S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 15, стр. 1158–1173 (2018) УДК 517.95
DOI 10.17377/semi.2018.15.094 MSC 35K92

ANISOTROPIC VANISHING DIFFUSION METHOD APPLIED TO GENUINELY NONLINEAR FORWARD-BACKWARD ULTRA-PARABOLIC EQUATIONS

I.V. KUZNETSOV, S.A. SAZHENKOV

Abstract. The results formulated in (I.V. Kuznetsov, Sib. Elect. Math. Rep. **14** (2017), 710–731) are extended onto the multi-time case. We prove existence and uniqueness of kinetic solutions to genuinely nonlinear forward-backward ultra-parabolic equations and show that kinetic solutions do not depend on the anisotropic elliptic regularization.

Keywords: forward-backward ultra-parabolic equation, entropy solution, kinetic solution

Introduction

In the present paper, the results presented in [19] are extended onto the multi-time case. We obtain a kinetic solution and, correspondingly, an equivalent entropy solution of the forward-backward ultra-parabolic equation. We prove existence of solutions using a sequence of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ to the regularized Problem $\mathbf{\Pi}_\varepsilon$, as $\varepsilon \to 0+$. To this end, we use the vanishing diffusion method incorporating the anisotropic $\boldsymbol{p}$-Laplacian, $\boldsymbol{p} = (p_1, \ldots, p_k) \in \mathbb{R}^k$, $p_i > 1$, $i = 1, \ldots, k$. The physical meaning of the vanishing anisotropic diffusion method is that we take into account fast and slow diffusive regimes when $\sum\limits_{i=1}^{k} |p_i - 2| \neq 0$, see [5, Chapter 5]. We show

Kuznetsov, I.V., Sazhenkov, S.A., Anisotropic vanishing diffusion method applied to genuinely nonlinear forward-backward ultra-parabolic equations.

© 2018 Kuznetsov I.V, Sazhenkov S.A.

The work was supported by the Federal Agency for Scientific Organizations of the Russian Federation (project no. III.22.4.2) and by the Russian Foundation for Basic Research (grant no. 18-01-00649).

Received June, 26, 2018, published October, 15, 2018.

that the existence of kinetic solutions does not depend on the anisotropic elliptic regularization.

The technique of elliptic regularization was invented in [22] during the study of degenerate parabolic equations, and subsequently adapted for the Navier-Stokes equations [35], and for forward-backward parabolic equations [7, 27]. Furthermore, singular limits of anisotropic elliptic perturbations were obtained in [8, 9]. Also, for hyperbolic equations the vanishing viscosity method with gradient dependent viscosity was applied in [25, 26].

This paper is organized as follows. In Section 1, we formulate the problem $\mathbf{\Pi}_0$. Section 2 is devoted to the anisotropic elliptic regularization $\mathbf{\Pi}_\varepsilon$. The main result of this paper is Theorem 1, see Section 3. The proof of Theorem 1 is given in Section 4. Moreover, we deduce boundary entropy conditions $(3.2\text{b})_{i=1}^k$ and $(3.2\text{c})_{i=1}^k$.

The obtained results can be also applied to 'forward' ultra-parabolic equations, see Remark 2 and Example 1 in Section 1.

1. Forward-backward ultra-parabolic equations

Let vector functions $\boldsymbol{a}(z) = (a_1(z), \ldots, a_k(z))$ and $\boldsymbol{\varphi}(z) = (\varphi_1(z), \ldots, \varphi_d(z))$ satisfy the following conditions.

Conditions on $\boldsymbol{a}$&$\boldsymbol{\varphi}$. *Let $a_i \in C^2(\mathbb{R})$, $i = 1, \ldots, k$, $\boldsymbol{a}(0) = \mathbf{0}$, $\varphi_j \in C^2(\mathbb{R})$, $j = 1, \ldots, d$, $\boldsymbol{\varphi}(0) = \mathbf{0}$. At least one component of vector function $\boldsymbol{a}$ is non-monotone function on $\mathbb{R}$. Moreover, $\boldsymbol{a}'$ satisfies the genuine nonlinearity condition:*

$$\operatorname{mes}\left\{\lambda \in \mathbb{R} : \sum_{i=1}^{k} a_i'(\lambda)\xi_i = 0\right\} = 0 \tag{1.1}$$

for every $(\xi_1, \ldots, \xi_k) \in \mathbb{S}^{k-1}$.

Remark 1. *Condition* (1.1) *can be generalized in the following way: the set*

$$\left\{\lambda \in \mathbb{R} : \sum_{i=1}^{k} a_i'(\lambda)\xi_i = 0\right\} \tag{1.1$'$}$$

has the empty interior for each $(\xi_1, \ldots, \xi_k) \in \mathbb{S}^{k-1}$, see [31].

Furthermore, in order to guarantee the stability principle (3.3) *for entropy solutions u_1 and u_2 formulated in Section 3, we need to imply an addition condition on function $\boldsymbol{\varphi}$:*

$$\|\boldsymbol{\varphi}'\|_{C(-\widetilde{M},\widetilde{M})} \, \|\nabla_x \xi_P\|_{C(\Omega)} < 1, \tag{1.2}$$

where

$$\widetilde{M} = \max_{\substack{i=1,\ldots,k\\ j=1,2}} \left(\left\| u_{j,0}^{(i)} \right\|_{L^\infty(\Xi^i)}, \left\| u_{j,T_i}^{(i)} \right\|_{L^\infty(\Xi^i)} \right),$$

$(u_{1,0}^{(i)}, u_{1,T_i}^{(i)})$, $(u_{2,0}^{(i)}, u_{2,T_i}^{(i)})$ $(i = 1, \ldots, k)$ *are two sets of initial and final data w.r.t. the time-like variables t_i (for details, see in* (1.3b) *in the formulation of problem $\mathbf{\Pi}_0$ below), ξ_P is the solution of the homogeneous Dirichlet problem for Poisson's equation in Ω:*

$$\Delta_x \xi_P(\boldsymbol{x}) = -1, \quad \boldsymbol{x} \in \Omega, \quad \xi_P(\boldsymbol{x})\Big|_{\partial\Omega} = 0.$$

Inequality (1.2) *is identical to inequality (45) from [19].*

Under Conditions on $\boldsymbol{a}\&\boldsymbol{\varphi}$, we formulate the boundary value Problem $\mathbf{\Pi}_0$ immediately below. Let

$$\begin{aligned}
&G_{T_1,\ldots,T_k} = \Omega \times (0,T_1) \times \ldots \times (0,T_k),\\
&\Gamma_0^1 = \overline{\Omega} \times \{t_1 = 0\} \times [0,T_2] \times \ldots \times [0,T_k],\\
&\Gamma_{T_1}^1 = \overline{\Omega} \times \{t_1 = T_1\} \times [0,T_2] \times \ldots \times [0,T_k],\\
&\ldots,\\
&\Gamma_0^k = \overline{\Omega} \times [0,T_1] \times \ldots \times [0,T_{k-1}] \times \{t_k = 0\},\\
&\Gamma_{T_k}^k = \overline{\Omega} \times [0,T_1] \times \ldots \times [0,T_{k-1}] \times \{t_k = T_k\};\\
&\Xi^1 = \Omega \times (0,T_2) \times \ldots \times (0,T_k),\\
&\ldots,\\
&\Xi^k = \Omega \times (0,T_1) \times \ldots \times (0,T_{k-1});\\
&\Gamma_l = \partial\Omega \times [0,T_1] \times \ldots \times [0,T_k],
\end{aligned}$$

a bounded domain $\Omega \subset \mathbb{R}^d$ ($\operatorname{mes}\Omega < \infty$) has a smooth boundary $\partial\Omega$.

Problem $\mathbf{\Pi}_0$. *For arbitrary initial and final data $u_0^{(i)}, u_{T_i}^{(i)} \in L^\infty(\Xi^i) \cap C_0^2(\Xi^i)$, $i = 1,\ldots,k$, it is necessary to find a function $u : G_{T_1,\ldots,T_k} \mapsto \mathbb{R}$ satisfying*

$$\operatorname{div}_t \boldsymbol{a}(u) + \operatorname{div}_x \boldsymbol{\varphi}(u) = \Delta_x u, \quad (\boldsymbol{x},\boldsymbol{t}) \in G_{T_1,\ldots,T_k}, \tag{1.3a}$$

$$u|_{\Gamma_0^i} \approx u_0^{(i)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i), \quad u|_{\Gamma_{T_i}^i} \approx u_{T_i}^{(i)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i), \quad (\boldsymbol{x},\widehat{\boldsymbol{t}}_i) \in \Xi^i, \quad i = 1,\ldots,k, \tag{1.3b}$$

$$u|_{\Gamma_l} = 0, \tag{1.3c}$$

in the form given in Definition 3. The sign $\approx$ means the equality only on a part of the boundary, $\widehat{\boldsymbol{t}}_i = (t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_k)$, $i = 2,\ldots,k-1$, $\widehat{\boldsymbol{t}}_1 = (t_2,\ldots,t_k)$, $\widehat{\boldsymbol{t}}_k = (t_1,\ldots,t_{k-1})$.

Remark 2. *We formulate equation* (1.3a) *in the sense of distributions. Since at least one component of vector function $\boldsymbol{a}(z)$ is non-monotone on the interval $(-M,M)$, equation* (1.3a) *is a forward-backward ultra-parabolic equation. Here*

$$M = \max_{i=1,\ldots,k} \left(\left\| u_0^{(i)} \right\|_{L^\infty(\Xi^i)}, \left\| u_{T_i}^{(i)} \right\|_{L^\infty(\Xi^i)} \right). \tag{1.4}$$

Moreover, a weak solution u can deviate from initial and final data $u_0^{(i)}, u_{T_i}^{(i)}$, $i = 1,\ldots,k$. Therefore, the difficulty of Problem $\mathbf{\Pi}_0$ is that equation (1.3a) *and initial and final conditions* (1.3b) *must be reformulated in the form of kinetic equalities, see in Definition 3. It is important to note that, under Conditions on $\boldsymbol{a}\&\boldsymbol{\varphi}$, equation* (1.3a) *can be also merely a 'forward' ultra-parabolic equation, see Example 1.*

Remark 3. *With the help of Theorem 1 formulated in Section 3 we can decrease the smoothness of initial and final data $u_0^{(i)}, u_{T_i}^{(i)} \in L^\infty(\Xi^i) \cap C_0^2(\Xi^i)$, $i = 1,\ldots,k$. Therefore, we can deal with the case $u_0^{(i)}, u_{T_i}^{(i)} \in L^\infty(\Xi^i)$, $i = 1,\ldots,k$ only. See Remark 6 in Section 3.*

Example 1. *Let $k = 2$, $a_1(\lambda) = \lambda$, $a_2(\lambda) = -f(\lambda)$, $\boldsymbol{\varphi}(\lambda) \equiv \mathbf{0}$, $\lambda \in \mathbb{R}$. Here $f(\lambda)$ is a smooth function satisfying the genuine nonlinearity condition*

$$\operatorname{mes}\{\lambda \in \mathbb{R} :\ \xi_1 - f'(\lambda)\xi_2 = 0\} = 0 \tag{1.5a}$$

for every $(\xi_1, \xi_2) \in \mathbb{S}^1$. *We deal with the Dirichlet–Cauchy problem*

$$\partial_{t_1} u = \Delta_x u + \partial_{t_2} f(u), \quad (\boldsymbol{x}, t_1, t_2) \in G_{T_1,T_2}, \tag{1.5b}$$

$$u|_{t_1=0} = u_0^{(1)}(\boldsymbol{x}, t_2),\ (\boldsymbol{x}, t_2) \in \Xi^1 = \Omega \times (0, T_2), \tag{1.5c}$$

$$u|_{t_2=0} \approx u_0^{(2)}(\boldsymbol{x}, t_1),\ u|_{t_2=T_2} \approx u_{T_2}^{(2)}(\boldsymbol{x}, t_1),\ (\boldsymbol{x}, t_1) \in \Xi^2 = \Omega \times (0, T_1), \tag{1.5d}$$

$$u|_{\Gamma_l} = 0. \tag{1.5e}$$

Equation (1.5b) *is a strongly degenerate anisotropic convection-diffusion equation. In [1, 13, 38] the Cauchy problem is treated. Therefore, instead of the genuine nonlinearity condition it is assumed in [13] that the function* f *is locally Lipschitz continuous real function.*

2. Anisotropic elliptic regularization

We are going to construct an entropy solution as a singular limit of weak solutions u_ε for non-homogeneous Dirichlet problem $\boldsymbol{\Pi}_\varepsilon$, as $\varepsilon \to 0+$.

Problem $\boldsymbol{\Pi}_\varepsilon$. *For arbitrary initial and final conditions* $u_0^{(i)}, u_{T_i}^{(i)} \in L^\infty(\Xi^i) \cap C_0^2(\Xi^i)$, $i = 1, \dots, k$, *it is necessary to find a function* $u_\varepsilon \colon G_{T_1,\dots,T_k} \mapsto \mathbb{R}$ *solving the boundary value problem*

$$\operatorname{div}_t \boldsymbol{a}(u_\varepsilon) + \operatorname{div}_x \boldsymbol{\varphi}(u_\varepsilon) = \Delta_x u_\varepsilon + \varepsilon \Delta_{\boldsymbol{p},t} u_\varepsilon,\ (\boldsymbol{x}, \boldsymbol{t}) \in G_{T_1,\dots,T_k}, \tag{2.1a}$$

$$u_\varepsilon|_{\Gamma_0^i} = u_0^{(i)}, \quad u_\varepsilon|_{\Gamma_{T_i}^i} = u_{T_i}^{(i)}, \quad i = 1, \dots, k, \quad u_\varepsilon|_{\Gamma_l} = 0, \tag{2.1b}$$

in the weak sense, see Definition 1 in Section 2.

We denote here

$$\Delta_{\boldsymbol{p},t} u_\varepsilon = \sum_{i=1}^{k} \partial_{t_i} \left(|\partial_{t_i} u_\varepsilon|^{p_i - 2} \partial_{t_i} u_\varepsilon \right), \quad \boldsymbol{p} = (p_1, \dots, p_k), \quad \varepsilon \in (0, 1].$$

Remark 4. *The open question is how the results affirmed in Theorem 1 in Section 3 can be valid for a singular limit of weak solutions to problem* $\boldsymbol{\Pi}_\varepsilon$ *with equation*

$$\operatorname{div}_t \boldsymbol{a}(u_\varepsilon) + \operatorname{div}_x \boldsymbol{\varphi}(u_\varepsilon) = \Delta_x u_\varepsilon + \varepsilon \operatorname{div}_t \left(|\nabla_t u_\varepsilon|^{p-2} \nabla_t u_\varepsilon \right),\ (\boldsymbol{x}, \boldsymbol{t}) \in G_{T_1,\dots,T_k} \tag{2.1c}$$

instead of (2.1a). *In* (2.1c) *exponent* p *is greater than one and not equal to two.*

We deal with anisotropic Sobolev spaces, see, for example, [16]. The anisotropic Sobolev space $W_0^{1,\boldsymbol{P}}(G_{T_1,\dots,T_k})$ is equipped with the norm

$$\|u\|_{W_0^{1,\boldsymbol{P}}(G_{T_1,\dots,T_k})} = \sum_{j=1}^{d} \|\partial_{x_j} u\|_{L^2(G_{T_1,\dots,T_k})} + \sum_{i=1}^{k} \|\partial_{t_i} u\|_{L^{p_i}(G_{T_1,\dots,T_k})} + \|u\|_{L^1(G_{T_1,\dots,T_k})},$$

where $\boldsymbol{P} = (\underbrace{2, \dots, 2}_{d \text{ times}}, p_1, \dots, p_k) \in \mathbb{R}^{d+k}$, $p_i > 1$, $i = 1, \dots, k$.

Let $V^{\boldsymbol{P}}(G_{T_1,\dots,T_k}) = \{v \in W^{1,\boldsymbol{P}}(G_{T_1,\dots,T_k}) : \ v|_{\Gamma_l} = 0\}$. Let $\hat{u} \in L^\infty(G_{T_1,\dots,T_k}) \cap V^{\boldsymbol{P}}(G_{T_1,\dots,T_k})$ be some extension of the data $u_0^{(i)}, u_{T_i}^{(i)}$ $(i = 1, \dots, k)$ into $G_{T_1,\dots,T_k}$. For example, $\hat{u}$ is the solution of the Dirichlet problem for Laplace's equation $\Delta_{x,t} \hat{u} = 0$ (in $G_{T_1,\dots,T_k}$) with boundary data $\hat{u}\big|_{\Gamma_l} = 0$, $\hat{u}\big|_{\Gamma_0^i} = u_0^{(i)}$, $\hat{u}\big|_{\Gamma_{T_i}^i} = u_{T_i}^{(i)}$

($i = 1, \dots, k$). This $\hat{u}$ belongs, in fact, to the space $C^2(\overline{G}_{T_1,\dots,T_k})$ and is represented in terms of Green's function [14, Sec. 2.2.4, Th. 12].

Definition 1. *Function $u_\varepsilon \in L^\infty(G_{T_1,\dots,T_k}) \cap V^{\boldsymbol{P}}(G_{T_1,\dots,T_k})$ is called a weak solution to problem $\boldsymbol{\Pi}_\varepsilon$ if the following demands hold.*

1. *Difference $u_\varepsilon - \hat{u}$ belongs to $L^\infty(G_{T_1,\dots,T_k}) \cap W_0^{1,\boldsymbol{P}}(G_{T_1,\dots,T_k})$.*

2. *The following equality holds*

$$\int_{G_{T_1,\dots,T_k}} \Big(-\boldsymbol{a}(u_\varepsilon) \cdot \nabla_t \phi - \boldsymbol{\varphi}(u_\varepsilon) \cdot \nabla_x \phi + \nabla_x u_\varepsilon \cdot \nabla_x \phi + \varepsilon \sum_{i=1}^{k} |\partial_{t_i} u_\varepsilon|^{p_i - 2} \, \partial_{t_i} u_\varepsilon \partial_{t_i} \phi \Big) \, d\boldsymbol{x} d\boldsymbol{t} = 0 \tag{2.2a}$$

for every $\phi \in L^\infty(G_{T_1,\dots,T_k}) \cap W_0^{1,\boldsymbol{P}}(G_{T_1,\dots,T_k})$.

Remark 5. *We can reformulate* (2.2a) *in the equivalent way:*

$$\int_{G_{T_1,\dots,T_k}} \Big(\mathrm{div}_t \boldsymbol{a}(u_\varepsilon) \phi + \mathrm{div}_x \boldsymbol{\varphi}(u_\varepsilon) \phi + \nabla_x u_\varepsilon \cdot \nabla_x \phi + \varepsilon \sum_{i=1}^{k} |\partial_{t_i} u_\varepsilon|^{p_i - 2} \, \partial_{t_i} u_\varepsilon \partial_{t_i} \phi \Big) \, d\boldsymbol{x} d\boldsymbol{t} = 0. \tag{2.2b}$$

Proposition 1. *Under Conditions on $\boldsymbol{a}\&\boldsymbol{\varphi}$, problem $\boldsymbol{\Pi}_\varepsilon$ has at least one weak solution u_ε for all $u_0^{(i)}, u_{T_i}^{(i)} \in L^\infty(\Xi^i) \cap C_0^2(\Xi^i)$, $i = 1, \dots, k$. Moreover, the maximum principle*

$$\|u_\varepsilon\|_{L^\infty(G_{T_1,\dots,T_k})} \le M \quad (M \textit{ is defined by } (1.4)), \tag{2.3}$$

and the energy estimate

$$\int_{G_{T_1,\dots,T_k}} \left(|\nabla_x u_\varepsilon|^2 + \varepsilon \sum_{i=1}^{k} |\partial_{t_i} u_\varepsilon|^{p_i} \right) \, d\boldsymbol{x} d\boldsymbol{t} \le C \tag{2.4}$$

hold. The constant $C = C(\boldsymbol{a}, \boldsymbol{\varphi}, \hat{u}, \Omega, T_1, \dots, T_k)$ does not depend on $\varepsilon \in (0, 1]$.

This proposition can be proved with the help of methods presented in [3–5, 19]. In this article, we use notions of kinetic and entropy solutions of Problem $\boldsymbol{\Pi}_0$.

3. Kinetic and entropy formulations of forward-backward ultra-parabolic equations

In this section we deal with the kinetic and entropy formulations of forward-backward ultra-parabolic equations.

Consider the function χ which is defined in the following way

$$\chi(\lambda; \upsilon) = \begin{cases} +1, & \text{for } 0 < \lambda < \upsilon, \\ -1, & \text{for } \upsilon < \lambda < 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Definition 2. *Let N be a positive integer, $\mathcal{O}$ be an open set of $\mathbb{R}^N$ and the function $h \in L^\infty(\mathcal{O} \times (-L, L))$ satisfy $0 \le h(\boldsymbol{z}, \lambda)\mathrm{sgn}(\lambda) \le 1$ for almost every $(\boldsymbol{z}, \lambda) \in \mathbb{R}^{N+1}$. It is said that h is a χ-function if there exists a function $\upsilon \in L^\infty(\mathcal{O})$ such that*

$$h(\boldsymbol{z}, \lambda) = \chi(\lambda; \upsilon(\boldsymbol{z}))$$

for a.e. $\boldsymbol{z} \in \mathcal{O}$. *Note that* $\upsilon(\boldsymbol{z}) = \int_{-L}^{L} h(\boldsymbol{z}, \lambda)\, d\lambda = \int_{-L}^{L} \chi(\lambda; \upsilon(\boldsymbol{z}))\, d\lambda$.

The following lemma formulated and proved in [39] guarantees the link between sequences of χ-functions and their limits.

Lemma 1. *Let* $\mathcal{O}$ *be an open set of* $\mathbb{R}^N$ *and* $h_n \in L^\infty(\mathcal{O} \times (-L, L))$ *be a sequence of* χ*-functions converging weakly to* $h \in L^\infty(\mathcal{O} \times (-L, L))$. *We define* $\upsilon_n(\cdot) = \int_{-L}^{L} h_n(\cdot, \lambda)\, d\lambda$ *and* $\upsilon(\cdot) = \int_{-L}^{L} h(\cdot, \lambda)\, d\lambda$. *Then the three assertions are equivalent:*

- h_n *converges strongly to* h *in* $L^1_{\mathrm{loc}}(\mathcal{O} \times (-L, L))$,
- υ_n *converges strongly to* υ *in* $L^1_{\mathrm{loc}}(\mathcal{O})$,
- h *is a* χ*-function.*

Definition 3. *Function* $u \in L^\infty(G_{T_1,\dots,T_k}) \cap L^2((0, T_1) \times \ldots \times (0, T_k); W_0^{1,2}(\Omega))$ *is called a kinetic solution of problem* $\mathbf{\Pi}_0$ *if it satisfies the following equations:*

1. *(Kinetic equation)*

$$\boldsymbol{a}'(\lambda) \cdot \nabla_t \chi(\lambda; u(\boldsymbol{x}, \boldsymbol{t})) + \boldsymbol{\varphi}'(\lambda) \cdot \nabla_x \chi(\lambda; u(\boldsymbol{x}, \boldsymbol{t})) = \Delta_x \chi(\lambda; u(\boldsymbol{x}, \boldsymbol{t})) + \partial_\lambda (m(\boldsymbol{x}, \boldsymbol{t}, \lambda) + n(\boldsymbol{x}, \boldsymbol{t}, \lambda)), \tag{3.1a}$$

2. *(Kinetic boundary conditions)*

$$a_i'(\lambda)\big(\chi(\lambda; u_0^{\tau,(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i)) - \chi(\lambda; u_0^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i))\big) - \delta_{(\lambda = u_0^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i))}\big(a_i(u_0^{\tau,(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i)) - a_i(u_0^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i))\big) = \partial_\lambda \mu_0^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i, \lambda), \tag{3.1b}$$

$$a_i'(\lambda)\big(\chi(\lambda; u_{T_i}^{\tau,(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i)) - \chi(\lambda; u_{T_i}^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i))\big) - \delta_{(\lambda = u_{T_i}^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i))}\big(a_i(u_{T_i}^{\tau,(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i)) - a_i(u_{T_i}^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i))\big) = -\partial_\lambda \mu_{T_i}^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i, \lambda), \tag{3.1c}$$

where $m, n \in \mathcal{M}^+(G_{T_1,\dots,T_k} \times (-M, M))$, $n = \delta_{(\lambda = u)} |\nabla_x u|^2$, $\mu_0^{(i)}, \mu_{T_i}^{(i)} \in \mathcal{M}^+(\Xi^i \times (-M, M))$, $i = 1, \dots, k$. *Here* $\mathcal{M}^+$ *denotes the space of finite positive Radon measures,* $u_0^{\tau,(i)}, u_{T_i}^{\tau,(i)}$, $i = 1, \dots, k$, *are strong traces of a kinetic solution, constant* M *is defined in* (1.4).

It is important to note that the existence of strong traces for $k = 2$ was proved in [19] with the help of the kinetic formulation of entropy solutions. Moreover, the kinetic boundary conditions are equivalent to those formulated in [10, 20]. Also, we need to mention the other equivalent formulations of entropy boundary conditions [6, 28]. For the kinetic formulation of sub and super entropy solutions for initial-boundary value problem for first-order hyperbolic equations see [15, 33].

Definition 4. *Function* $u \in L^\infty(G_{T_1,\dots,T_k}) \cap L^2((0, T_1) \times \ldots \times (0, T_k); W_0^{1,2}(\Omega))$ *is called an entropy solution of problem* $\mathbf{\Pi}_0$ *if it satisfies the following demands:*

1. *(Entropy condition)*

$$\mathrm{div}_t \boldsymbol{q}_a(u) + \mathrm{div}_x \boldsymbol{q}_\varphi(u) - \Delta_x \eta(u) \le -\eta''(u) |\nabla_x u|^2, \tag{3.2a}$$

2. *(Entropy boundary conditions,* $i = 1, \dots, k$*)*

$$q_{a_i}(u_0^{\tau,(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i)) - q_{a_i}(u_0^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i)) - \eta'(u_0^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i))(a_i(u_0^{\tau,(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i)) - a_i(u_0^{(i)}(\boldsymbol{x}, \widehat{\boldsymbol{t}}_i))) \le 0, \tag{3.2b}$$

(3.2c) $$q_{a_i}(u^{\tau,(i)}_{T_i}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i)) - q_{a_i}(u^{(i)}_{T_i}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i)) - \eta'(u^{(i)}_{T_i}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i))(a_i(u^{\tau,(i)}_{T_i}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i)) - a_i(u^{(i)}_{T_i}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i))) \geq 0,$$

for a.e. $(\boldsymbol{x},\widehat{\boldsymbol{t}}_i) \in \Xi^i$ *and for any convex entropy flux triple* $(\eta, \boldsymbol{q}_a, \boldsymbol{q}_\varphi)$*:*

$$\boldsymbol{q}'_a(z) = \boldsymbol{a}'(z)\eta'(z), \quad \boldsymbol{q}'_\varphi(z) = \boldsymbol{\varphi}'(z)\eta'(z), \quad \eta''(z) \geq 0, \quad z \in \mathbb{R}.$$

In [19] it was shown that when $k = 2$, Definitions 3 and 4 are equivalent. The equivalence for any $k > 2$ can be proved similarly.

The main result of the present paper is the following theorem.

Theorem 1. *The set* $\{u_\varepsilon\}$ *is pre-compact and converges strongly to an entropy solution* u *of problem* (1.3a)–(1.3c)*, as* $\varepsilon \to 0$*.*

This theorem is proved in Section 4.

Proposition 2. *Under Conditions on* $\boldsymbol{a}\&\boldsymbol{\varphi}$*, Problem* $\mathbf{\Pi}_0$ *has the unique kinetic solution* u *for all* $u^{(i)}_0, u^{(i)}_{T_i} \in L^\infty(\Xi^i)$*,* $i = 1,\dots,k$*. Moreover, the* L^1 *stability holds:*

(3.3) $$\|u_1 - u_2\|_{L^1(G_{T_1,\dots,T_k})} \leq C\sum_{i=1}^{k}\Big(\|u^{(i)}_{1,0} - u^{(i)}_{2,0}\|_{L^1(\Xi^i)} + \|u^{(i)}_{1,T_i} - u^{(i)}_{2,T_i}\|_{L^1(\Xi^i)}\Big).$$

The proof of this proposition is analogous to the proof of Theorem 1 formulated in [19], see Section 4. We use the methods created in [10,11,18–21,24,29–32,34,37,39].

Remark 6. *It is important to note that* (3.3) *enables to decrease the smoothness of* $u^{(i)}_0$ *and* $u^{(i)}_{T_i}$*,* $i = 1,\dots,k$*, assumed in problem* $\mathbf{\Pi}_0$*. See Remark 3 in Section 1.*

4. The scheme of the proof of Theorem 1

Using kinetic formulation of (1.3a), the genuine nonlinearity condition (1.1), the precomactness of $\{\chi(\lambda; u_\varepsilon)\}_{\varepsilon>0}$, see [21], and Lemma 1, we can prove that $\{u_\varepsilon\}_{\varepsilon>0}$ is a precompact set in $L^1(G_{T_1,\dots,T_k})$.

By analogy with [17,19], we can prove that a kinetic solution has traces $u^{\tau,(i)}_0, u^{\tau,(i)}_{T_i}$, $i = 1,\dots,k$, in the L^1-sense.

The entropy inequality (3.2a) is valid for every convex entropy flux triple $(\eta, \boldsymbol{q}_a, \boldsymbol{q}_\varphi)$. We get this inequality from

$$\mathrm{div}_t\boldsymbol{q}_a(u_\varepsilon) + \mathrm{div}_x\boldsymbol{q}_\varphi(u_\varepsilon) = \Delta_x\eta(u_\varepsilon) + \varepsilon\sum_{i=1}^{k}\partial_{t_i}(|\partial_{t_i}u_\varepsilon|^{p_i-2}\partial_{t_i}\eta(u_\varepsilon)) - \eta''(u_\varepsilon)\left(|\nabla_x u_\varepsilon|^2 + \varepsilon\sum_{i=1}^{k}|\partial_{t_i}u_\varepsilon|^{p_i}\right),$$

which is deduced by putting $\phi = \eta'(u_\varepsilon)\gamma$ in (2.2b), where γ is an arbitrary nonnegative finite test function in $G_{T_1,\dots,T_k}$.

In this paper we are going to deduce the entropy boundary conditions $(3.2\text{b})_{i=1}^k$ and $(3.2\text{c})_{i=1}^k$. Therefore, with above mentioned results we can prove Theorem 1 as in [19]. The main idea is that we use the precompactness of $\{u_\varepsilon\}_{\varepsilon>0}$ and the existence of traces $u^{\tau,(i)}_0$ and $u^{\tau,(i)}_{T_i}$, $i = 1,\dots,k$, in order to prove the following lemma which is similar to Lemma 1.1 in [12].

Lemma 2. *For any function $\theta_i \in C_0^2(\Xi^i)$ it follows that*

$$-\lim_{\varepsilon\to 0+} \varepsilon \int_{\Xi^i} \theta_i(\boldsymbol{x},\widehat{\boldsymbol{t}}_i) \left|\partial_{t_i} u_\varepsilon(\boldsymbol{x},\widehat{\boldsymbol{t}}_{0,i})\right|^{p_i-2} \partial_{t_i} u_\varepsilon(\boldsymbol{x},\widehat{\boldsymbol{t}}_{0,i})\, d\boldsymbol{x} d\widehat{\boldsymbol{t}}_i = \int_{\Xi^i} \theta_i(\boldsymbol{x},\widehat{\boldsymbol{t}}_i)\big(a_i(u_0^{\tau,(i)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i)) - a_i(u_0^{(i)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i))\big)\, d\boldsymbol{x} d\widehat{\boldsymbol{t}}_i, \tag{4.1a}$$

$$-\lim_{\varepsilon\to 0+} \varepsilon \int_{\Xi^i} \theta_i(\boldsymbol{x},\widehat{\boldsymbol{t}}_i) \left|\partial_{t_i} u_\varepsilon(\boldsymbol{x},\widehat{\boldsymbol{t}}_{T_i,i})\right|^{p_i-2} \partial_{t_i} u_\varepsilon(\boldsymbol{x},\widehat{\boldsymbol{t}}_{T_i,i})\, d\boldsymbol{x} d\widehat{\boldsymbol{t}}_i = \int_{\Xi^i} \theta_i(\boldsymbol{x},\widehat{\boldsymbol{t}}_i)\big(a_i(u_{T_i}^{\tau,(i)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i)) - a_i(u_{T_i}^{(i)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_i))\big)\, d\boldsymbol{x} d\widehat{\boldsymbol{t}}_i, \tag{4.1b}$$

where $\widehat{\boldsymbol{t}}_{0,i} = (t_1,\dots,t_{i-1},0,t_{i+1},\dots,t_k)$, $\widehat{\boldsymbol{t}}_{T_i,i} = (t_1,\dots,t_{i-1},T_i,t_{i+1},\dots,t_k)$, $i = 2,\dots,k-1$; $\widehat{\boldsymbol{t}}_{0,1} = (0,t_2,\dots,t_k)$, $\widehat{\boldsymbol{t}}_{0,k} = (t_1,\dots,t_{k-1},0)$, $\widehat{\boldsymbol{t}}_{T_1,1} = (T_1,t_2,\dots,t_k)$, $\widehat{\boldsymbol{t}}_{T_k,k} = (t_1,\dots,t_{k-1},T_k)$.

Proof. We integrate by parts the following expression in the t_1 variable:

$$\begin{aligned} \varepsilon \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} &\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\partial_{t_1}(|\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\, d\boldsymbol{x} dt_1 d\widehat{\boldsymbol{t}}_1 = \\ &- \varepsilon \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)\rho'_\delta(t_1)\, d\boldsymbol{x} dt_1 d\widehat{\boldsymbol{t}}_1 \\ &- \varepsilon \int_{\Xi_1} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\partial_{t_1} u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1} u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)\, d\boldsymbol{x} d\widehat{\boldsymbol{t}}_1, \end{aligned} \tag{4.2}$$

where $\rho_\delta \in C^2(\mathbb{R}^+)$:

$$\rho_\delta(t_1) = 0,\ t_1 > \delta,\ \rho_\delta(0) = 1,\ |\rho'_\delta(t_1)| \le \frac{c}{\delta},\ t_1 \in (0,\delta),\ c > 0, \qquad \lim_{\delta\to 0+} \int_0^\delta \Phi(t_1)\rho'_\delta(t_1)\, dt_1 = -\Phi(0).$$

The traces $\big(|\partial_{t_1} u_\varepsilon|^{p_1-2}\partial_{t_1} u_\varepsilon\big)\big|_{\Gamma^1_{t_1}}$, $t_1 \in [0,T_1]$ are understood in the weak sense, see Proposition A.1 and Remark A.2 in Appendix A in the end of the article for more explanations.

From (2.4) it follows that $\forall\delta > 0$:

$$\begin{aligned} \lim_{\varepsilon\to 0+} \varepsilon \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)\rho'_\delta(t_1)\, d\boldsymbol{x} dt_1 d\widehat{\boldsymbol{t}}_1 = \\ \lim_{\varepsilon\to 0+} \varepsilon^{\frac{1}{p_1}} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\varepsilon^{\frac{1}{p_1}}\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\varepsilon^{\frac{1}{p_1}}\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)\rho'_\delta(t_1)\, d\boldsymbol{x} dt_1 d\widehat{\boldsymbol{t}}_1 = 0. \end{aligned} \tag{4.3}$$

In the limit as $\varepsilon \to 0+$, equation (4.2) reads

$$\begin{aligned} \lim_{\varepsilon\to 0+} \varepsilon \int_{\Xi^1} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\partial_{t_1} u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1} u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)\ d\boldsymbol{x} d\widehat{\boldsymbol{t}}_1 = \\ -\lim_{\varepsilon\to 0+} \varepsilon \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\partial_{t_1}(|\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1} u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\, d\boldsymbol{x} dt_1 d\widehat{\boldsymbol{t}}_1 =: \\ A_1(\delta). \end{aligned} \tag{4.4}$$

Function $A_1(\delta)$ can be rewritten in the following way

$$A_1(\delta) = -\lim_{\varepsilon\to 0+} \varepsilon \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\partial_{t_1}\big(|\partial_{t_1}u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1}u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)\big)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1$$

$$= \lim_{\varepsilon\to 0+} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_1(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho'_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1 + \int_{\Xi^1} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_1(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\;d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1$$

$$+\sum_{i=2}^k \lim_{\varepsilon\to 0+} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \partial_{t_i}\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_i(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1$$

$$+\lim_{\varepsilon\to 0+} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \nabla_x\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\cdot\boldsymbol{\varphi}(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1$$

$$+\lim_{\varepsilon\to 0+} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \Delta_x\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1$$

$$-\sum_{i=2}^k \lim_{\varepsilon\to 0+} \varepsilon^{\frac{1}{p_i}}\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \partial_{t_i}\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\varepsilon^{\frac{1}{p_i}}\partial_{t_i}u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_i-2}\varepsilon^{\frac{1}{p_i}}\partial_{t_i}u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1$$

$$= \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_1(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho'_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1 + \int_{\Xi^1} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_1(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\;d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1$$

$$+\sum_{i=2}^k \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \partial_{t_i}\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_i(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1$$

$$+\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \nabla_x\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\cdot\boldsymbol{\varphi}(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1$$

$$+\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \Delta_x\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1.$$

With the help of

$$\lim_{\delta\to+0} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \partial_{t_i}\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_i(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1 = 0,\quad i=2,\ldots,k,$$

$$\lim_{\delta\to+0} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \nabla_x\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\cdot\boldsymbol{\varphi}(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1 = 0,$$

$$\lim_{\delta\to+0} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \Delta_x\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1 = 0$$

and

$$\lim_{\delta\to 0+} \int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_1(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho'_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1 = -\int_{\Xi^1} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)a_1(u_0^{\tau,(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\;d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1,$$

we obtain

$$\lim_{\varepsilon\to 0+} \varepsilon \int_{\Xi^1} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\partial_{t_1}u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1}u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)\;d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1 = \lim_{\delta\to 0+} A_1(\delta)$$

$$= \int_{\Xi^1} \theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\big(a_1(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)) - a_1(u_0^{\tau,(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\big)\;d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1.$$

It is obvious that $(4.1\text{a})_{i=2}^{k}$ and $(4.1\text{b})_{i=1}^{k}$ can be deduced in a similar way. □

With the help of this lemma we can deduce $(3.2\text{b})_{i=1}$ in the following way. Let $\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)$ be a nonnegative test function. We put

$$\phi(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)=\eta'(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)$$

in (2.2b) and modify:

$$\begin{aligned}
&\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}\partial_{t_1}q_{a_1}(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1=\\
&-\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}q_{a_1}(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho'_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&-\int_{\Xi^1}q_{a_1}(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\,d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1=\\
&\sum_{i=2}^{k}\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}q_{a_i}(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\partial_{t_i}\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&+\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}\boldsymbol{q}_\varphi(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\cdot\nabla_x\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&+\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}\Delta_x\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\eta(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&-\varepsilon\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}|\partial_{t_1}u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1}\eta(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho'_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&-\varepsilon\int_{\Xi^1}\eta'(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\partial_{t_1}u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1}u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)\,d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1\\
&-\varepsilon\sum_{i=2}^{k}\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}|\partial_{t_i}u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_i-2}\partial_{t_i}\eta(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\partial_{t_i}\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&-\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}\eta''(u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\big(|\nabla_x u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^2\\
&\qquad+\varepsilon\sum_{i=1}^{k}|\partial_{t_i}u_\varepsilon(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1)|^{p_i}\big)\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1.
\end{aligned}\tag{4.5}$$

Using $(4.1\text{a})_{i=1}$, in the limit, as $\varepsilon\to+0$, we get

$$\begin{aligned}
&-\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}q_{a_1}(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho'_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&-\int_{\Xi^1}q_{a_1}(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\,d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1\le\\
&\sum_{i=2}^{k}\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}q_{a_i}(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\partial_{t_i}\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&+\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}\boldsymbol{q}_\varphi(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\cdot\nabla_x\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
&+\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}\Delta_x\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\eta(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1
\end{aligned}\tag{4.6}$$

$$-\lim_{\varepsilon\to+0}\varepsilon\int_{\Xi^1}\eta'(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)|\partial_{t_1}u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)|^{p_1-2}\partial_{t_1}u_\varepsilon(\boldsymbol{x},0,\widehat{\boldsymbol{t}}_1)\,d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1=\dots$$

We put $\theta_1(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)=\eta'(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)$ in $(4.1\text{a})_{i=1}$.

$$\begin{aligned}\dots=\sum_{i=2}^{k}\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}q_{a_i}(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\partial_{t_i}\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
+\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}\boldsymbol{q}_{\varphi}(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\cdot\nabla_x\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
+\int_{\Omega\times(0,\delta)\times(0,T_2)\times\ldots\times(0,T_k)}\Delta_x\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\eta(u(\boldsymbol{x},t_1,\widehat{\boldsymbol{t}}_1))\rho_\delta(t_1)\,d\boldsymbol{x}dt_1d\widehat{\boldsymbol{t}}_1\\
+\int_{\Xi^1}\eta'(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\omega(\boldsymbol{x},\widehat{\boldsymbol{t}}_1)\big(a_1(u_0^{\tau,(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))-a_1(u_0^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{t}}_1))\big)\,d\boldsymbol{x}d\widehat{\boldsymbol{t}}_1.\end{aligned}$$

Finally, the boundary condition $(3.2\text{b})_{i=1}$ follows from (4.6) as $\delta\to 0+$. The entropy boundary conditions $(3.2\text{b})_{i=2}^k$ and $(3.2\text{c})_{i=1}^k$ can be deduced in a similar way.

Appendix A. Weak traces of $|\partial_{t_i}u_\varepsilon|^{p_i-2}\,\partial_{t_i}u_\varepsilon$ on $\Gamma^i_{\tilde t_i\pm0}$

For $\tilde t_i\in[0,T_i]$ denote

$\Gamma^1_{\tilde t_1\pm0}:=\overline{\Omega}\times\{t_1=\tilde t_1\pm0\}\times[0,T_2]\times\ldots\times[0,T_k]$,

$\vdots$

$\Gamma^j_{\tilde t_j\pm0}:=\overline{\Omega}\times[0,T_1]\times\ldots\times[0,T_{j-1}]\times\{t_j=\tilde t_j\pm0\}\times[0,T_{j+1}]\times\ldots\times[0,T_k]$,

$\vdots$

$\Gamma^k_{\tilde t_k\pm0}:=\overline{\Omega}\times[0,T_1]\times\ldots\times[0,T_{k-1}]\times\{t_k=\tilde t_k\pm0\}$.

In order to estimate weak traces of $|\partial_{t_i}u_\varepsilon|^{p_i-2}\,\partial_{t_i}u_\varepsilon$ on $\Gamma^i_{\tilde t_i\pm0}$ $(i=1,\ldots,k)$ we recall the notion of the Sobolev–Slobodeckii norm for Sobolev spaces of fractional order and make use of the Lions–Magenes Trace Theorem.

Let $\mathcal{O}\subset\mathbb{R}^N_z$ be a bounded Lipschitz domain with a boundary $\partial\mathcal{O}$.

Definition A.1. *[23, Prop. 2.7], [36, Sec. 2.2]. For arbitrary $r\in(1,+\infty)$ and $s\in(0,1)$ by $W^{s,r}(\partial\mathcal{O})$ we denote the Sobolev space of functions $\phi=\phi(\boldsymbol{z})$, $\boldsymbol{z}\in\partial\mathcal{O}$ equipped with the Sobolev–Slobodeckii norm*

$$\|\phi\|_{W^{s,r}(\partial\mathcal{O})}=\left(\int_{\partial\mathcal{O}}|\phi(\boldsymbol{z})|^r d\sigma_z+\int_{\partial\mathcal{O}}\int_{\partial\mathcal{O}}\frac{|\phi(\boldsymbol{z}')-\phi(\boldsymbol{z}'')|^r}{|\boldsymbol{z}'-\boldsymbol{z}''|^{N+rs}}d\sigma_{z'}d\sigma_{z''}\right)^{1/r}. \tag{A.1}$$

Definition A.2. *For $\phi\in\mathcal{D}(\overline{\mathcal{O}})$ the mapping γ_0^{int} defined by the formula*

$$\gamma_0^{int}\phi(\boldsymbol{z}):=\lim_{\substack{\tilde{\boldsymbol{z}}\to\boldsymbol{z}\\ \tilde{\boldsymbol{z}}\in\mathcal{O}\\ \boldsymbol{z}\in\partial\mathcal{O}}}\phi(\boldsymbol{z})\quad\textit{for } \boldsymbol{z}\in\partial\mathcal{O} \tag{A.2}$$

is called the interior boundary trace operator.

Theorem A.1 (The Lions–Magenes Trace Theorem). *[23, Th. 5.1].*

(i) *For any $r \in (1, +\infty)$ the interior boundary trace operator γ_0^{int}, defined by* (A.2) *for $\phi \in \mathcal{D}(\overline{\mathcal{O}})$, admits a continuous extension*

$$\gamma_0^{int} \in \mathcal{L}(W^{1,r}(\mathcal{O}), W^{\frac{1}{r'},r}(\partial\mathcal{O})) \quad (1/r + 1/r' = 1),$$

hereby there is a constant $c_T > 0$ such that

$$\|\gamma_0^{int}\phi\|_{W^{\frac{1}{r'},r}(\partial\mathcal{O})} \le c_T\|\phi\|_{W^{1,r}(\mathcal{O})}, \quad \forall\, \phi \in W^{1,r}(\mathcal{O}). \tag{A.3}$$

(Constant c_T is independent of ϕ.)

(ii) *For any $r \in (1, +\infty)$ the interior boundary trace operator γ_0^{int} has a continuous right inverse operator (called the lift operator) $\mathcal{E} \in \mathcal{L}(W^{\frac{1}{r'},r}(\partial\mathcal{O}), W^{1,r}(\mathcal{O}))$ satisfying $\gamma_0^{int}\mathcal{E}\psi = \psi$ for all $\psi \in W^{\frac{1}{r'},r}(\partial\mathcal{O})$ as well as*

$$\|\mathcal{E}\psi\|_{W^{1,r}(\mathcal{O})} \le c_{IT}\|\psi\|_{W^{\frac{1}{r'},r}(\partial\mathcal{O})}, \quad \forall\, \psi \in W^{\frac{1}{r'},r}(\partial\mathcal{O}). \tag{A.4}$$

Definition A.3. *[23, Sec. 4.3], [36, Sec. 2.2]. For $s \in (0,1)$ and $r \in (1,+\infty)$ the space $W^{-s,r'}(\partial\mathcal{O})$ is defined as the dual space of $W^{s,r}(\partial\mathcal{O})$. Hereby we have $1/r + 1/r' = 1$, and the associated norm is*

$$\|\Lambda\|_{W^{-s,r'}(\partial\mathcal{O})} = \sup_{\substack{\phi\, \in\, W^{s,r}(\partial\mathcal{O})\\ \phi \ne 0}} \frac{|\langle \Lambda, \phi\rangle_{\partial\mathcal{O}}|}{\|\phi\|_{W^{s,r}(\partial\mathcal{O})}}. \tag{A.5}$$

By $\langle\cdot,\cdot\rangle_{\partial\mathcal{O}}$ we denote the duality bracket between $W^{s,r}(\partial\mathcal{O})$ and $W^{-s,r'}(\partial\mathcal{O})$.

Remark A.1. *Consider a cylindrical domain $\mathcal{O} = \mathcal{B}\times(b_1, b_2)$, where $\mathcal{B}$ is a bounded Lipschitz domain in $\mathbb{R}^{N-1}_{z_1,\dots,z_{N-1}}$ and $-\infty < b_1 < b_2 < +\infty$. For the sections $\mathcal{B} \times \{z_N = b_1\} \subset \partial\mathcal{O}$ and $\mathcal{B}\times\{z_N = b_2\} \subset \partial\mathcal{O}$ we can naturally define $W^{s,r}(\mathcal{B}\times\{z_N = b_1\})$, $W^{s,r}(\mathcal{B} \times \{z_N = b_2\})$, $W^{-s,r'}(\mathcal{B} \times \{z_N = b_1\})$, and $W^{-s,r'}(\mathcal{B} \times \{z_N = b_2\})$ with the respective induced norms* (A.1) *and* (A.5).

Furthermore, on the strength of Definition A.2 and Theorem A.1, the restriction of γ_0^{int}: $\mathcal{O} \mapsto \partial\mathcal{O}$ is well-defined for $\mathcal{B}\times\{z_N = b_1\}$ and $\mathcal{B}\times\{z_N = b_2\}$, and one has

$$(\text{restriction of } \gamma_0^{int}) = \big(\phi \mapsto \phi|_{\mathcal{B}\times\{z_N=b_j\}}\big) \in \mathcal{L}(W^{1,r}(\mathcal{O}), W^{\frac{1}{r'},r}(\mathcal{B} \times \{z_N = b_j\})),$$

$$\big\|\phi|_{\mathcal{B}\times\{z_N=b_j\}}\big\|_{W^{\frac{1}{r'},r}(\mathcal{B}\times\{z_N=b_j\})} \le c_T\|\phi\|_{W^{1,r}(\mathcal{O})}, \quad \forall\, \phi \in W^{1,r}(\mathcal{O}), \quad j = 1, 2. \tag{A.6}$$

Proposition A.1. *Denote $p_* = \max\{2, p_1, \dots, p_k\}$. For $i = 1, \dots, k$ and for any fixed $\varepsilon \in (0, 1]$ one has*

$$\big(|\partial_{t_i} u_\varepsilon|^{p_i-2}\, \partial_{t_i} u_\varepsilon\big)\Big|_{\Gamma^i_{\tilde{t}_i+0}} \in W^{-\frac{1}{p'_*},p'_*}(\Gamma^i_{\tilde{t}_i+0}), \quad \tilde{t}_i \in [0, T_i), \tag{A.7}$$

$$\big(|\partial_{t_i} u_\varepsilon|^{p_i-2}\, \partial_{t_i} u_\varepsilon\big)\Big|_{\Gamma^i_{\tilde{t}_i-0}} \in W^{-\frac{1}{p'_*},p'_*}(\Gamma^i_{\tilde{t}_i-0}), \quad \tilde{t}_i \in (0, T_i], \tag{A.8}$$

$$\big(|\partial_{t_i} u_\varepsilon|^{p_i-2}\, \partial_{t_i} u_\varepsilon\big)\Big|_{\Gamma^i_{\tilde{t}_i+0}} = \big(|\partial_{t_i} u_\varepsilon|^{p_i-2}\, \partial_{t_i} u_\varepsilon\big)\Big|_{\Gamma^i_{\tilde{t}_i-0}}, \quad \tilde{t}_i \in (0, T_i), \tag{A.9}$$

$1/p_* + 1/p'_* = 1$.

Remark A.2. *In line with Proposition A.1, we write simply*

$$\Big|\partial_{t_1} u_\varepsilon(\boldsymbol{x}, \tilde{t}_1, \widehat{\boldsymbol{t}}_1)\Big|^{p_1-2}\partial_{t_1} u_\varepsilon(\boldsymbol{x}, \tilde{t}_1, \widehat{\boldsymbol{t}}_1)$$

for $(\boldsymbol{x}, \tilde{t}_1, \widehat{\boldsymbol{t}}_1) \in \Gamma^1_{\tilde{t}_1}$, $\tilde{t}_1 \in [0, T_1]$, *instead of* $\big(|\partial_{t_1} u_\varepsilon|^{p_1-2}\,\partial_{t_1} u_\varepsilon\big)\Big|_{\Gamma^1_{\tilde{t}_1 \pm 0}}$, *everywhere in Section 4.*

Proof of Proposition A.1. **(1)** Using arguments from [2, Ch. 3, Sec. 1.1], on the strength of estimates (2.3) and (2.4) and the notion of generalized derivative, from (2.2a) we deduce that

$$|\partial_{t_i} u_\varepsilon|^{p_i-2}\,\partial_{t_i} u_\varepsilon \quad \text{is weakly differentiable w.r.t. } t_i,$$

$$\partial_{t_i}(|\partial_{t_i} u_\varepsilon|^{p_i-2}\,\partial_{t_i} u_\varepsilon) \in W^{-1,\boldsymbol{P}'}(G_{T_1,\dots,T_k}), \tag{A.10}$$

$$\text{where } \boldsymbol{P}' = (\underbrace{2,\dots,2}_{d \text{ times}}, p_1', \dots, p_k') \text{ is the multi-index, } 1/p_i + 1/p_i' = 1,$$

$$t_i \mapsto |\partial_{t_i} u_\varepsilon|^{p_i-2}\,\partial_{t_i} u_\varepsilon \quad \text{is a weakly absolutely continuous mapping from } [0, T_i] \text{ into } L^{p_i'}(\Xi^i). \tag{A.11}$$

Therefore the weak traces $\big(|\partial_{t_i} u_\varepsilon|^{p_i-2}\,\partial_{t_i} u_\varepsilon\big)\Big|_{\Gamma^i_{\tilde{t}_i \pm 0}}$ and the integrals

$$\int_{\Xi^i} \varepsilon\,\big(|\partial_{t_i} u_\varepsilon|^{p_i-2}\,\partial_{t_i} u_\varepsilon\big)\Big|_{\Gamma^i_{\tilde{t}_i \pm 0}} \phi(\boldsymbol{x}, t_i, \widehat{\boldsymbol{t}}_i)\,d\boldsymbol{x}\,d\widehat{\boldsymbol{t}}_i$$

are well-defined for $\phi \in W^{1,\boldsymbol{P}}(G_{T_1,\dots,T_k})$ $(i = 1, \dots, k)$; and (A.9) holds true.

(2) By means of technique from [2, Ch. 3, Sec. 1.1], thanks to (A.10)and (A.11), we establish that (2.2a) is equivalent to the integral equality

$$\begin{aligned}
&\int_{\Omega\times(t_1',t_1'')\times(t_2',t_2'')\times\dots\times(t_k',t_k'')} \Big(-\boldsymbol{a}(u_\varepsilon)\cdot\nabla_t\phi - \boldsymbol{\varphi}(u_\varepsilon)\cdot\nabla_x\phi \\
&\qquad + \nabla_x u_\varepsilon\cdot\nabla_x\phi + \varepsilon\sum_{i=1}^k |\partial_{t_i} u_\varepsilon|^{p_i-2}\,\partial_{t_i} u_\varepsilon \partial_{t_i}\phi\Big)\,d\boldsymbol{x}\,d\boldsymbol{t} \\
&= \varepsilon\int_{\Omega\times(t_2',t_2'')\times\dots\times(t_k',t_k'')} \Big[\big(|\partial_{t_1} u_\varepsilon|^{p_1-2}\,\partial_{t_1} u_\varepsilon\big)\Big|_{\Gamma^1_{t_1''}} \phi(\boldsymbol{x}, t_1'', \widehat{\boldsymbol{t}}_1) \\
&\qquad - \big(|\partial_{t_1} u_\varepsilon|^{p_1-2}\,\partial_{t_1} u_\varepsilon\big)\Big|_{\Gamma^1_{t_1'}} \phi(\boldsymbol{x}, t_1', \widehat{\boldsymbol{t}}_1)\Big]\,d\boldsymbol{x}\,d\widehat{\boldsymbol{t}}_1 \\
&+\dots+\varepsilon\int_{\Omega\times(t_1',t_1'')\times\dots\times(t_{k-1}',t_{k-1}'')} \Big[\big(|\partial_{t_k} u_\varepsilon|^{p_k-2}\,\partial_{t_k} u_\varepsilon\big)\Big|_{\Gamma^k_{t_k''}} \phi(\boldsymbol{x}, \widehat{\boldsymbol{t}}_k, t_k'') \\
&\qquad - \big(|\partial_{t_k} u_\varepsilon|^{p_k-2}\,\partial_{t_k} u_\varepsilon\big)\Big|_{\Gamma^k_{t_k'}} \phi(\boldsymbol{x}, \widehat{\boldsymbol{t}}_k, t_k')\Big]\,d\boldsymbol{x}\,d\widehat{\boldsymbol{t}}_k \\
&+ \int_{\Omega\times(t_2',t_2'')\times\dots\times(t_k',t_k'')} \big[-a_1(u_\varepsilon)|_{\Gamma^1_{t_1''}}\phi(\boldsymbol{x}, t_1'', \widehat{\boldsymbol{t}}_1) + a_1(u_\varepsilon)|_{\Gamma^1_{t_1'}}\phi(\boldsymbol{x}, t_1', \widehat{\boldsymbol{t}}_1)\big]\,d\boldsymbol{x}\,d\widehat{\boldsymbol{t}}_1 \\
&+\dots+\int_{\Omega\times(t_1',t_1'')\times\dots\times(t_{k-1}',t_{k-1}'')} \big[-a_k(u_\varepsilon)|_{\Gamma^k_{t_k''}}\phi(\boldsymbol{x}, \widehat{\boldsymbol{t}}_k, t_k'') + a_k(u_\varepsilon)|_{\Gamma^k_{t_k'}}\phi(\boldsymbol{x}, \widehat{\boldsymbol{t}}_k, t_k')\big]\,d\boldsymbol{x}\,d\widehat{\boldsymbol{t}}_k,
\end{aligned} \tag{A.12}$$

where $t_i', t_i'' \in [0, T_i]$ are arbitrary such that $t_i' \le t_i''$ $(i = 1, \dots, k)$, and $\phi \in V^{\boldsymbol{P}}(G_{T_1,\dots,T_k})$ is an arbitrary test-function.

(3) In the rest of the proof we confine ourselves to the case $i = 1$ in the formulation of Proposition A.1. The cases $i = 2, 3, \dots, k$ are treated quite similarly.

Fix $\tilde{t}_1 \in (0, T_1]$ arbitrarily. For the sake of brevity denote

$$G_{\tilde{t}_1, T_2, \ldots, T_k} := \Omega \times (0, \tilde{t}_1) \times (0, T_2) \times \ldots \times (0, T_k).$$

Consider $\psi \in W^{\frac{1}{p'_*}, p_*}(\partial G_{\tilde{t}_1, T_2, \ldots, T_k})$ such that

$$\psi \in W_0^{\frac{1}{p'_*}, p_*}(\Gamma^1_{\tilde{t}_1}), \quad \psi \equiv 0 \text{ on } \partial G_{\tilde{t}_1, T_2, \ldots, T_k} \setminus \Gamma^1_{\tilde{t}_1}. \tag{A.13}$$

In (A.12) take $t'_i = 0$, $t''_i = T_i$ $(i = 2, \ldots, k)$, $t'_1 = 0$, $t''_1 = \tilde{t}_1$, and insert $\phi = \mathcal{E}\psi$, where $\mathcal{E} \in \mathcal{L}(W^{\frac{1}{p'_*}, p_*}(\partial G_{\tilde{t}_1, T_2, \ldots, T_k}), W^{1, p_*}(G_{\tilde{t}_1, T_2, \ldots, T_k}))$ is the lift operator defined in Theorem A.1. Since $W^{1, p_*}(G_{\tilde{t}_1, T_2, \ldots, T_k})$ is continuously embedded into $W^{1, \boldsymbol{P}}(G_{\tilde{t}_1, T_2, \ldots, T_k})$, such choice of test-function in (A.12) is legitime. With this choice of ϕ and values t'_i and t''_i and with the help of estimates (2.3), (2.4), and (A.4), from the integral equality (A.12) we derive that

$$\left| \int_{\Xi_1} \left(|\partial_{t_1} u_\varepsilon|^{p_1 - 2} \partial_{t_1} u_\varepsilon \right) \Big|_{\Gamma^1_{\tilde{t}_1}} \psi(\boldsymbol{x}, \tilde{t}_1, \widehat{\boldsymbol{t}}_1) \, d\boldsymbol{x} d\widehat{\boldsymbol{t}}_1 \right| \tag{A.14}$$
$$\leq \frac{1}{\varepsilon} \int_{G_{\tilde{t}_1, T_2, \ldots, T_k}} \left(|\boldsymbol{a}(u_\varepsilon)| \, |\nabla_t(\mathcal{E}\psi)| + |\boldsymbol{\varphi}(u_\varepsilon)| \, |\nabla_x(\mathcal{E}\psi)| + |\nabla_x u_\varepsilon| \, |\nabla_x(\mathcal{E}\psi)| \right) d\boldsymbol{x} d\boldsymbol{t}$$
$$+ \sum_{i=1}^{k} \int_{G_{\tilde{t}_1, T_2, \ldots, T_k}} |\partial_{t_i} u_\varepsilon|^{p_i - 1} \, |\partial_{t_i}(\mathcal{E}\psi)| \, d\boldsymbol{x} d\boldsymbol{t} + \frac{1}{\varepsilon} \int_{\Xi^1} \left| a_1(u_\varepsilon)|_{\Gamma_{\tilde{t}_1}} \right| \left| \psi(\boldsymbol{x}, \tilde{t}_1, \widehat{\boldsymbol{t}}_1) \right| \, d\boldsymbol{x} d\widehat{\boldsymbol{t}}_1$$
$$\leq c_1(\varepsilon) \, \|\mathcal{E}\psi\|_{W^{1, p_*}(G_{\tilde{t}_1, T_2, \ldots, T_k})} + \frac{c_2(a_1)}{\varepsilon} \|\psi\|_{W^{\frac{1}{p'_*}, p_*}(\Gamma^1_{\tilde{t}_1})} \leq c_3(\varepsilon) \|\psi\|_{W^{\frac{1}{p'_*}, p_*}(\Gamma^1_{\tilde{t}_1})},$$

where $c_3(\varepsilon) = c_{IT} c_1(\varepsilon) + \dfrac{c_2(a_1)}{\varepsilon}$, c_{IT} is the constant from (A.4), that is, c_{IT} is the norm of $\mathcal{E}$.

Due to arbitrariness of ψ on $\Gamma^1_{\tilde{t}_1}$, bound (A.14) immediately implies (A.7) and (A.8) for $\tilde{t}_1 \in (0, T_1]$. In order to establish (A.7) for $\tilde{t}_1 = 0$, it suffices to substitute $G_{\tilde{t}_1, T_2, \ldots, T_k}$ by $G_{T_1, \ldots, T_k}$, and the rest of the proof becomes quite analogous to the above one.

Proposition A.1 is proved. □

Conclusion

In the present paper, we have extended the results presented in [19] on the case when temporal artificial diffusion coefficients depend on partial derivatives of u_ε in $t_1, \ldots, t_k$ variables. Namely, we have shown that various choices of temporal artificial diffusion would lead to the same kinetic solution of Problem $\mathbf{\Pi}_0$.

Acknowledgements

Appendix A has been added to the original manuscript thanks to valuable remarks of the Referee Incognito. The authors are very grateful to the Referee Incognito for his attention to the work and many useful remarks aimed towards improvement of the text.

The authors also are very grateful to Professor Stanislav N. Antontsev (CMAF-CIO, Universidade de Lisboa, Portugal) for many fruitful discussions.

References

[1] A.L. Amadori and R. Natalini, *Entropy solutions to a strongly degenerate anisotropic convection–diffusion equation with application to utility theory*, J. Math. Anal. Appl. **284** (2003), 511–531. MR1998649

[2] S.N. Antontsev, A.V. Kazhikhov, and V.N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, Amsterdam: North-Holland Publishing Co., 1990. MR1035212

[3] S.N. Antontsev and I.V. Kuznetsov, *Singular perturbations of forward-backward p-parabolic equations*, JEPE, **2** (2016), 357–370. MR3645952

[4] S.N. Antontsev and I.V. Kuznetsov, *Existence of entropy measure-valued solutions for forward-backward p-parabolic equations*, Sib. Elect. Math. Rep., **14** (2017), 774–793. MR3693744

[5] S. Antontsev and S. Shmarev, *Evolution PDEs with Nonstandard Growth Conditions: Existence, Uniqueness, Localization, Blow-up*, Paris: Atlantis Press, 2015. MR3328376

[6] C. Bardos, A.Y. Le Roux, and J.-C. Nedelec, *First order quasilinear equations with boundary conditions*, Communications in Partial Differential Equations, **4**:9 (1979), 1017–1034. MR0542510

[7] O.B. Bocharov, *On the first boundary value problem for the forward-backward parabolic equation*, Dynamika Sploshnoy Sredy, **37** (1978), 27–39 (in Russian). MR0539019

[8] M. Chipot, S. Guesmia, and A. Sengouga, *Singular perturbations of some nonlinear problems*, J. Math. Sc., **176** (2011), 828–843. MR2838978

[9] M.M. Chipot, *Asymptotic Issues for Some Partial Differential Equations*, Zurich: Imperial College Press, 2016. MR3526204

[10] G.M. Coclite, K.H. Karlsen, and Y.-S. Kwon, *Initial-boundary value problems for conservation laws with source terms and the Degasperis-Procesi equation*, J. Func. Anal., **257** (2009), 3823–3857. MR2557726

[11] C. De Lellis, F. Otto, and M. Westdickenberg, *Structure of entropy solutions for multi-dimensional scalar conservation laws*, Arch. Ration. Mech. Anal., **170** (2003), 137–184. MR2017887

[12] F. Dubois and Ph. LeFloch, *Boundary conditions for nonlinear hyperbolic systems of conservation laws*, J. Differ. Equ., **71** (1988), 93–122. MR0922200

[13] M. Escobedo, J.L. Vázquez, and E. Zuazua, *Entropy solutions for diffusion-convection equations with partial diffusivity*, Trans. Amer. Math. Soc., **343**:2 (1994), 829–842. MR1225573

[14] L.C. Evans, *Partial Differential Equations*, Second ed., Graduate Studies in Mathematics, **19**, AMS, 2010. MR2597943

[15] C. Imbert and J. Vovelle, *Kinetic formulation for multidimensional scalar conservation laws with boundary conditions and applications*, SIAM J. Math. Anal., **36**:1 (2004), 214–232. MR2083859

[16] S.N. Kruzhkov and I.M. Kolodii, *On the theory of embedding of anisotropic Sobolev spaces*, Russ. Math. Surv., **38** (1983), 188–189. MR0695478

[17] I.V. Kuznetsov, *Traces of entropy solutions to second order forward-backward parabolic equations*, J. Math. Sc., **211** (2015), 767–788. MR3424178

[18] I.V. Kuznetsov, *Kinetic formulation of forward-backward parabolic equations*, Sib. Elect. Math. Rep., **13** (2016), 930–949. MR3576020

[19] I.V. Kuznetsov, *Genuinely nonlinear forward-backward ultra-parabolic equations*, Sib. Elect. Math. Rep., **14** (2017), 710–731. MR3693743

[20] Y.-S. Kwon, *Well-posedness for entropy solution of multidimensional scalar conservation laws with strong boundary condition*, J. Math. Anal. Appl., **340** (2008), 543–549. MR2376175

[21] M. Lazar and D. Mitrovic, *Velocity averaging — a general framework*, Dyn. Partial Differ. Equ., **9** (2012), 239–260. MR2975108

[22] J.-L. Lions, *Perturbations Singuliéres dans les Problémes aux Limites et en Contrôl Optimal*, Berlin: Springer, 1973. MR0600331

[23] J.-L. Lions and E. Magenes, *Problemi ai limiti non omogenei (III)*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^{e} série, **15**:1–2 (1961), 41–103. MR0146526

[24] P.-L. Lions, B. Perthame, and E. Tadmor, *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Am. Math. Soc., **7** (1994), 169–191. MR1201239

[25] P. Marcati and R. Natalini, *Convergence of the pseudo-viscosity approximation for conservation laws*, Nonlinear Anal., **23**:5 (1994), 621–628. MR1297281
[26] A. Matas and J. Merker, *The limit of vanishing viscosity for doubly nonlinear parabolic equations*, Electron. J. Qual. Theory Differ. Equ., **8** (2014), 1–14. MR3183606
[27] V.N. Monakhov, *Reciprocal flows in boundary layer*, Dynamika Sploshnoy Sredy, **113** (1998), 107–113 (in Russian). MR1769268
[28] F. Otto, *Initial-boundary value problem for a scalar conservation law*, Comptes Rendus de l'Academie des Sciences. Serie I. Mathematique, **322**:8 (1996), 729–734. MR1387428
[29] E.Yu. Panov, *Existence of strong traces for generalized solutions of multidimensional scalar conservation laws*, J. Hyperbolic Differ. Equ., **2**:4 (2005), 885–908. MR2195985
[30] E.Yu. Panov, *Existence of strong traces for quasi-solutions of multidimensional conservation laws*, J. Hyperbolic Differ. Equ., **4**:4 (2007), 729–770. MR2374223
[31] E.Yu. Panov, *Ultra-parabolic equations with rough coefficients. Entropy solutions and strong precompactness property*, J. Math. Sc., **159**:2 (2009), 180–228. MR2544036
[32] E.Yu. Panov, *Ultra-parabolic H-measures and compensated compactness*, Ann. I. H. Poincaré – AN, **28** (2011), 47–62. MR2765509
[33] E.Yu. Panov, *On the Dirichlet problem for first order quasilinear equations on a manifold*, Trans. Amer. Math. Soc., **363**:5 (2011), 2393–2446. MR2763721
[34] B. Perthame, *Kinetic Formulation of Conservation Laws*, Oxford: Ox. Univ. Press, 2002. MR2064166
[35] R. Salvi, *On the existence of periodic weak solutions of Navier-Stokes equations in regions with periodically moving boundaries*, Acta Appl. Math., **37**:1–2 (1994), 169–179. MR1308755
[36] O. Steinbach, *Numerical Approximation Methods for Elliptic Boundary Value Problems, Finite and Boundary Elements*, Springer, 2003. Zbl 1130.65001
[37] L. Tartar, *The General Theory of Homogenization. A Personalized Introduction*, Berlin: Springer-Verlag, 2009. MR2582099
[38] Al.S. Tersenov, *On the global solvability of the Cauchy problem for a quasilinear ultraparabolic equation*, Asymptot. Anal., **82** (2013), 295–314. MR3086568
[39] A. Vasseur, *Strong traces for solutions of multidimensional scalar conservation laws*, Arch. Ration. Mech. Anal., **160** (2001), 181–193. MR1869441

Ivan V. Kuznetsov
Novosibirsk State University,
Pirogova st., 2,
630090, Novosibirsk, Russia
and
Lavrentyev Institute of Hydrodynamics,
Siberian Division of the Russian Academy of Sciences,
pr. Acad. Lavrentyeva 15,
630090, Novosibirsk, Russia
E-mail address: kuznetsov_i@hydro.nsc.ru

Sergey A. Sazhenkov
Novosibirsk State University,
Pirogova st., 2,
630090, Novosibirsk, Russia
and
Lavrentyev Institute of Hydrodynamics,
Siberian Division of the Russian Academy of Sciences,
pr. Acad. Lavrentyeva 15,
630090, Novosibirsk, Russia
E-mail address: sazhenkovs@yandex.ru