COUNTING SPANNING TREES IN COBORDISM OF TWO CIRCULANT GRAPHS

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Abstract. We consider a family of graphs $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$ that is a generalisation of the family of $I$-graphs, which, in turn, includes the generalized Petersen graphs. We present an explicit formula for the number $\tau(n)$ of spanning trees in these graphs in terms of the Chebyshev polynomials and find its asymptotics. Also, we show that the number of spanning trees can be represented in the form $\tau(n) = p \cdot a(n)^2$, where $a(n)$ is an integer sequence and $p$ is a prescribed integer depending on the number of even elements in the sequence $s_1, \ldots, s_k, t_1, \ldots, t_\ell$ and the parity of $n$.

Keywords: circulant graph, $I$-graph, Petersen graph, spanning tree, Chebyshev polynomial, Mahler measure.

1. Introduction

A tree is an undirected graph in which any two vertices are connected by exactly one path. In other words, a connected graph without cycles is a tree.

A spanning tree $T$ in a graph $G$ is a subgraph that is a tree which includes all of the vertices of $G$.

The number $\tau(G)$ of spanning trees in a connected graph $G$ is a well studied invariant. In some simplest cases it can be calculated directly:

- if $G$ is itself a tree, then $\tau(G) = 1$;
- if $G$ is the cycle graph $C_n$ with $n$ vertices, then $\tau(G) = n$;

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• if $G$ is a complete graph $K_n$ with $n$ vertices, then by Caley’s formula $\tau(G) = n^{n-2}$ (see, e.g., [1], pp. 141–146);
• if $G$ is the complete bipartite graph $K_{p,q}$, then $\tau(G) = p^{p-1}q^{q-1}$ (see, e.g., [7], p. 100);
• if $G$ is the $n$-dimensional hypercube graph $Q_n$, then $\tau(G) = 2^{2^n-n-1}$.

More complicated formulas for the number of spanning trees are known for some special graphs, such as the wheel, fan, ladder, Möbius ladder [3], grids [13], lattices [15], prism and anti-prism [16]. We mention that the number of spanning trees for special graphs, such as the wheel, fan, ladder, Möbius ladder [3], grids [13], lattices [15], prism and anti-prism [16].

Let $s_1, s_2, \ldots, s_k$ be integers such that $1 \leq s_1 < s_2 < \ldots < s_k \leq \frac{n}{2}$. The graph $C_n(s_1, s_2, \ldots, s_k)$ with $n$ vertices $0, 1, 2, \ldots, n-1$ is called a circulant graph if the vertex $i, 0 \leq i \leq n-1$ is adjacent to the vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k$ (mod $n$). All vertices of the graph are of even degree $2k$. If $n$ is even and $s_k = \frac{n}{2}$, then the vertices $i$ and $i + s_k$ are connected by two edges.

Let $G = C_n(s_1, s_2, \ldots, s_k)$ and $G' = C_n(t_1, t_2, \ldots, t_\ell)$ be circulant graphs. A cobordism $H(G,G')$ of graphs $G$ and $G'$ is the graph with the following vertex set and edge set:

$V(H(G,G')) = \{u_i, v_i | i = 1, 2, \ldots, n\}$,

$E(H(G,G')) = \{u_iu_{i+s_1}, u_iv_i, v_iu_{i+t_\ell} | i = 1, 2, \ldots, n, j = 1, 2, \ldots, k, h = 1, 2, \ldots, \ell\}$

where all subscripts are given modulo $n$. An example of cobordism $H(G,G')$ of graphs $G = C_6(1)$ and $G' = C_6(1,2)$ is shown in Fig. 1.

![Cobordism of graphs $C_6(1)$ and $C_6(1,2)$](image)

To emphasize the dependence of $H(G,G')$ on the parameters, we also will write it in the form $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$. In the above definition, all vertices $u_i$ are of valency $2k + 1$, while all vertices $v_i$ are of valency $2\ell + 1$. In the case of even $n$ when at least one of $s_j$ or $t_k$ is equal to $n/2$, the graph under consideration has multiple edges. Repeating the arguments from the papers [2], [8], [14] we conclude that the graph $H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell)$ is connected if and only if greatest common divisor $\gcd(n, s_1, \ldots, s_k, t_1, \ldots, t_\ell) = m > 1$ then...
The degree $V_A$ matrix $\in u, v$ identity of the first and second kind respectively. the circulant matrix $C$ number and provide its asymptotic behavior. the graph $H$ operator $T$ to find the number of spanning trees for $H$ graph can be computed through eigenvalues of its Laplacian matrix, it is not easy in [9] and [12] respectively. Even though the number of spanning trees in a given graph was investigated in [9] and [12] respectively. Even though the number of spanning trees in a given graph can be computed through eigenvalues of its Laplacian matrix, it is not easy to find the number of spanning trees for $H_n(s_1, . . . , s_k; t_1, . . . , t_L)$ using them.

In this paper, we obtain a closed formula for the number of spanning trees in the graph $H_n(s_1, . . . , s_k; t_1, . . . , t_L)$, investigate some arithmetical properties of this number and provide its asymptotic behavior.

2. Basic definitions and preliminary facts

We need the following basic properties of Chebyshev polynomials.

Let $T_n(z) = \cos(n \arccos z)$ and $U_{n-1}(z) = \sin(n \arccos z)$ be the Chebyshev polynomials of the first and second kind respectively.

Then $T_n'(z) = nU_{n-1}(z)$, $T_n(1) = 1$, $U_{n-1}(1) = n$. For $z \neq 0$ we have the following identity $T_n\left(\frac{1}{2}(z + z^{-1})\right) = \frac{1}{2}(z^n + z^{-n})$.

Also, the polynomials $T_n(z)$ and $U_{n-1}(z)$ admit the following well-known presentation $T_n(z) = (q^n + q^{-n})/2$ and $U_{n-1}(z) = (q^n - q^{-n})/(q - q^{-1})$, where $q = z + \sqrt{z^2 - 1}$. See monograph [10] for other properties.

We denote the vertex and edge set of $G$ by $V(G)$ and $E(G)$ respectively. Given $u, v \in V(G)$, we denote by $a_{uv}$ the number of edges between vertices $u$ and $v$. The matrix $A = A(G) = \{a_{uv}\}_{u, v \in V(G)}$ is called the adjacency matrix of the graph $G$. The degree $d(v)$ of a vertex $v \in V(G)$ is defined by $d(v) = \sum u a_{uv}$. Let $D = D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv} = d(v)$. Matrix $L = L(G) = D(G) - A(G)$ is called the Laplacian matrix, or simply Laplacian, of the graph $G$.

In what follows, by $I_n$ we denote the identity matrix of order $n$.

We refer to an $n \times n$ matrix to be circulant, and denote it by $\text{circ}(a_0, a_1, \ldots, a_{n-1})$ if it is of the form

\[
\text{circ}(a_0, a_1, \ldots, a_{n-1}) = \begin{pmatrix}
a_0 & a_1 & a_2 & \cdots & a_{n-1} \\
a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\
& \vdots & \ddots & \vdots & \vdots \\
a_1 & a_2 & a_3 & \cdots & a_0
\end{pmatrix}.
\]

Recall [4] that the eigenvalues of matrix $C = \text{circ}(a_0, a_1, \ldots, a_{n-1})$ are given by the following simple formulas $\lambda_j = p(\varepsilon_n^j)$, $j = 0, 1, \ldots, n - 1$, where $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$ and $\varepsilon_n$ is the order $n$ primitive root of the unity. Moreover, the circulant matrix $C = p(T_n)$, where $T_n = \text{circ}(0, 1, 0, \ldots, 0)$ is the matrix shift operator $T_n : (x_0, x_1, \ldots, x_{n-2}, x_{n-1}) \rightarrow (x_1, x_2, \ldots, x_{n-1}, x_0)$.

Denote by $L = L(H(G, G'))$ the Laplacian of $H(G, G')$, where $G = C_n(s_1, s_2, \ldots, s_k)$ and $G' = C_n'(t_1, t_2, \ldots, t_L)$.
Then we have
\[
L = \begin{pmatrix}
(2k + 1)I_n - \sum_{j=1}^{k} (T_n^{s_j} + T_n^{-s_j}) & -I_n \\
-I_n & (2l + 1)I_n - \sum_{h=1}^{l} (T_n^{t_h} + T_n^{-t_h})
\end{pmatrix}.
\]

3. Counting the number of spanning trees

The main result of this section is the following theorem.

**Theorem 1.** The number of spanning trees in the graph \( H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell) \) is given by the formula
\[
\tau(n) = \frac{n}{q} \prod_{s=1}^{k} |2T_n(w_s) - 2|,
\]
where \( w_s, s = 1, 2, \ldots, s_k + t_\ell - 1 \) are different from 1 roots of the algebraic equation
\[
(2k + 1 - \sum_{j=1}^{k} 2T_{s_j}(w))(2\ell + 1 - \sum_{h=1}^{\ell} 2T_{t_h}(w)) = 1,
\]
\[
q = k\sum_{j=1}^{k} s_j^2 + \sum_{h=1}^{\ell} t_h^2 \text{ and } T_n(x) \text{ is the Chebyshev polynomial of the first kind.}
\]

**Proof.** By the Kirchhoff theorem, the number of spanning trees \( \tau(n) \) is equal to the product of nonzero eigenvalues of the Laplacian of graph \( H(G, G') \) divided by the number of its vertices \( 2n \). To investigate the spectrum of Laplacian matrix
\[
L = \begin{pmatrix}
(2k + 1)I_n - \sum_{j=1}^{k} (T_n^{s_j} + T_n^{-s_j}) & -I_n \\
-I_n & (2l + 1)I_n - \sum_{j=1}^{l} (T_n^{t_j} + T_n^{-t_j})
\end{pmatrix},
\]
we note that the eigenvalues of circulant matrix \( T_n \) are \( \varepsilon_n \), where \( \varepsilon_n = e^{\frac{2\pi i}{n}} \). Since all eigenvalues of \( T_n \) are distinct, the matrix \( T_n \) is conjugate to the diagonal matrix \( T_n = \text{diag}(1, \varepsilon_n, \ldots, \varepsilon_n^{n-1}) \), where diagonal entries of \( \text{diag}(1, \varepsilon_n, \ldots, \varepsilon_n^{n-1}) \) are \( 1, \varepsilon_n, \ldots, \varepsilon_n^{n-1} \). To find spectrum of \( L \), without loss of generality, one can assume that \( T_n = T_n \). In this case, all \( n \times n \) blocks of \( L \) are diagonal matrices. If \( \lambda \) is eigenvalue of \( L \) and \((x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \) is the corresponding eigenvector, we have the following system of equations
\[
\begin{align*}
A(T_n)x - y &= \lambda x \\
-x + B(T_n)y &= \lambda y,
\end{align*}
\]
where \( A(z) = 2k + 1 - \sum_{j=1}^{k} (z^{s_j} + z^{-s_j}) \) and \( B(z) = 2\ell + 1 - \sum_{h=1}^{\ell} (z^{t_h} + z^{-t_h}) \).

From the first equation we have \( y = A(T_n)x - \lambda x = (A(T_n) - \lambda)x \). Substituting \( y \) in the second equation we obtain \((A(T_n) - \lambda)(B(T_n) - \lambda) = 0 \). Recall that the matrices under consideration are diagonal and the \((j+1, j+1)\)-th entry of \( T_n \) is equal to \( \varepsilon_n^{j}\). Hence, \((A(\varepsilon_n) - \lambda)(B(\varepsilon_n) - \lambda) - 1 \) \( x_{j+1} = 0 \) and \( y_{j+1} = (A(\varepsilon_n) - \lambda)x_{j+1} \).

As a result, for any \( j = 0, \ldots, n-1 \) the matrix \( L \) has exactly two eigenvalues \( \lambda_{1,j} \) and \( \lambda_{2,j} \), which are the roots of quadratic equation \((A(\varepsilon_n) - \lambda)(B(\varepsilon_n) - \lambda) - 1 = 0 \).
The following identity holds
\[ x = e_j + 1 = (0, \ldots, 1, \ldots, 0) \]
and \( y = (A(z_n) - \lambda)e_j + 1 \). In particular, if \( j = 0 \) for \( \lambda_1, \lambda_2, 0 \) we have \((1 - \lambda)(1 - \lambda) - 1 = \lambda(\lambda - 2) = 0 \). That is, \( \lambda_1, 0 = 0 \) and \( \lambda_2, 0 = 2 \). Since \( \lambda_1, 0 \) and \( \lambda_2, 0 \) are roots of the same quadratic equation, we obtain \( \lambda_1, 0 \lambda_2, 0 = P(\varepsilon_n^j) \), where

\[ P(z) = A(z)B(z) - 1 = (2k + 1 - \sum_{j=1}^{k} (z^s + z^{-s})) (2\ell + 1 - \sum_{h=1}^{\ell} (z^s + z^{-s})) - 1. \]

Now we have
\[ \tau(n) = \frac{1}{2n} \lambda_{2, 0} \prod_{j=1}^{n-1} \lambda_{1, j} \lambda_{2, j} = \frac{1}{n} \prod_{j=1}^{n-1} \lambda_{1, j} \lambda_{2, j} = \frac{1}{n} \prod_{j=1}^{n-1} P(\varepsilon_n^j). \]

To continue we need the following lemma.

**Lemma 1.** The following identity holds \( P(z) = Q(w) \), where
\[ Q(w) = (2k + 1 - \sum_{j=1}^{k} 2T_s(w)) (2\ell + 1 - \sum_{h=1}^{\ell} 2T_h(w)) - 1, \]
\( T_m(w) \) is the Chebyshev polynomial of the first kind and \( w = \frac{1}{2}(z + z^{-1}) \).

Moreover, if \( \gcd(s_1, \ldots, s_k, t_1, \ldots, t_\ell) = 1 \) then all the roots of the Laurent polynomial \( P(z) \) counted with multiplicities are \( 1, 1, z_1, 1/z_1, \ldots, z_{s_k + t_\ell - 1}, 1/z_{s_k + t_\ell - 1} \), where we have \( |z_s| \neq 1, s = 1, 2, \ldots, s_k + t_\ell - 1 \). Polynomial \( Q(w) \) has the roots \( w_1, \ldots, w_{s_k + t_\ell - 1} \), where \( w_s \neq 1 \) for all \( s = 1, 2, \ldots, s_k + t_\ell - 1 \).

**Proof.** Let us substitute \( z = e^{i\varphi} \). It is easy to see that \( w = \frac{1}{2}(z + z^{-1}) = \cos \varphi \). Then the first statement of the lemma follows from the identity \( T_s(w) = \cos(n \arccos w) = \cos(n \varphi) \).

To prove the second statement of the lemma we suppose that the Laurent polynomial \( P(z) \) has a root \( z_0 \) such that \( |z_0| = 1 \). Then \( z_0 = e^{i\varphi_0}, \varphi_0 \in \mathbb{R} \). Now we have
\[ (2k + 1 - \sum_{j=1}^{k} 2 \cos(s_j \varphi_0)) (2\ell + 1 - \sum_{h=1}^{\ell} 2 \cos(t_h \varphi_0)) - 1 = 0. \]

Since \( 2k + 1 - \sum_{j=1}^{k} 2 \cos(s_j \varphi_0) \geq 1 \) and \( 2\ell + 1 - \sum_{h=1}^{\ell} 2 \cos(t_h \varphi_0) \geq 1 \) the equation holds if and only if \( \cos(s_j \varphi_0) = 1, j = 1, \ldots, k \) and \( \cos(t_h \varphi_0) = 1, h = 1, \ldots, \ell \). So \( s_j \varphi_0 = 2\pi m_j \) and \( t_h \varphi_0 = 2\pi n_h \) for some integer \( m_j \) and \( n_h \). As \( \gcd(s_1, \ldots, s_k, t_1, \ldots, t_\ell) = 1 \) there exist two integer sequences \( p_j \) and \( q_k \) such that \( s_1 p_1 + \cdots + s_k p_k + t_1 q_1 + \cdots + t_\ell q_\ell = 1 \). Hence, \( \varphi_0 = \varphi_0(s_1 p_1 + \cdots + s_k p_k + t_1 q_1 + \cdots + t_\ell q_\ell) = 2\pi (m_1 p_1 + \cdots + m_k p_k + n_1 q_1 + \cdots + n_\ell q_\ell) = 2\pi \zeta \in \mathbb{Z} \). As a result, \( z_0 = e^{i\varphi_0} = 1 \). Now we have to show that the multiplicity of the root \( z_0 = 1 \) is 2. Indeed, \( P(1) = P'(1) = 0 \) and \( P''(1) = -2(s_1^2 + \cdots + s_k^2 + t_1^2 + \cdots + t_\ell^2) \neq 0 \).

Set \( H(z) = \prod_{s=1}^{m} (z - z_s)(z - z_s^{-1}) \), where \( m = s_k + t_\ell - 1 \) and \( z_s \) are roots of \( P(z) \) different from 1. Then by Lemma 1, we have \( P(z) = \frac{(z-1)^2}{z_{s_k+t_\ell}} H(z) \).

The following lemma has been proved in ([9], Lemma 5.3.)
Lemma 2. Let \( H(z) = \prod_{s=1}^{m} (z - z_s)(z - z_s^{-1}) \) and \( H(1) \neq 0 \). Then

\[
\prod_{j=1}^{n-1} H(z_j) = \prod_{s=1}^{m} T_n(w_s) - 1 \frac{w_s - 1}{w_s - 1},
\]

where \( w_s = \frac{1}{2}(z_s + z_s^{-1}) \), \( s = 1, \ldots, m \) and \( T_n(x) \) is the Chebyshev polynomial of the first kind.

Proof. Taking into account Lemma 1 and Lemma 2, from (2) we get

\[
\tau(n) = \frac{1}{n} \prod_{j=1}^{n-1} P(e_j^n) = \frac{1}{n} \prod_{j=1}^{n-1} \frac{\epsilon_j^n - 1}{\epsilon_j^n s_k + t_\ell} H(e_j^n) = \frac{(-1)^{n-1}}{n} \prod_{j=1}^{n-1} H(e_j^n)
\]

(3)

\[
= (-1)^{(n-1)(s_k + t_\ell)} n \prod_{s=1}^{n-1} T_n(w_s) - 1 \frac{w_s - 1}{w_s - 1}.
\]

Since \( \tau(n) \) the left hand side of equation (3) is a positive number, we obtain

\[
\tau(n) = n \prod_{s=1}^{n-1} \left| \frac{T_n(w_s) - 1}{w_s - 1} \right| = n \prod_{s=1}^{n-1} \left| T_n(w_s) - 1 \right| / \prod_{s=1}^{n-1} \left| w_s - 1 \right|.
\]

Now we evaluate the product \( \prod_{s=1}^{n-1} \left| w_s - 1 \right| \). We note that \( Q(w) = (2k + 1 - 2 \sum_{j=1}^{n} T_{s_j}(w))(2\ell + 1 - 2 \sum_{h=1}^{s_k} T_{s_h}(w)) - 1 \) is an integer polynomial with the leading coefficient \( a_0 = 2^{s_k + t_\ell} \). We have \( Q(1) = 0 \) and \( Q'(1) = -2q \), where \( q = \sum_{j=1}^{s_k} s_j^2 + \sum_{h=1}^{s_k} t_h^2 \).

Herefrom we obtain

\[
\prod_{s=1}^{n-1} \left| w_s - 1 \right| = \left| \frac{1}{a_0} Q'(1) \right| = \frac{2q}{2^{s_k + t_\ell}} = \frac{q}{2^{s_k + t_\ell} - 1}.
\]

Combining equations (4) and (5) we finish the proof of the theorem.

Theorem 2. Let \( \tau(n) \) be the number of spanning trees of the graph \( H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell) \). Denote by \( s \) and \( t \) the number of odd numbers in the sequences \( s_1, \ldots, s_k \) and \( t_1, \ldots, t_\ell \) respectively. Let \( p \) is the square-free part of the number \( u = s + t + 4st \). Then there exists an integer sequence \( a(n), n \in \mathbb{N} \) such that

1. \( \tau(n) = n a(n)^2 \), if \( n \) is odd.
2. \( \tau(n) = n^2 a(n^2)^2, \) if \( n \) is even.

Proof. Recall that all nonzero eigenvalues are \( \{\lambda_{2,0}, \lambda_{1,j}, \lambda_{2,j} : j = 1, 2, \ldots, n-1\} \), where \( \lambda_{2,0} = 2 \). By the Kirchoff theorem we have \( 2n\tau(n) = \lambda_{2,0} \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j} = 2 \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j} \). Hence \( n\tau(n) = \prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j} \). Note that \( \lambda_{1,j} \lambda_{2,j} = P(e_j^n) = P(e_j^{n-j}) = \lambda_{1,n-j} \lambda_{2,n-j} \), where \( P(z) \) is given by the formula (1). Therefore, we have \( n\tau(n) = (\prod_{j=1}^{n-1} \lambda_{1,j} \lambda_{2,j})^2 \) if \( n \) is odd, and \( n\tau(n) = \lambda_{1,j} \lambda_{2,j}^2 (\prod_{j=1}^{n/2-1} \lambda_{1,j} \lambda_{2,j})^2 \) if \( n \) is even. Moreover, if \( n \) is even we get
By Theorem 1 we have

\[ \lambda_1 \cdot \lambda_2 = P(-1) \]

\[ = \left( 2k + 1 - \sum_{j=1}^{k} \left( (-1)^{s_j} + (-1)^{-s_j} \right) \right) \left( 2\ell + 1 - \sum_{h=1}^{\ell} \left( (-1)^{t_h} + (-1)^{-t_h} \right) \right) - 1 \]

\[ = 1 + 4 \sum_{j=1}^{k} \frac{1 - (-1)^{s_j}}{2} \left( 1 + 4 \sum_{h=1}^{\ell} \frac{1 - (-1)^{t_h}}{2} \right) - 1 \]

\[ = (1 + 4s)(1 + 4t) - 1 = 4(s + t + 4st). \]

Let \( u = s + t + 4st \). We represent \( u \) in the form \( u = pr^2 \), where \( p \) is the square-free part of \( u \).

Note that graph \( H_u(s_1, \ldots, s_k; t_1, \ldots, t_\ell) \) admits a cyclic group of automorphisms isomorphic to \( \mathbb{Z}_n \) acting freely on the set of spanning trees. Thus, \( \frac{\tau(n)}{n_s} \) is an integer. We set \( 1^o \) rational numbers are integer. We set \( \tau_1, \tau_2, \) are integer numbers. From here we conclude that in equalities \( 1^o, 2^o \) the squared value a rational number. Because of \( \frac{\tau(n)}{n} \) is integer and \( p \) is a square-free, all these rational numbers are integer. We set \( a(n) = \Pi_{j=1}^{n-1/2} \lambda_{1,j} \lambda_{2,j} \) if \( n \) is odd and \( a(n) = 2r \Pi_{j=1}^{n/2-1} \lambda_{1,j} \lambda_{2,j} \) if \( n \) is even to finish the proof.

4. Asymptotics for the number of spanning trees

The asymptotic formula for the number of spanning trees in the graph \( H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell) \) is given in the following theorem.

**Theorem 3.** Suppose that \( \gcd(s_1, \ldots, s_k, t_1, \ldots, t_\ell) = 1 \). Then the number \( \tau(n) \) of spanning trees of the graph \( H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell) \) has the following asymptotics

\[ \tau(n) \sim \frac{n A^n}{\sum_{j=1}^{k} s_j^2 + \sum_{h=1}^{\ell} t_h^2}, \quad n \to \infty, \]

where \( A = \prod_{P(z)=0, |z|>1} |z| \) is the Mahler measure of the Laurent polynomial

\[ P(z) = (2k + 1 - \sum_{j=1}^{k} (z^{s_j} + z^{-s_j}))(2\ell + 1 - \sum_{h=1}^{\ell} (z^{t_h} + z^{-t_h})) - 1. \]

**Proof.** By Theorem 1 we have

\[ \tau(n) = \frac{n}{q} \prod_{s=1}^{n} |2T_n(w_s) - 2|, \]
where \( q = \sum_{j=1}^{k} s_j^2 + \sum_{h=1}^{\ell} t_h^2 \) and \( w_s, s = 1, 2, \ldots, s_k + t_\ell - 1 \) are different from 1 roots of the polynomial

\[
Q(w) = (2k + 1 - 2 \sum_{j=1}^{k} T_{s_j}(w))(2\ell + 1 - 2 \sum_{h=1}^{\ell} T_{t_h}(w)) - 1.
\]

By Lemma 1, \( T_n(w_s) = \frac{1}{2}(z_s^n + z_s^{-n}) \), where the \( z_s \) and \( 1/z_s \) are roots of the polynomial \( P(z) \) with the property \( |z_s| \neq 1, s = 1, 2, \ldots, s_k + t_\ell - 1 \). Replacing \( z_s \) by \( 1/z_s \), if it is necessary, we can assume that all \( |z_s| > 1 \) for all \( s = 1, 2, \ldots, s_k + t_\ell - 1 \). Then \( T_n(w_s) \sim \frac{1}{2} z_s^n \) and \( |2T_n(w_s) - 2| \sim |z_s|^n \) as \( n \to \infty \). Hence

\[
\frac{n}{q} \prod_{s=1}^{s_k+t_\ell-1} |2T_n(w_s) - 2| \sim \frac{n}{q} \prod_{s=1}^{s_k+t_\ell-1} |z_s|^n = \frac{n}{q} \prod_{P(z)=0, |z|>1} |z|^n = \frac{nA^n}{q}.
\]

The theorem is proved.

**Remark:** It is known ([5], p. 67) that the Mahler measure \( A = \prod_{P(z)=0, |z|>1} |z| \) of the polynomial \( P(z) \) can be calculated by the formula

\[
A = \exp \left( \int_0^1 \log |P(e^{2\pi it})| dt \right).
\]

5. **Examples and Tables**

1° The graph \( H_n(1, 2; 1) \). By Theorem 1 we have the following formula for the number of spanning trees

\[
\tau(n) = \frac{n}{6} |(2T_n(\sqrt{5}/2) - 2)(2T_n(\sqrt{5}/2) - 2)|.
\]

By Theorem 2, there is an integer sequence \( a(n) \) such that \( \tau(n) = n a(n)^2 \) if \( n \) odd and \( \tau(n) = 6 n a(n)^2 \) if \( n \) is even. Also, by Theorem 3 we get \( \tau(n) \sim \frac{\sqrt{5}}{6}(4 + \sqrt{15})^n, n \to \infty \).

2° The graph \( H_n(1, 2; 1, 2) \). By Theorem 1 we have the formula

\[
\tau(n) = \frac{n}{10} (2T_n(\frac{3}{2}) - 2)(2T_n(\frac{1 + \sqrt{33}}{4}) - 2)(2T_n(\frac{1 - \sqrt{33}}{4}) - 2)).
\]

By Theorem 2, one can find an integer sequence \( a(n) \) such that \( \tau(n) = n a(n)^2 \) if \( n \) odd and \( \tau(n) = 6 n a(n)^2 \) if \( n \) is even. Also, by Theorem 3 we get \( \tau(n) \sim \frac{\sqrt{5}}{6} A^n, n \to \infty \), there \( A = \frac{1}{6}(3 + \sqrt{5})(4 + \sqrt{3} + \sqrt{15} + 8\sqrt{3}) \approx 14.53515 \).

3° For the graph \( H_n(1, 3; 1) \) there exists an integer sequence \( a(n) \) such that \( \tau(n) = n a(n)^2 \) if \( n \) odd and \( \tau(n) = 11 n a(n)^2 \) if \( n \) is even. By Theorem 1, the number of spanning trees is given by the formula

\[
\tau(n) = \frac{n}{11} (2T_n(w_1) - 2)(2T_n(w_2) - 2)(2T_n(w_3) - 2)),
\]
where \( w_j, j = 1, 2, 3 \) are the roots of the cubic equation \( 8w^3 - 4w^2 - 8w - 7 = 0 \). We get \( \tau(n) \sim \frac{n}{A^n}, n \to \infty \), where \( A \approx 8.62367 \) is a suitable root of the algebraic equations \( z^4 - 9z^3 + 3z^2 + 2z + 1 = 0 \).

4\textsuperscript{o} Graph \( H_n(1, 3; 1, 3) \). We have \( \tau(n) = na(n)^2 \), if \( n \) is odd and \( \tau(n) = 5na(n)^2 \), if \( n \) is even, where \( a(n) \) is an integer sequence.

**Tables of complexity for graphs \( H_n(s_1, \ldots, s_k; t_1, \ldots, t_\ell) \).**

**Table 1.** Graph \( H_n(1, 2; 1) \)

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<th>( a(n) )</th>
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Recall that \( \tau(n) = na(n)^2 \) if \( n \) odd and \( \tau(n) = 6na(n)^2 \) if \( a \) is even. Moreover, \( a(n) = U_{2m}\left(\sqrt{\frac{5}{2}}\right) \) if \( n = 2m + 1 \) and \( a(n) = U_{m-1}^2\left(\sqrt{\frac{5}{2}}\right) \) if \( n = 2m \). In particular, the following identity holds \( a(2m + 1)^2 = a(2(2m + 1)) \).
Table 2. Graph $H_n(1, 2; 1, 2)$

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One has $\tau(n) = n a(n)^2$ if $n$ odd and $\tau(n) = 6 n a(n)^2$ if $a$ is even.
Table 3. Graph $H_n(1,3;1)$

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Here $\tau(n) = n a(n)^2$ if $n$ odd and $\tau(n) = 11 n a(n)^2$ if $a$ is even. Also, for any positive integer $m$ we have

1. $a(3m)$ is divisible by 5;
2. $a(5m)$ and $a(7m)$ are divisible by 71;
3. $a(6m)$ and $a(11m)$ are divisible by 11.
The authors are grateful to Prof. A.D. Mednykh who has attracted our attention to the problem.

References


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