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ON RESOLVABILITY OF LINDELÖF GENERATED SPACES

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Abstract. In this paper we study the properties of $\mathscr{P}$ generated spaces (by analogy with compactly generated). We prove that a regular Lindelöf generated space with uncountable dispersion character is resolvable. It is proved that Hausdorff hereditarily L-spaces are L-tight spaces which were defined by István Juhász, Jan van Mill in (*Variations on countable tightness*, arXiv:1702.03714v1). We also prove ω-resolvability of regular L-tight space with uncountable dispersion character.

Keywords: resolvable space, k-space, tightness, ω-resolvable space, Lindelöf generated space, $\mathscr{P}$ generated space, $\mathscr{P}$-tightness.

1. Introduction

A Hausdorff space is said to be a k-space (also called compactly generated) if it has the final topology with respect to all inclusions $K \mapsto X$ of compact subspaces K of X, so that a set A in X is closed in X if and only if $A \cap K$ is closed in K for all compact subspaces K of X. A space is Fréchet-Uryson if, whenever a point x is in the closure of a subset A, there is a sequence from A converging to x; it is proved in [1] that a space is hereditarily k, i.e., every subspace is a k-space, if and only if it is Fréchet-Uryson. For examples, any locally compact, or the first countably Hausdorff spaces are k-spaces.

An interesting common generalizations of k-spaces and the notion of tightness was given by A.V. Arhangel'skii and D.N. Stavrova in [2]. They defined the k, k_1,

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k^*-tightness as follows: the k-tightness of X does not exceed τ ($t_k(X) \leq \tau$) if and only if for every $A \subseteq X$ that is not closed there exists a $\tau-$compact $B \subseteq X$ for which $A \cap B$ is not closed in X (a set B is called $\tau-$compact if $B = \bigcup\{B_\alpha : \alpha \in \tau\}$, where B_α is a compact subset of X for all $\alpha \in \tau$); the k_1-tightness of X does not exceed τ ($t_{k_1}(X) \leq \tau$) if and only if for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a $\tau-$compact $B \subseteq X$ such that $x \in \overline{A \cap B}$; the k^*-tightness of X does not exceed τ ($t^*_k(X) \leq \tau$) if and only if for every $A \subseteq X$ and every $x \in \overline{A}$ there exists a $\tau-$compact $B \subseteq A$ such that is $x \in \overline{B}$.

In [4] István Juhász and Jan Van Mill defined and studied nine attractive natural tightness conditions for topological spaces. They called a space $\mathscr{P}-$tight, if for all $x \in X$ and $A \subseteq X$ such that $x \in \overline{A}$, there exists $B \subseteq A$ such that $x \in \overline{B}$ and B has the property $\mathscr{P}$.

István Juhász and Jan Van Mill considered in [4] the following properties $\mathscr{P}$ that a subspace of a topological space might have :

ωD	Countable discrete;
ωN	Countable and nowhere dense;
C_2	Second-countable;
ω	Countable;
hL	Hereditarily Lindelöf;
σ-cmp	$\sigma-$compact;
ccc	The countable chain condition;
L	Lindelöf;
wL	Weakly Lindelöf.

Inspired by the researches above, in the first part of this paper we introduce and study $\mathscr{P}$ generated spaces (by analogy with k-spaces). Since the space with property $\mathscr{P}$ is not necessarily closed, there are two ways to determine the $\mathscr{P}$ generated spaces.

Definition 1. *A topological space X is $\mathscr{P}-$space ($\mathscr{P}$ generated space) if a subspace A is closed in X if and only if $A \cap P$ is closed in P for any subspace $P \subseteq X$ which has the property $\mathscr{P}$.*

Definition 2. *A topological space X is $\mathscr{P}c-$space if a set A is closed in X if and only if $A \cap P$ is closed in P (or, is the same one, in X) for any closed subspace $P \subseteq X$ which has the property $\mathscr{P}$.*

In this paper we will consider the property $\mathscr{P} \in \wp := \{\omega\text{N}, C_2, \omega, hL, \sigma\text{-}cmp, ccc, L, wL\}$ because each ωD-space is discrete. Note that every space with countable tightness (i.e. ω-tight) is $\omega-$space [11].

The class of L-spaces generalizes the class of spaces with countable tightness and the class of k-spaces. Below (Theorem 1) we get the example of the space with countable tightness, but is not Lc-space.

Note that the relationships between the spaces of $\mathscr{P}-$tight where the property $\mathscr{P} \in \wp \cup \{\omega\text{D}\}$ were considered in [4].

We summarize the relationships between $\mathscr{P}-$spaces ($\mathscr{P}-$s) and $\mathscr{P}-$tight ($\mathscr{P}-$t) where the property $\mathscr{P} \in \wp \setminus \{\omega\text{N}, C_2\}$ in next diagram.

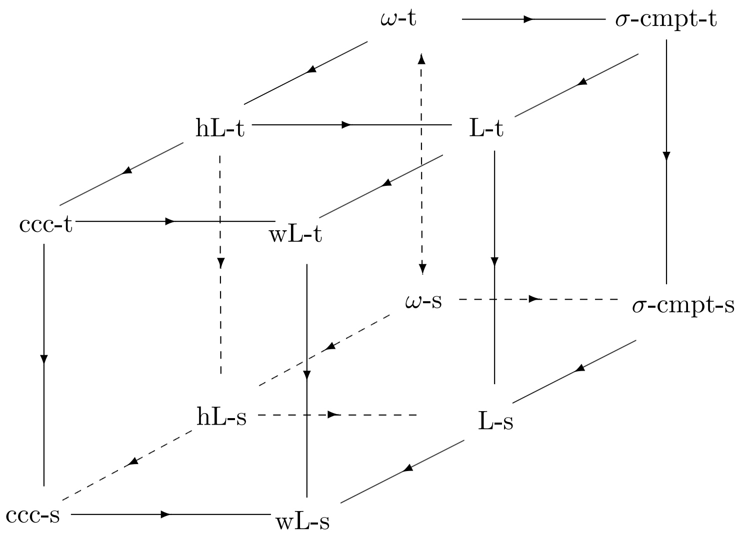

Fig. 1. The Diagram of the relationships between
$\mathscr{P}-$spaces and $\mathscr{P}-$tight.

In 1943 E. Hewitt [5] called a topological space τ-resolvable, if it can be represented as a union of τ dense disjoint subsets. A 2-resolvable space is called resolvable, and irresolvable space is one which is not resolvable. He also defined the dispersion character $\Delta(X)$ of a space X as the smallest size of a non-empty open subset of X. A topological space X is called maximally resolvable if it is $\Delta(X)-$resolvable.

The ω-resolvability of Lindelöf spaces whose dispersion character is uncountable was proved by I. Juhasz, L. Soukup, Z. Szentmiklossy in [6].

E. Hewitt in [5] constructed an example of countable irresolvale normal space, so condition $\Delta(X) > \omega$ is natural. V.I. Malykhin constructed an example of irresolvale Hausdorf Lindelöf space with uncountable dispersion character in [7], therefore, resolvability of regular Lindelöf spaces was studied.

The resolvability of locally compact spaces was proved by Hewitt in 1943 [5]. The resolvability (maximal resolvability) of k-spaces was proved by N.V. Velichko in 1976 [8] (E.G. Pytkeev in 1983 [9]). In connection with these results, the question of the resolvability of L-spaces is natural.

Second part of this paper is devoted to resolvability of regular L-spaces of uncountable dispersion character. We also prove ω-resolvability of regular hereditarily L-spaces of uncountable dispersion character.

Throughout this paper the symbol ω denotes the smallest infinite cardinal, ω_1 stands for the smallest uncountable cardinal. For a subset A of a topological space X, the closure and interior set of A are respectively denotes by $\overline{A}$ (or $[A]$) and $Int(A)$. We assume that all spaces are Hausdorff. Notation and terminology are taken from [10].

2. $\mathscr{P}$ generated space

By Definitions 1 and 2, a $\mathscr{P}c-$space is a $\mathscr{P}-$space. The following example shows that the converse is not true.

Theorem 1. *There exists a space X such that X is a $\omega-$space and $t_k(X) = t_{k_1}(X) = t_k^*(X)) = t(X) = \omega$.*

Proof. Let $X = \beta\mathbb{N}$, $M = \beta\mathbb{N} \setminus \mathbb{N}$. In a $x \in X$ we put the base of neighborhoods $\mathscr{B}(x) = \{x\}$, if $x \in \mathbb{N}$ and $\mathscr{B}(x) = \{(U(x) \setminus M) \cup \{x\}\}$ where $U(x)$ is open in $\beta\mathbb{N}$, if $x \in M$.

Now we show that X is a $\omega-$space. Consider the set $A \subset X$ for which $A \cap P$ is closed in P for any countable subspace $P \subseteq X$. Let $x \in \overline{A}$. The set $P_1 = (A \cap \mathbb{N}) \cup \{x\}$ is a countable. Then $A \cap P_1$ is closed in P_1 therefore $x \in A$.

Let us note that if B is infinity subset of $\mathbb{N}$ then $\overline{B}$ contains uncountable discrete space. Consequently $\overline{B}$ is not a countable space. Therefore, if $B \subset X$ is closed countable space, then the intersection $\mathbb{N} \cap B$ is finite, and, consequently, closed in X. But $\mathbb{N}$ is not closed in X, hence, X is not $\omega c-$space.

Being the $\omega-$space, X is hL-space, σ-*cmp*-space, *ccc*-space, L-space and wL-space. It is clear that X is not $hLc-$space, $\sigma-$cmp$c-$space, $cccc-$space, $Lc-$space, $wLc-$space.

The equalities $t_k(X) = t_{k_1}(X) = t_k^*(X)) = t(X) = \omega$ are obvious. Being $\omega-$tight a space X is wL-tight too. □

Corollary 1. *There exists a space X such that X is a $\omega-$space, but is not $\omega c-$space.*

It is clear that any $k-$space is $Lc-$space (hence, L-space), and every space with countable tightness is hL-space (therefore $\sigma-$cmp-space, *ccc*-space, L-space and wL-space). The converse statements are not true. Theorem 1 shows that even $wLc-$spaces do not generalize a notion of countable tightness. Thus, a natural generalization of spaces with countable tightness and k-spaces is $\mathscr{P}-$spaces.

Theorem 2. *There exists a space X such that X is a L-space, but is not hL-tight.*

Proof. For example, let D be the infinite discrete space of cardinality continuum $\mathfrak{c}$, and $X = \beta D$ be the Čech-Stone compactification of D. Being the compact space, X is a L-space. Consider $p \in X \setminus \{\bigcup \overline{A} : A \in [D]^{\leq\omega}\}$. Since the space D has not an uncountable Lindelöf subspaces, there is no Lindelöf subspace $B \subset D$ such that $p \in \overline{B}$.

□

In what follows we shall concentrate on the study of some properties of $\mathscr{P}-$spaces. Most of all we are interested in the resolvability of L-space. The following statements will be useful for these purpose.

Theorem 3. *Let the property $\mathscr{P} \in \wp \setminus \{C_2\}$. Then the following properties of a space X are equivalent.*

(i) X is a $\mathscr{P}-$space;

(ii) a set $A \subset X$ is non-closed in X if and only if there exists $P \subseteq X$ which have a property $\mathscr{P}$ such that $P \cap A$ is non-closed in X.

Proof. $(i) \to (ii)$. Let X is a $\mathscr{P}-$space, $A \subset X$ is non-closed in X. Then there exists $P \subseteq X$ which have a property $\mathscr{P}$ such that $P \cap A$ is non-closed in P, therefore $P \cap A$ is non-closed in X. Let for a set A there exists $P \subseteq X$ which have a property $\mathscr{P}$ such that $P \cap A$ is non-closed in X. Consider $x \in \overline{P \cap A} \setminus (P \cap A) \subseteq \overline{P} \cap \overline{A} \setminus (P \cap A)$. If $x \in A$ then $x \in \overline{P} \setminus P$, i.e. $P \cap A$ is non-closed in P, consequently a set A is non-closed in X. If $x \in P$ then $x \in \overline{A} \setminus A$, i.e. a set A is non-closed in X.

$(ii) \to (i)$. Consider $A \subset X$ such that $A \cap P$ is closed in P for any subspace $P \subseteq X$ which has the property $\mathscr{P}$. Suppose that $A \neq \overline{A}$. Then exists $P \subseteq X$ which have a property $\mathscr{P}$ such that $P \cap A$ is non-closed in X. Consider $x \in \overline{P \cap A} \setminus (P \cap A)$. Let $P_1 = P \cup \{x\}$. Then P_1 has the property $\mathscr{P}$ and $P_1 \cap A$ is not closed in P_1, contradiction. □

Corollary 2. *A space X is $\mathscr{P}-$space if and only if for any A non-closed in X there are $x \in \overline{A} \setminus A$ and $P \subseteq X$ with a property $\mathscr{P}$ such that $x \in \overline{P \cap A}$.*

Corollary 3. *For any non-isolated point x in $\mathscr{P}-$space X there is a subspace $P \subseteq X$ with a property $\mathscr{P}$ such that $x \in \overline{P \setminus \{x\}}$.*

We give the following definition.

Definition 3. *The property $\mathscr{P}$ is ICS (independent of the containing subspace) in X if $P \subseteq X$ has a property $\mathscr{P}$ in X if and only if P has a property $\mathscr{P}$ in Y for any subset Y such that $P \subseteq Y$.*

The next theorem shows that hereditarily $\mathscr{P}-$space is $\mathscr{P}-$tight if property $\mathscr{P}$ is ICS in X.

Theorem 4. *Let property $\mathscr{P}$ be ICS in X. A space X is hereditarily $\mathscr{P}-$space if and only if for any $A \subset X$ and $x \in \overline{A} \setminus A$ there exists $P \subseteq A$ which have a property $\mathscr{P}$ such that $x \in \overline{P}$.*

Proof. Let X be a hereditarily $\mathscr{P}-$space, $A \subset X$ and $x \in \overline{A} \setminus A$. The subspace $B = A \cup \{x\}$ is $\mathscr{P}-$space, x is non-isolated point in B. Then there is $P \subseteq A$ with a property $\mathscr{P}$ such that $x \in \overline{P}$.

Converse, let $B \subset X$, $A \subseteq B$ and $A \cap P$ is closed in B for any $P \subseteq B$ which have a property $\mathscr{P}$. If $x \in \overline{A} \setminus A$ then there exists $P \subseteq A$ which have a property $\mathscr{P}$ such that $x \in \overline{P}$, i.e. $A \cap P$ is non-closed in B. □

Corollary 4. *Let property $\mathscr{P}$ be ICS in X. A space X is $\mathscr{P}-$tight if and only if each subspace Y of X is $\mathscr{P}-$tight.*

Note that ωN-tight is not hereditarily property. The real numbers $\mathbb{R}$ is ωN-tight, but it subspace $Y = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is not ωN-tight.

3. On resolvability of L-spaces

The maximal resolvability of $\omega-$tight spaces whose dispersion character is uncountable was proved by E.G. Pytkeev in [9]. Therefore $\omega-$spaces with uncountable dispersion character are maximally resolvable. A. Bella and V.I. Malykhin was proved that ωN-tight spaces are maximally resolvable [12].

E. Hewitt in [5] constructed an example of countable irresolvable normal space. Such space is obviously $\omega-$tight (hL-, σ-*cmp*-, *ccc*-, L-, wL-tight) space with countable dispersion character. Therefore, in studying the resolvability of these spaces, it is necessarily to consider the spaces with uncountable dispersion character.

In this section we prove $\omega-$resolvability of regular L-tight spaces which dispersion character is uncountable. Consequently, hL-, σ-*cmp*-, *ccc*-, wL-tight spaces are also $\omega-$resolvable (if X is regular space then properties *ccc*-tight, L-tight, wL-tight are equivalent).

The $\omega-$resolvability of ωD-spaces was proved by P.L. Sharma and S. Sharma in [13]. V.I. Malykhin constructed an example of irresolvable Hausdorf Lindelöf space with uncountable dispersion character [7].

It is clear that any Lindelöf space is L-space. Therefore, the resolvability of L-spaces must be investigated in the class of regular spaces.

We will use a following

Theorem 5 (Hewitt's criterion of resolvability [12])**.** *A topological space X is resolvable (τ-resolvable) if and only if for any nonempty open subset U of X there exist nonempty resolvable (τ-resolvable) subspace of U.*

A.G. El'kin proved the following elegant result

Theorem 6 (Theorem 1 in [3])**.** *Let X be a collectionwise Hausdorff σ-discrete normal space that satisfies the following condition:*

(*) *For each point $x \in X$ there exists a discrete set $D \subset X$ such that $x \in \overline{D} \setminus D$. Then the space X is ω-resolvable.*

Recall that a set $D \subseteq X$ is strongly discrete if there exists a pairwise disjoint family $\mathfrak{U}_D = \{U_x : x \in D, U_x$ is an open neighborhood of $x\}$ (i.e. $U_x \cap U_y = \emptyset$ for $x \neq y$, $x \in D$, $y \in D$).

A point x of a space X is called lsd-point (*a limit point of a strongly discrete subspace*) if there exists a strongly discrete subspace $D \subset X$ such that $x \in \overline{D} \setminus D$ [7].

By using idea of the proof of El'kin's Theorem 6, P.L. Sharma and S. Sharma proved the following result [13].

Theorem 7 (P.L. Sharma, S. Sharma)**.** *Let X be a T_1-space. If each $x \in X$ is a limit point of a strongly discrete subspace of X then X is a $\omega-$resolvable.*

Definition 4. *A discrete set D of cardinality ω_1 is a correct discrete, if any $Y \subseteq D$ such that $|Y| = \omega_1$ has a ω_1- accumulation point.*

The following lemma is a generalized version of the lemma 2.1 in [15].

Lemma 1. *Let X be a regular space such that for each $x \in X$ there exists a correct discrete set D_x of cardinality ω_1 such that x is a ω_1-accumulation point of D_x. Then X is resolvable.*

Proof. By Theorem 5, it suffices to prove that each non-empty open set V of X contains dense-in-itself Y such that $\overline{Y}$ is resolvable.

Denote by $P(D)$ the set of ω_1-accumulation points of a correct discrete set D. Let us note that $\overline{P(D)} = P(D)$ and for any open set U such that $P(D) \subset U$, the set $D \setminus U$ is at most countable.

Let $y \in V$, $D_y \subset V$ is a correct discrete set of cardinality ω_1, such that $y \in P(D_y)$. Let $Y_1 = D_y$, $Y_n = \bigcup\{D_y : y \in Y_{n-1}$, where D_y is correct discrete set, y is ω_1-accumulation of $D_y\}$, we put $Y = \bigcup\limits_{n=1}^{\infty} Y_n$. It is clear that $|Y| = \omega_1$ and any $y \in Y$ is a ω_1-accumulation point of correct (in $\overline{Y}$) discrete subset D_y of Y.

Now we prove that $\overline{Y}$ is resolvable.

First, we renumber the space $Y = \{y_\alpha : \alpha < \omega_1\}$.

For $\alpha < \omega_1$ we construct sets $A^i_\alpha \subset \overline{Y}$, $A_i = \bigcup\limits_{\alpha<\omega_1} A^i_\alpha$ where $i = 1, 2$, such that $A_1 \cap A_2 = \emptyset$ and $\overline{A_1} = \overline{A_2} \supset Y$.

Consider y_1. Let D_1 be a correct (in X) discrete space of cardinality ω_1, such that $y_1 \in \overline{P(D_1)}$. Let $A^1_1 = D_1$, $A^2_1 = P(D_1)$, $P_1 = P(D_1)$. Obviously, $A^1_1 \cap A^2_1 = \emptyset$ and $\overline{A^1_1} \cap \overline{A^2_1} \supset P_1$.

Suppose that for each $\alpha < \beta < \omega_1$ we construct sets $P_\alpha \subset \overline{Y}$, $D_\alpha \subset Y$, A^1_α, A^2_α with the following properties.

1. If $y_\alpha \in \left[\bigcup\limits_{\eta<\alpha} P_\eta\right]$, then $P_\alpha = D_\alpha = \varnothing$, $A^i_\alpha = \bigcup\limits_{\eta<\alpha} A^i_\eta$, $i = 1, 2$.

2. If $y_\alpha \notin \left[\bigcup\limits_{\eta<\alpha} P_\eta\right]$, but $y_\alpha \in \left[\bigcup\limits_{\eta<\alpha} A^1_\eta\right] \bigcap \left[\bigcup\limits_{\eta<\alpha} A^2_\eta\right]$, then $P_\alpha = \{y_\alpha\}$, $D_\alpha = \varnothing$, $A^1_\alpha = \bigcup\limits_{\eta<\alpha} A^1_\eta$, $A^2_\alpha = \bigcup\limits_{\eta<\alpha} A^2_\eta$.

3. If $y_\alpha \notin \left[\bigcup\limits_{\eta<\alpha} P_\eta\right]$ and $y_\alpha \notin \left[\bigcup\limits_{\eta<\alpha} A^1_\eta\right] \bigcap \left[\bigcup\limits_{\eta<\alpha} A^2_\eta\right]$, then D_α is a correct discrete set of cardinality ω_1, such that $y_\alpha \in P(D_\alpha)$, $(\bigcup\limits_{\eta<\alpha} D_\eta) \bigcap D_\alpha = \emptyset$, $P_\alpha = P(D_\alpha)$. In this case $A^1_\alpha = \bigcup\limits_{\eta<\alpha} A^1_\eta \bigcup P_\alpha$, $A^2_\alpha = \bigcup\limits_{\eta<\alpha} A^2_\eta \bigcup D_\alpha$.

4. $A^1_\alpha \cap A^2_\alpha = \varnothing$; A^i_α form a monotonically nondecreasing sequence (by α), $\left[\bigcup\limits_{\eta\leq\alpha} P_\eta\right] \subset \left[A^i_\alpha\right]$, and $A^i_\alpha \subset \bigcup\limits_{\eta\leq\alpha} (P_\eta \cup D_\eta)$, $i = 1, 2$.

Consider y_β. If $y_\beta \in \left[\bigcup\limits_{\alpha<\beta} P_\alpha\right]$, we suppose $P_\beta = D_\beta = \varnothing$, $A^i_\beta = \bigcup\limits_{\alpha<\beta} A^i_\alpha$, $i = 1, 2$. It is easy to see $\left[\bigcup\limits_{\alpha\leq\beta} P_\alpha\right] \subset \left[A^i_\beta\right]$, $i = 1, 2$ and $A^1_\beta \cap A^2_\beta = \varnothing$.

Indeed, by construction, $\bigcup\limits_{\alpha\leq\beta} [P_\alpha] \subset \left[A^i_\beta\right]$, $i = 1, 2$, then $\left[\bigcup\limits_{\alpha\leq\beta} P_\alpha\right] \subset \left[\bigcup\limits_{\alpha\leq\beta} [P_\alpha]\right] \subset \left[A^i_\beta\right]$.

Let $y_\beta \notin \left[\bigcup\limits_{\alpha<\beta} P_\alpha\right]$, but $y_\beta \in \left[\bigcup\limits_{\alpha<\beta} A^1_\alpha\right] \bigcap \left[\bigcup\limits_{\alpha<\beta} A^2_\alpha\right]$, then $P_\beta = \{y_\beta\}$, $D_\beta = \varnothing$, $A^i_\beta = \bigcup\limits_{\alpha<\beta} A^i_\alpha$, $i = 1, 2$.

It is easy to see that A^i_β satisfy item **4**.

Finally the last case $y_\beta \notin \left[\bigcup\limits_{\alpha<\beta} P_\alpha\right]$, and $y_\beta \notin \left[\bigcup\limits_{\alpha<\beta} A^1_\alpha\right] \bigcap \left[\bigcup\limits_{\alpha<\beta} A^2_\alpha\right]$.

Suppose, for definiteness, $y_\beta \notin \left[\bigcup\limits_{\alpha<\beta} A^1_\alpha\right]$. Consider a neighborhood U_β of y_β, such that the closure of U_β is disjoint with $\left[\bigcup\limits_{\alpha<\beta} A^1_\alpha\right]$ and $\left[\bigcup\limits_{\alpha<\beta} P_\alpha\right]$.

In this neighborhood there is at most a countable of points of the set $\bigcup\limits_{\alpha<\beta} A^2_\alpha$ because it intersect with U_β can only with D_α, $\alpha<\beta$ (by **4**).

For each α such that $D_\alpha \neq \varnothing$ and D_α is a discrete, $P(D_\alpha) \subset \overline{Y} \setminus [U_\beta]$, hence, $D_\alpha \bigcap U_\beta$ is at most a countable. It follows that $\left(\bigcup\limits_{\alpha<\beta} D_\alpha\right) \bigcap U_\beta$ has a cardinality $< \omega_1$ and, by **4**, $\left(\bigcup\limits_{\alpha<\beta} A^2_\alpha\right) \bigcap U_\beta$ is countable. Let D_β be a correct discrete set such that $D_\beta \subset U_\beta$, $y_\beta \in P(D_\beta)$ and $D_\beta \cap \left(\bigcup\limits_{\alpha<\beta} A^2_\alpha\right) = \emptyset$.

Let $P_\beta = P(D_\beta)$, $A^1_\beta = D_\beta \bigcup \left(\bigcup\limits_{\alpha<\beta} A^1_\alpha\right)$, $A^2_\beta = P_\beta \bigcup \left(\bigcup\limits_{\alpha<\beta} A^2_\alpha\right)$.

By construction, A^1_β, A^2_β are disjoint sets and has the property **4**.

We prove that $Y \subset \left[\bigcup\limits_{\alpha<\omega_1} P_\alpha\right]$. Suppose that $Y \setminus \left[\bigcup\limits_{\alpha<\omega_1} P_\alpha\right] \neq \varnothing$, consider $y_\gamma \in Y \setminus \left[\bigcup\limits_{\alpha<\omega_1} P_\alpha\right]$. Then $P_\gamma = \varnothing$, because if $P_\gamma \neq \varnothing$, then $y_\gamma \in P_\gamma$. By **1, 2, 3**, we have $y_\gamma \in \left[\bigcup\limits_{\alpha<\gamma} P_\alpha\right] \subset \left[\bigcup\limits_{\alpha<\omega_1} P_\alpha\right]$, a contradiction.

Let $A_i = \bigcup\limits_{\alpha<\omega_1} A^i_\alpha$, $i=1,2$. By construction, $A_1 \cap A_2 = \varnothing$ and $[A_1] = [A_2] \supset Y$. So we have that $\overline{Y}$ is resolvable.

□

Corollary 5. *Let X be a regular space, $Y = \{x \in X : \exists$ countable discrete set D_x such that $x \in \overline{D_x} \setminus D_x$ or $\exists$ correct discrete D_x of cardinality ω_1 such that $x \in \overline{D_x} \setminus D_x\}$ and $\overline{Y} = X$. Then X is resolvable.*

Proof. Let $A = \{x \in X : \exists$ countable discrete set D_x such that $x \in \overline{D_x} \setminus D_x\}$ and $B = Y \setminus A$. By Theorem 7, the set $Int(A)$ is resolvable. Hence, there are disjoint sets A_1, $A_2 \subset Int(A)$ such that $\overline{A_i} \supset Int(A)$ for $i=1,2$. If $Int(A) = \emptyset$ then we put $A_i = \emptyset$ for $i = 1,2$.

Now we prove that if $Int(B) \neq \emptyset$ then $Int(B)$ is resolvable. Let W be an non-empty open set such that $\overline{W} \subset Int(B)$. Note that there is a set $Y \subset W$ such that $|Y| = \omega_1$ and any $y \in Y$ is a ω_1-accumulation point of correct (in $\overline{Y}$ and also in $\overline{W}$) discrete subset D_y of $\overline{Y}$. Then, by Lemma 1, $Int(B)$ is resolvable.

Hence, there are disjoint sets B_1, $B_2 \subset Int(B)$ such that $\overline{B_i} \supset Int(B)$ for $i=1,2$. If $Int(B) = \emptyset$ then we put $B_i = \emptyset$ for $i=1,2$.

Let $Z = X \setminus (\overline{Int(A)} \cup \overline{Int(B)})$, $A_z = A \cap Z$, $B_z = B \cap Z$. Consider $C = Int(A_z \cup B_z)$. Let $C_1 = A \cap C$, $C_2 = B \cap C$ for $C \neq \emptyset$. Note that $A \cap B = \emptyset$, hence, $C_1 \cap C_2 = \emptyset$, $Int(C_1) = Int(C_2) = \emptyset$ and $C = C_1 \cup C_2$. It follows that $C \subset \overline{C_i}$ for $i=1,2$. If $C = \emptyset$ then we assume $C_1 = C_2 = \emptyset$.

Let $D = X \setminus (\overline{Int(A)} \cup \overline{Int(B)} \cup \overline{C})$. If $D = \emptyset$ we assume $D_1 = (A \cup B) \cap D$ and $D_2 = (X \setminus (A \cup B)) \cap D$. Note that D_1 and D_2 are disjoint sets and are dense in D. If $D = \emptyset$ then we assume $D_1 = D_2 = \emptyset$. Note that $X = \overline{Int(A)} \cup \overline{Int(B)} \cup \overline{C} \cup \overline{D}$.

Finally, let $X_i = A_i \cup B_i \cup C_i \cup D_i$ for $i = 1, 2$. It is clear $X = \overline{X_i}$ for $i = 1, 2$ and $X_1 \cap X_2 = \emptyset$.

□

The following theorem is the main result of this work.

Theorem 8. *The regular L-space with uncountable dispersion character is resolvable.*

Proof. Let U be an open subset of X such that each Lindelöf subspace of U at most countable. Note that if $U \neq \emptyset$ then U is maximally resolvable [9] because it is a ω-tight and has uncountable dispersion character.

Consider a set $Z = \bigcup\{U$, where U is an open subset of X such that each Lindelöf subspace of U at most countable $\}$. Then the set Z is resolvable (as the union of resolvable subspaces [14]) or $Z = \emptyset$.

Let $Y = X \setminus [Z]$. By definition of Y and Corollary 3, any open subset of Y contains uncontable Lindelöf subspace L. If L is a heriditaraly Lindelöf then it contains subspace M of uncountable dispersion character. Then M is resolvable (see [6]). Let $H = \bigcup\{M$, where M is a heriditaraly Lindelöf of uncountable dispersion character $\}$. The set H is resolvable (as the union of resolvable subspaces [14]) or $H = \emptyset$.

Let $K = Y \setminus \overline{H}$. It remains to prove that K is resolvable. Let us note that if a subspace $L \subset K$ is not heriditaraly Lindelöf, then it contains a closed set F which is not G_δ-set. Let $K \neq \emptyset$.

We claim that K has the conditions of Corollary 5. Let V be an open set such that $\overline{V} \subset K$. Consider $L \subset \overline{V}$ such that L is Lindelöf, but is not heriditaraly Lindelöf and $F \subset L$ such that F is a closed set of L, but is not G_δ-set in L.

By induction on α, we construct the required discrete $D \subset L$.

Let $x_1 \in L \setminus F$. By the regularity of L, there are an open sets U_1 and V_1 such that $\overline{U_1}^L \cap \overline{V_1}^L = \emptyset$, $x_1 \in U_1$ and $F \subset V_1$.

Let we constructed x_α, V_α, U_α for $\alpha < \beta$ such that $\overline{U_\alpha}^L \cap \overline{V_\alpha}^L = \emptyset$, $x_\alpha \in U_\alpha$ and $F \subset V_\alpha$ and

1. $x_\alpha \in \bigcap\limits_{\gamma<\alpha} \overline{V_\gamma^L}$;
2. $x_\gamma \notin U_\alpha$ for $\alpha \neq \gamma$;
3. $x_\alpha \notin F$;
4. $\bigcup\limits_{\gamma<\alpha} \{x_\gamma\}$ are closed subsets of L for $\alpha < \beta$.

If $\beta = \omega_1$ or $\bigcup\limits_{\gamma<\alpha} \{x_\gamma\}$ is not closed set then inductive process is completed.

If $\bigcup\limits_{\gamma<\alpha} \{x_\gamma\}$ is closed set then there is $x_\beta \in \bigcap\limits_{\gamma<\beta} \overline{V_\gamma}^L \setminus F$. There are an open sets V_β and U_β such that $F \subset V_\beta$, $x_\beta \in U_\beta$, $\overline{U_\beta}^L \cap \overline{V_\beta}^L = \emptyset$, $\overline{V_\beta}^L \cap (\bigcup\limits_{\gamma<\beta} \{x_\gamma\}) = \emptyset$ and $\overline{U_\beta}^L \cap (\bigcup\limits_{\gamma<\beta} \{x_\gamma\}) = \emptyset$.

By construction, if $|D| = \omega$ then $\overline{D} \setminus D \neq \emptyset$.

If $|D| = \omega_1$, but D contains a countable is not closed (in K) subset D_1, then $D = D_1$.

If $|D| = \omega_1$ and each countable subset of D is closed set in K then D is a correct discrete set in Lindelöf space L and, hence, D is a correct discrete set in $\overline{V}$.

So K is resolvable. Then X is resolvable as the union of resolvable subspaces. □

Note that a regular wL-space is a L-space.

Corollary 6. *Let $\mathscr{P} \in \{\omega N,\ \omega,\ hL,\ \sigma\text{-}cmp,\ ccc,\ L,\ wL\}$. The regular $\mathscr{P}$-space with uncountable dispersion character is resolvable.*

Lemma 2. *Let X be a regular resolvable space and $\Delta(X) > \omega$. Then there are disjoint dense in X subsets Y_1, Y_2 of X such that $\Delta(Y_1) > \omega$.*

Proof. Let $X = X_1 \cup X_2$ and $\overline{X_i} = X$ for $i = 1, 2$. Suppose that $\Delta(X_1) = \Delta(X_2) = \omega$. Consider an open set U_1 of X such that $\Delta(U_1 \cap X_1) = \omega$. Then $\Delta(U_1 \cap X_2) > \omega$. Let $Z_1^1 = U_1 \cap X_2$, $Z_1^2 = U_1 \cap X_1$. If $\Delta(X_1 \setminus \overline{U_1}) > \omega$ then inductive process is completed. Suppose that $\Delta(X_1 \setminus \overline{U_1}) = \omega$. Let U_2 be an open set of X such that $\Delta(X_1 \setminus \overline{U_2}) = \omega$. $Z_2^1 = U_2 \cap X_2$, $Z_2^2 = U_2 \cap X_1$ and so on. By inductive process, we construct a disjoint family $\{U_\alpha, \alpha < \beta\}$ of open subsets of X such that $X = [\bigcup_{\alpha<\beta} U_\alpha]$ and disjoint sets $Z_\alpha^1 \subset U_\alpha$, $Z_\alpha^2 \subset U_\alpha$. Let $Y_1 = \bigcup_{\alpha<\beta} Z_\alpha^1$ and $Y_2 = \bigcup_{\alpha<\beta} Z_\alpha^2$. It is clear that $X = \overline{Y_i}$ for $i = 1, 2$ and $\Delta(Y_1) > \omega$. □

Theorem 9. *The regular L-tight space X of uncountable dispersion character is ω-resolvable.*

Proof. Let $X = Y_1 \cup X_1$, where X_1, Y_1 are disjoint and dense in X. Let $\Delta(Y_1) > \omega$. Let $Y_1 = Y_2 \cup X_2$, where X_2, Y_2 are disjoint dense in Y_1 and $\Delta(Y_2) > \omega$. A space Y_1 is dense in X, consequently X_2, Y_2 are dense in X too. And so on. By inductive process, we construct a countable disjoint family $\{X_n,\ n \in \omega\}$ of dense in X sets. □

Note that a regular wL-tight is a L-tight.

Corollary 7. *Let $\mathscr{P} \in \{\omega N,\ \omega,\ hL,\ \sigma\text{-}cmp,\ ccc,\ L,\ wL\}$. The regular $\mathscr{P}$-tight space with uncountable dispersion character is ω-resolvable.*

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