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ALL TIGHT DESCRIPTIONS OF 3-PATHS IN PLANE GRAPHS WITH GIRTH AT LEAST 9

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Abstract. Lebesgue (1940) proved that every plane graph with minimum degree δ at least 3 and girth g at least 5 has a path on three vertices (3-path) of degree 3 each. A description is tight if no its parameter can be strengthened, and no triplet dropped.

Borodin et al. (2013) gave a tight description of 3-paths in plane graphs with $\delta \geq 3$ and $g \geq 3$, and another tight description was given by Borodin, Ivanova and Kostochka in 2017.

Borodin and Ivanova (2015) gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, they proved that this set of tight descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, they characterized (2018) all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$.

Recently, several tight descriptions of 3-paths were obtained for plane graphs with $\delta = 2$ and $g \geq 4$ by Jendrol', Maceková, Montassier, and Soták, four of which descriptions are for $g \geq 9$.

In this paper, we prove ten new tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$ and show that no other tight descriptions exist.

Keywords: plane graph, structure properties, tight description, 3-path, minimum degree, girth.

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1. Introduction

Throughout the paper, G is a plane graph. Let $\delta(G)$ be the minimum vertex degree and let $w_k(G)$ be the minimum degree-sum of a path on k vertices in G. We will drop the argument when G is clear from context. The degree of a vertex v or a face f, that is the number of edges incident with v or f, is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc. An edge uv is an *(i,j)-edge* if $d(u) \leq i$ and $d(v) \leq j$. A path uvw is a *path of type* (i,j,k) or *(i,j,k)-path* if $d(u) \leq i$, $d(v) \leq j$, and $d(w) \leq k$.

Already in 1904, Wernicke [23] proved that every G with $\delta = 5$ has a $(5,6)$-edge, and Franklin [12] strengthened this to the existence of at least two 6^--neighbors of a 5^--vertex; this implies that $w_3 \leq 17$, which bound is sharp.

It follows from Lebesgue's [22] results in 1940 that each G with $\delta \geq 3$ satisfies $w_2 \leq 14$. For 3-connected plane graphs, Kotzig [21] proved a precise result: $w_2 \leq 13$.

In 1972, Erdős (see [13]) conjectured that Kotzig's bound $w_2 \leq 13$ holds for all plane graphs with $\delta \geq 3$. Barnette (see [13]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [3]. More generally, Borodin [4,5] proved that every G with $\delta \geq 3$ contains a $(3,10)$-, or $(4,7)$-, or $(5,6)$-edge, which description is tight.

In 1993, Ando, Iwasaki, Kaneko [2] proved that every 3-connected G satisfies $w_3 \leq 21$, which is sharp due to the Jendrol' construction in [14]. This was refined by Borodin [6] in 1997 as follows: every 3-connected G has: (i) either $w_3 \leq 18$ or a vertex of degree ≤ 15 adjacent to two 3-vertices, and (ii) either $w_3 \leq 17$ or $w_2 \leq 7$. Here, the bounds $w_3 \leq 21$ and $w_3 \leq 17$ were known to be tight long ago, and the sharpness of $w_3 \leq 18$ was recently confirmed by Borodin et al. [10].

Back in 1997, Jendrol' [15] gave an approximate description of 3-paths: every G with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(10,3,10)$, $(7,4,7)$, $(6,5,6)$, $(3,4,15)$, $(3,6,11)$, $(3,8,5)$, $(3,10,3)$, $(4,4,11)$, $(4,5,7)$, or $(4,7,5)$.

A description of 3-paths is *tight* if no its parameter can be strengthened and no term dropped. Borodin et al. [10] gave the first tight description of 3-paths: every G with $\delta \geq 3$ and $g \geq 3$ has a 3-path of one of the following types: $(3,4,11)$, $(3,7,5)$, $(3,10,4)$, $(3,15,3)$, $(4,4,9)$, $(6,4,8)$, $(7,4,7)$, $(6,5,6)$. Another similar tight description for $\delta \geq 3$ and $g \geq 3$ was given by Borodin, Ivanova and Kostochka [11] in 2017.

In 2015, Borodin and Ivanova [7] gave seven tight descriptions of 3-paths when $\delta \geq 3$ and $g \geq 4$. Furthermore, they proved that this set of descriptions is complete, which was a result of a new type in the structural theory of plane graphs. Also, they characterized [9] all one-term tight descriptions if $\delta \geq 3$ and $g \geq 3$. The problem of producing all tight descriptions for $g \geq 3$ remains widely open even for $\delta \geq 3$. Other results on k-paths with $k \geq 3$ and $\delta \geq 3$ can be found in surveys Borodin, Ivanova [8] and Jendrol', Voss [20].

Recently, several tight descriptions of 3-paths were obtained for $\delta = 2$ and $g \geq 4$ by Jendrol', Maceková, Montassier, and Soták [16–19], four of which descriptions are for $g \geq 9$ (for details, see Theorem 1 below). In [1], we proved precise upper bounds for w_3 in several natural classes of plane graphs with $\delta = 2$ and $5 \leq g \leq 7$ and disproved a conjecture by Jendrol' and Maceková [16] concerning the case $g = 5$.

The purpose of our paper is to list all tight descriptions of 3-paths for $\delta = 2$ and $g \geq 9$.

Theorem 1. *There exist precisely these tight descriptions of* 3*-paths in plane graphs with minimum degree* 2 *and girth* g *at least* 9*:*

(A) $g \geq 16$*:* $\{(2,2,2)\}$ *(folklore);*

(B) $11 \leq g \leq 15$*:* $\{(2,2,3)\}$ *(Jendrol' et al. [17]) and* $\{(2,3,2)\}$*;*

(C) $g = 10$*:* $\{(2,2,3),(2,3,2)\}$ *(Jendrol' and Maceková [16], the tightness shown in Jendrol' et al. [17]),* $\{(2,4,2)\}$ *(Jendrol' et al. [17]),* $\{(2,3,3)\}$*,* $\{(2,2,4),(3,2,3)\}$*, and* $\{(3,2,4)\}$*;*

(D) $g = 9$*:* $\{(2,2,5),(2,3,2)\}$ *(Jendrol' et al. [18]),* $\{(2,5,2),(2,2,3)\}$*,* $\{(2,2,5),(3,2,3)\}$*,* $\{(2,5,3)\}$*,* $\{(2,3,5)\}$*, and* $\{(3,2,5)\}$*.*

2. Proving Theorem 1(B)

We first prove that $\{(2,3,2)\}$ is a description, then that $\{(2,3,2)\}$ is tight, and finally that there are no tight descriptions other than $\{(2,2,3)\}$ and $\{(2,3,2)\}$.

2.1. **Proving that** $(2,3,2)$ **is a description.** Let G avoid $(2,3,2)$-paths. Without loss of generality, we can assume that G is connected. Let V, E, and F be the sets of vertices, edges and faces of G, respectively. Euler's formula $|V| - |E| + |F| = 2$ for G may be rewritten as

$$\sum_{x \in V \cup F} (d(x) - 4) = -8. \tag{1}$$

Every vertex and face $x \in V \cup F$ contributes the *charge* $\mu(x) = d(x) - 4$ to (1), so only the charges of 3^--vertices are negative. We define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -8.

We apply the following rules of discharging.

R1. *Every face* f *gives* v*:*
(a) $\frac{3}{4}$ *to each incident* 2*-vertex having a* 2*-neighbor, and*
(b) $\frac{1}{2}$ *to each other incident vertex.*

R2. *Every* 3^+*-vertex gives* $\frac{1}{2}$ *to each* 2*-neighbor.*

We now check $\mu'(x) \geq 0$ whenever $x \in V \cup F$.

Case 1. $v \in V$.

Subcase 1.1. $d(v) = 2$. If v belongs to a $(2,2)$-path, then v receives $2 \times \frac{3}{4}$ from the incident faces by R1 and $\frac{1}{2}$ from the 3^+-neighbor by R2 due to the absence of (2,3,2)-path, so $\mu'(v) = 2 - 4 + 2 \times \frac{3}{4} + \frac{1}{2} = 0$. Otherwise, we have $\mu'(v) = -2 + 4 \times + \frac{1}{2} = 0$.

Subcase 1.2. $d(v) \geq 3$. If $d(v) = 3$ then v gives away $\frac{1}{2}$ by R2 at most once due to the absence of $(2,3,2)$-paths in G, so $\mu'(v) \geq 3 - 4 - \frac{1}{2} + 3 \times \frac{3}{4} = 0$ in view of R1. If $d(v) \geq 4$ then $\mu'(v) \geq d(v) - 4 - d(v) \times \frac{1}{2} + d(v) \times \frac{3}{4} \geq 0$ by R1, R2.

Case 2. $f \in F$. Note that f is incident with at most $\lfloor \frac{d(f)}{3} \rfloor$ $(2,2)$-paths since there are no $(2,2,2)$-paths.

If $d(f) = 11$ then there are at most three $(2, 2)$-paths in the boundary of f, which implies $\mu'(f) \geq 11 - 4 - 6 \times \frac{3}{4} - (11 - 6) \times \frac{1}{2} = 0$ by R1, R2.

If $d(f) \geq 12$ then we similarly have $\mu'(f) \geq d(f) - 4 - \lfloor \frac{d(f)}{3} \rfloor \times (2 \times \frac{3}{4}) - (d(f) - 2 \times \lfloor \frac{d(f)}{3} \rfloor) \times \frac{1}{2} \geq \frac{d(f)}{2} - 4 - \frac{1}{2} \times \lfloor \frac{d(f)}{3} \rfloor \geq \frac{d(f)-12}{3} \geq 0$, as desired.

2.2. **Proving the tightness of** $\{(2, 3, 2)\}$**.** We now construct a graph G_{10+k} with $g = 10 + k$ whenever $1 \leq k \leq 5$ that avoids all "smaller" 3-paths, which are only $(2, 2, 2)$-paths, as follows.

In particular, to obtain a G_{15}, it suffices to put two 2-vertices on each edge of the dodecahedron. In the general case, we first split the edges of the dodecahedron into matchings $M_1, \dots, M_5$ so that each face in incident with representatives of all matchings, which is easy. Then, we put two 2-vertices on all edges from $M_1, \dots, M_k$ and one 2-vertex on each of the other edges, which yields a desired G_{10+k}.

2.3. **Proving that there are no tight descriptions other than** $\{(2, 2, 3)\}$ **and** $\{(2, 3, 2)\}$**.** Suppose $D = \{(x_1, y_1, z_1), \dots, (x_k, y_k, z_k)\}$ is a tight description of 3-paths in plane graphs with $\delta = 2$ and $11 \leq g \leq 15$. This means that

(1) every such graph has a (x_i, y_i, z_i)-path for at least one i with $1 \leq i \leq k$, and

(2) if we delete any term (x_i, y_i, z_i) from D or decrease any parameter in D by one without changing the other $3k - 1$ parameters, then the new description is not satisfied by at least one graph of the given class.

Note that, due to its tightness, the description D cannot have triplets (X, Y, Z) and (X', Y', Z') such that $X \leq X'$, $Y \leq Y'$, and $Z \leq Z'$, for $D' = D \setminus \{(X, Y, Z)\}$ is equivalent to D but shorter. In particular, D has no term $(2, 2, 2)$ due to the graph G_{15} above, so suppose $x_1 y_1 z_1$ has an element 3^+.

If $(x_1, y_1, z_1) = (2^+, 3^+, 2^+)$, then $D = \{(2, 3, 2)\}$ since $\{(2, 3, 2)\}$ is known to be a tight description. If $(x_1, y_1, z_1) = (2^+, 2^+, 3^+)$, then $D = \{(2, 2, 3)\}$ by the same reason.

3. Proving Theorem 1(C)

First note that $\{(2, 3, 3)\}$ was not declared to be a tight description in Jendrol' and Maceková [16] and Jendrol' et al. [17], although this fact follows easily from the tight description $\{(2, 2, 3), (2, 3, 2)\}$ obtained in [16] and whose tightness was proved in [17].

Indeed, $\{(2, 3, 3)\}$ is a description as $\{(2, 2, 3), (2, 3, 2)\}$ is a stronger description. On the other hand, each weakest strengthening of $\{(2, 3, 3)\}$, that is $\{(2, 2, 3)\}$ and $\{(2, 3, 2)\}$, is not anymore a description since $\{(2, 2, 3), (2, 3, 2)\}$ is known to be tight; this means that $\{(2, 3, 3)\}$ is tight.

We next prove that $\{(2, 2, 4), (3, 2, 3)\}$ is a description and then that it is tight. The former fact implies that $\{(3, 2, 4)\}$ is also a tight description, as explained in the last two paragraphs. Finally, we will show that there are no tight descriptions for $g = 10$ other than those five listed in Theorem 1(C).

3.1. **Proving that** $\{(2, 2, 4), (3, 2, 3)\}$ **is a description.** Suppose on the contrary that G has neither $(2, 2, 4)$- nor $(3, 2, 3)$-paths and contract all 2-vertices in G to obtain a graph G^* with $\delta(G^*) \geq 3$. As follows from Lebesgue's Theorem [22], G^* has a face f^* that is either a 3-face incident with a 5^--vertex, or a 4-face of one of the types $(3, 3, 5, \infty)$, $(3, 4, 4, 5)$, or else a $(3, 3, 3, 3, 5)$-face.

The pre-image f of f^* under contraction is a 10^+-face in G by assumption; in particular, f must be incident with at least five 2-vertices.

If $d(f^*) = 3$, then f must actually be incident with at least seven 2-vertices, so we should have a $(2,2,2)$-path in G; a contradiction.

Suppose $d(f^*) = 4$. Note that at most one 2-vertex can be put on each of at least three edges incident in the boundary of f with 4^--vertices (whose number is at least two due to Lebesgue's Theorem [22]) when going back from G^* to G due to the absence of $(2,2,4)$-paths in G. Since the forth edge of f may receive at most two 2-vertices, we have $d(f) \le 4+3+2 < 10$, a contradiction.

Finally, suppose $d(f^*) = 5$, where $f^* = v_1 \dots v_5$ with $d(v_1) = \dots = d(v_4) = 3$. Now at most one 2-vertex may be put on each of the edges v_1v_2 and v_1v_5 and no 2-vertex may appear on the other three edges incident with f due to the absence of $(3,2,3)$- and $(2,2,4)$-paths in G, respectively. Hence $d(f) \le 7$, a contradiction.

3.2. **Proving the tightness of** $\{(2,2,4),(3,2,3)\}$**.** We must show that neither $\{(2,2,4)\}$ nor $\{(3,2,3)\}$ is a description (of plane graphs with $g = 10$).

The former triplet fails to describe a graph H_1 obtained from the dodecahedron by putting a 2-vertex on every edge.

For rejecting the latter, we take the $(3,4,4,4)$ Archimedean solid, which is a plane quadrangulation such that every face is incident with a 3-vertex and three 4-vertices, and put one 2-vertex on every edge incident with a 3-vertex and two 2-vertices on all other edges to obtain a graph H_2 avoiding $(3,2,3)$-paths.

3.3. **Proving the non-existence of tight descriptions other than** $\{(2,2,3),(2,3,2)\}$**,** $\{(2,4,2)\}$**,** $\{(2,3,3)\}$**,** $\{(2,2,4),(3,2,3)\}$**, and** $\{(3,2,4)\}$**.**

Suppose $D = \{(x_1,y_1,z_1),\dots,(x_k,y_k,z_k)\}$ is a tight description of 3-paths in plane graphs with $\delta = 2$ and $g = 10$. By symmetry, we can assume that $x_i \le z_i$ whenever $1 \le i \le k$.

Case 1. $\max\{y_1,z_1,\dots,y_5,z_5\} \ge 4$.

If, say, $y_1 \ge 4$ then $D = \{(2,4,2)\}$ since $\{(2,4,2)\}$ is tight.

Suppose $z_1 \ge 4$. Now $y_1 = 2$ since otherwise there is a stronger tight description $\{(2,3,3)\}$, a contradiction. If $x_1 \ge 3$ then $D = \{(3,2,4)\}$ since $\{(3,2,4)\}$ is tight.

Thus we have $(x_1,y_1,z_1) = (2,2,4^+)$. Note that this term fails to describe the graph H_1 mentioned above, which has no $(2,2)$-paths. It further follows from H_1 that D must have a term, say (x_2,y_2,z_2), which is either $(3^+,2^+,3^+)$ or $(2^+,3^+,2^+)$.

However, if $(x_2,y_2,z_2) = (2^+,3^+,2^+)$ then we have a stronger tight description $\{(2,2,3),(2,3,2)\}$ than D, a contradiction. It remains to assume that $(x_2,y_2,z_2) = (3^+,2^+,3^+)$, in which case $D = \{(2,2,4),(3,2,3)\}$, as desired.

Case 2. Each entry of D is 2 or 3.

To be able to describe the graph H_1, our D must have a term, say (x_1,y_1,z_1), which is either $(3,2^+,3)$ or $(2^+,3,2^+)$. Not to majorize the description $\{(2,3,3)\}$, this term should in fact be either $(3,2,3)$ or $(2,3,2)$.

However, if $(x_1,y_1,z_1) = (2,3,2)$ then it follows from the tight description $\{(2,2,3),(2,3,2)\}$ that there should exist a term $(x_2,y_2,z_2) = (2^+,2^+,3)$ in D, and so $D = \{(2,2,3),(2,3,2)\}$.

Hence we can further assume that $y_1 = \dots = y_k = 2$ and $(x_1,y_1,z_1) = (3,2,3)$. It follows that in fact $D = \{(2,2,3),(3,2,3)\}$, but then D fails to describe 3-paths in the graph H_2 mentioned in Section 3.2, a contradiction.

4. Proving Theorem 1(D)

Note that the existence and tightness of the three one-term descriptions $\{(2,5,3)\}$, $\{(2,3,5)\}$, and $\{(3,2,5)\}$ follow easily from the same properties of corresponding two-term descriptions, in the same fashion as in Section 3. For example, $\{(2,5,3)\}$ follows from $\{(2,5,2),(2,2,3)\}$.

We first prove that $\{(2,5,2),(2,2,3)\}$ and $\{(2,2,5),(3,2,3)\}$ are descriptions and then show them to be tight. Finally, we show that there are no tight descriptions for $g = 9$ other than those six listed in Theorem 1(D).

4.1. **Proving that $\{(2,5,2),(2,2,3)\}$ and $\{(2,2,5),(3,2,3)\}$ are descriptions.** Suppose on the contrary that G does not obey one of these two descriptions. We consider the contracted graph G^* with $\delta(G^*) \geq 3$, as in Section 3, and its 5^--faces f^* implied by Lebesgue's Theorem [22].

If $d(f^*) = 3$, then $\partial(f)$ must have two 2-vertices between each two 3^+-vertices. In particular, this implies both $(2,5,2)$- and $(2,2,5)$-path in $\partial(f)$, a contradiction.

Suppose $d(f^*) = 4$ and $\partial(f)$ has three 5^--vertices v_1, v_2, v_3. It is not hard to see that to avoid $(2,2,5)$- or $(2,5,2)$-paths, our f can have at most two 2-vertices on each of the pairs of edges v_1v_2, v_1v_4 and v_2v_3, v_3v_4. However, then $d(f) \leq 8$, a contradiction.

Finally, suppose $d(f^*) = 5$, where $f^* = v_1 \dots v_5$ with $d(v_1) = \dots d(v_4) = 3$. Now for G not to obey $\{(2,5,2),(2,2,3)\}$ or $\{(2,2,5),(3,2,3)\}$, at most one 2-vertex may be put on each of the pairs of edges v_1v_2, v_1v_5 and v_3v_4, v_4v_5 and at most one 2-vertex on the edge v_2v_3 when going back from f^* to f. This implies $d(f) \leq 5 + 3 \times 1 < 9$, a contradiction.

4.2. **Proving the tightness of $\{(2,5,2),(2,2,3)\}$ and $\{(2,2,5),(3,2,3)\}$.** To see that neither $\{(2,4,2),(2,2,3)\}$ nor $\{(2,2,4),(3,2,3)\}$ is a description, it suffices to put two 2-vertices on every edge of the icosahedron and note that the graph H_3 obtained has no $(4,4,4)$-paths.

To reject $\{(2,5,2),(2,2,2)\}$, we reproduce the graph H_4 obtained in Jendrol' et al. [18]. Take concentric cycles $W_9 = w_1 \dots w_9$, $XY_{18} = x_1y_1 \dots x_9y_9$, $Z_9 = z_1 \dots z_9$, and add a path with two internal 2-vertices between w_i to x_i and also between y_i and z_i whenever $1 \leq i \leq 9$. It remains to observe that $g(H_4) = 9$ and H_4 has no $(2,5,2)$-paths.

Finally, $\{(2,2,5),(2,2,3)\}$ fails to describe the graph H_1 obtained from the dodecahedron by putting a 2-vertex on every edge.

4.3. **Proving the non-existence of tight descriptions other than those six in Theorem 1(D).** Suppose $D = \{(x_1,y_1,z_1),\dots,(x_k,y_k,z_k)\}$ is a tight description of 3-paths in plane graphs with $\delta = 2$ and $g = 9$. By symmetry, we can assume that $x_i \leq z_i$ whenever $1 \leq i \leq k$. It follows from the graph H_3 above that D must have an entry, say y_1 or z_1, not smaller than 5.

Case 1. $y_1 \geq 5$. Now $(x_1,y_1,z_1) = (2,5^+,2)$ since $\{(2,5,3)\}$ is a tight description. It follows from H_4 (which has no $(2,5,2)$-paths) that, say, $(x_2,y_2,z_2) = (2^+,2^+,3^+)$. Since $\{(2,5,2),(2,2,3)\}$ is known to be a tight description, we have $D = \{(2,5,2),(2,2,3)\}$.

Case 2. $z_1 \geq 5$. Now $(x_1,y_1,z_1) = (2,2,5^+)$ since $\{(3,2,5)\}$ and $\{(2,3,5)\}$ are tight descriptions. As we remember, H_1 has only $(3,2,3)$- and $(2,3,2)$-paths. This implies that either $(x_2,y_2,z_2) = (3^+,2^+,3^+)$ or $(x_2,y_2,z_2) = (3^+,2^+,3^+)$.

In the first case, we have $D = \{(2,2,5),(3,2,3)\}$ since $\{(2,2,5),(3,2,3)\}$ is a tight description. In the second case, we similarly have $D = \{(2,2,5),(2,3,2)\}$, as desired.

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