
10	1524700860	5041
11	13206085211	34649
12	113421319200	39690
13	967394085853	272791
14	8202127530324	312481
15	69187876305615	2147679
16	581029173657600	2460160
17	4860336387936977	16908641
18	40516248835180908	19368801
19	336705083493332419	133121449
20	2790393161407500000	152490250
21	23067154934435396421	1048062951
22	190255294472937724932	1200553201
23	1565962073278860529463	8251382159
24	12864827815145039462400	9451935360
25	105504765778756956276025	64962994321
26	863862754470795262834716	74414929681
27	7062760813242151477708587	511452572409
28	57664442640869593933840800	585867502090
29	470205167886859173668499629	4026657584951
30	3829569782144865232704422580	4612525087041

Recall that $\tau(n) = n\, a(n)^2$ if n odd and $\tau(n) = 6\, n\, a(n)^2$ if a is even. Moreover, $a(n) = U_{2m}(\sqrt{\frac{5}{2}})$ if $n = 2m + 1$ and $a(n) = U_{m-1}^2(\sqrt{\frac{5}{2}})$ if $n = 2m$. In particular, the following identity holds $a(2m + 1)^2 = a(2(2m + 1))$.

TABLE 2. Graph $H_n(1,2;1,2)$

n	$\tau(n)$	$a(n)$
3	768	16
4	13824	24
5	300125	245
6	5308416	384
7	93410863	3653
8	1566093312	5712
9	25841205504	53584
10	418786021500	83545
11	6712200103499	781153
12	106552904122368	1216512
13	1678996344098125	11364575
14	26291028053425044	17691479
15	409523810803296000	165231920
16	6350009347658317824	257188512
17	98072947641328297193	2401873627
18	1509407705902501527552	3738448512
19	23158747982940883103419	34912483051
20	354336008452953202368000	54339673080
21	5407872946852536547435776	507462076784
22	82347550238453649204376068	789838640207
23	1251342155345709255180159383	7376054546111
24	18979265265160930643291406336	11480437850112
25	287361050203120326345056265625	107212135545025
26	4343910657664044756716598697500	166869952774225
27	65567842092316209023787182254848	1558344825772432
28	988336237674399011162001607039488	2425480116529704
29	14878674098476633372116623274700661	22650777636400853
30	223721140474536510953649401044992000	35254719185642880

One has $\tau(n) = n\,a(n)^2$ if n odd and $\tau(n) = 6\,n\,a(n)^2$ if a is even.

TABLE 3. Graph $H_n(1,3;1)$

n	$\tau(n)$	$a(n)$
3	75	5
4	2156	7
5	25205	71
6	199650	55
7	2258368	568
8	22978648	511
9	212139225	4855
10	2063331710	4331
11	1973899753	42361
12	184093671300	37345
13	1722684608125	364025
14	16029281787904	322624
15	147891645009375	3139975
16	1360541896243376	2780351
17	12471100334325233	27084943
18	113847031491736950	23978845
19	1036309586795643331	233543593
20	9407868262133890780	206792257
21	85183976640063201600	2014045640
22	769573435857475857938	1783271017
23	6938304482233726496423	17368517849
24	62434807080101246267400	15378409585
25	560849846785647363921025	149779817971
26	5030058352762615644078750	132618311775
27	45045893638771128818307675	1291652596895
28	402848040074966548572274688	1143655622144
29	3598101448168589107717563869	11138781967969
30	32098788638431723521472706250	9862507616225

Here $\tau(n) = n\,a(n)^2$ if n odd and $\tau(n) = 11\,n\,a(n)^2$ if a is even. Also, for any positive integer m we have

(1) $a(3m)$ is divisible by 5;
(2) $a(5m)$ and $a(7m)$ are divisible by 71;
(3) $a(6m)$ and $a(11m)$ are divisible by 11.

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