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INITIAL BOUNDARY VALUE PROBLEM FOR A NONLOCAL IN TIME PARABOLIC EQUATION

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Abstract. This paper deals with a quasi-linear parabolic partial differential equation that includes a nonlocal in time term. This term contains the integral of the solution over the entire time interval, where the problem is considered. The weak solvability of the initial boundary value problem for this equation is proven.

Keywords: nonlocal in time parabolic equation, initial boundary value problem, solvability.

1. Introduction

This paper deals with the following parabolic differential equation:

$$\partial_t u - \Delta u + \varphi\Big(\int_0^T u(s)\,ds\Big)\,u = 0, \tag{1}$$

where $u = u(x,t)$ is the unknown function, $x = (x_1,\dots,x_n)$ the vector of the spacial variables in $\mathbb{R}^n$, $n \in \mathbb{N}$, $t \in (0,T)$, T a positive number, φ a scalar nonnegative increasing function such that $\varphi(0) = 0$. Further, we consider positive solutions of this equation, therefore, we may assume that φ is defined on $[0,+\infty)$. This equation is considered in a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. We suppose that the following boundary and initial conditions are satisfied:

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{2}$$

$$u(x,0) = u_0(x), \tag{3}$$

where u_0 is a non-negative function.

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Equation (1) contains the non-local in time t term with the integral over the whole interval $(0, T)$, where the problem is considered. In biological and medical applications, aqueous solutions of polymers are often encountered. This raises the problem of the dynamics of a single polymer molecule or, as it is also called, a polymer chain. Under the action of the surrounding fluid, the polymer chain commits a chaotic motion, therefore, in order to describe its dynamics, probabilistic characteristics should be involved. In particular, the average distance between the ends of the chain plays an important role. In the work [1], a biological nanosensor based on the measurement of this quantity is proposed. However, what is more important, there arose an interesting differential equation or even a new class of equations. To describe the position of a chain segment, the density of probability that the segment is in a certain region of the space is used. The density of probability satisfies, with a high accuracy, a certain parabolic equation in which there is a term responsible for the interaction of chain segments. The role of time in the equation is played by the arc length parameter along the chain. The interaction is effected through the surrounding fluid. Since each segment interacts with all other segments, this term contains an integral of the density of probability over the entire chain, i.e., over the time interval on which the problem is being considered. In fact, to determine all the coefficients in the equation, it is necessary to know the "future". It should be noted that the obtained equation cannot be reduced to known problems by any transformations.

Up to now, problems with memory have been studied for parabolic equations, which included the integral of the solution from the initial to the current time. There is an extensive literature on this subject and it is not difficult to find related works. It is also possible to find papers that study problems, where the "future" stands in the data (see, e.g., [2], [3], [4], [5]). In the works [3] and [4], parabolic equations with a combination of initial and final or intermediate data were considered. This type of problem is very different from ours. The paper [5] is devoted to the investigation of a system of equations with an integral over the entire time interval, but this nonlocality is easily eliminated, and the equation is reduced to an ordinary parabolic equation with nonlocal data as in [3, 4]. Really, suppose that the initial condition (3) is replaced by the following one:

$$u(x, T) - u(x, 0) = f(x)$$

with a prescribed function $f : \Omega \to \mathbb{R}$. The integration of (1) with respect to t from 0 to T gives the following standard elliptic boundary value problem:

$$\Delta v - \varphi(v)\, v = f, \quad v|_{\partial\Omega} = 0,$$

where $v(x) = \int_0^T u(x, s)\, ds$. After solving this problem, we can find u as a solution of the following linear problem:

$$\partial_t u - \Delta u + \varphi(v)\, u = 0, \quad u|_{\partial\Omega} = 0, \quad u|_{t=T} - u|_{t=0} = f,$$

studied in [3, 4].

In this paper, we prove the weak solvability of problem (1)–(3). Equation (1) is similar to that obtained in [1]. It is simpler, however, the coefficient at the solution contains the integral of the solution over the entire time interval. That is, the main novelty of the problem remains.

2. Solvability of the problem

In this section, we investigate problem (1) – (3). The problem relates to the chaotic dynamics of a polymer molecule with one end fixed and the function u, its solution, is the spatial probability density of the other segments. For this reason, we will look for nonnegative solutions. The function φ is the interaction energy potential. We will consider the most interesting case, where φ tends to infinity as its argument approaches a positive number γ. Suppose that $\varphi : [0, \gamma) \to [0, +\infty)$ satisfies the following conditions: $\varphi(0) = 0$, φ is differentiable and convex, $\varphi(s) \to +\infty$ as $s \to \gamma$. Such a choice of the function φ assumes that u is bounded. In [1], the Flory – Huggins potential is used which looks as follows:

$$\varphi_*(s) = -\log(1 - s) - s.$$

For φ_*, the number γ is equal to 1.

If we consider not only positive solutions, the function φ must be somehow extended to the negative part of the real axis. It is also possible to deal with the function φ that is defined on $[0, +\infty)$ and has a prescribed rate of growth at infinity. This case is, generally speaking, more simple and is not considered in the present paper.

For definiteness, we will assume that $\varphi(s) = +\infty$ for $s \geq \gamma$ and $\varphi(s) = 0$ for $s \leq 0$.

Not all of the standard methods can be employed to prove the solvability of the problem. For example, the semi-discretization method cannot be used since the equation contains the term with the integral over the whole interval $[0, T]$, where the solution is to be defined. The Galerkin method does not preserve the positiveness for the approximations of the solution. One can apply the semi-group theory, however, this approach leads to a quite complicated integral equation. We will make use of the Schauder fixed point theorem which states that, for a Banach space X and a closed convex bounded set $E \subset X$, if a mapping $\Psi : E \to E$ is completely continuous, then Ψ has at least one fixed point in E.

Further, we will use the standard Lebesgue and Sobolev spaces $L^p(\Omega)$, $p \in [1, \infty]$, and $H^1_0(\Omega)$. As usual, $H^{-1}(\Omega)$ is the dual space of $H^1_0(\Omega)$ with respect to the pivot space $L^2(\Omega)$. The norm in $L^2(\Omega)$ will be denoted by $\| \cdot \|$.

Theorem 1. *If $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$, and T is an arbitrary positive number, then problem* (1) – (3) *has a weak solution $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega))$ such that*

$$u \geq 0, \quad 0 \leq v \leq \gamma, \quad \text{where} \quad v = \int_0^T u\,dt,$$

$$\varphi(v) \in L^2(\Omega), \quad \sqrt{\varphi(v)}\,u \in L^2(\Omega_T), \quad \text{where} \quad \Omega_T = \Omega \times [0, T],$$

$$\partial_t u \in L^2(0, T; H^{-1}(\Omega)), \quad \text{and} \quad u \in C(0, T; L^2(\Omega)).$$

Proof. As already mentioned above, we employ the Schauder fixed point theorem. As the Banach space X, we take the space $L^2(\Omega)$ with the standard norm. Let us describe the construction of the mapping Ψ. For every function $w = w(x)$, define the function $v = v(x)$ as a solution of the following problem:

$$-\Delta v + \varphi(v)\,v + w - u_0 = 0, \quad v|_{\partial\Omega} = 0. \tag{4}$$

This problem is a result of the integration of equation (1) with respect to t from 0 to T. The functions v and w correspond to $\int_0^T u(\cdot, t)\,dt$ and $u(\cdot, T)$, respectively. After that, we define the function $u = u(x,t)$ as the solution of the following problem:

(5) $$\partial_t u - \Delta u + \varphi(v)\, u = 0, \quad u|_{t=0} = u_0, \quad u|_{\partial\Omega} = 0.$$

Finally, define $u_T(x) = u(x,T)$. Thus, we have constructed the mapping Ψ such that $u_T = \Psi(w)$. Whenever w is a fixed point of Ψ, the function u defined in (5) is the solution of the original problem (1) – (3). Our goal is to find a set $E \subset L^2(\Omega)$ such that $\Psi(E) \subset E$ and to verify the complete continuity of Ψ on E.

At first, we consider problem (4) with an arbitrary $w \in L^2(\Omega)$. It is not difficult to see that the equation in (4) is the Euler equation for the functional

$$\Phi(v) = \int_\Omega \Big(\frac{1}{2}\,|\nabla v|^2 + F(v) + (w - u_0)\,v\Big)\,dx,$$

where $F(v) = \int_0^v \varphi(s)\,s\,ds$ for $v < \gamma$ and $F(v) = +\infty$ for $v \ge \gamma$. The functional Φ is Gâteaux-differentiable at all points of its effective domain in $H_0^1(\Omega)$ and its minimizer in this space gives the solution of problem (4) (see [6]).

Lemma 1. *For every $w \in L^2(\Omega)$, there exists a unique function $v \in H_0^1(\Omega)$ such that*

(6) $$\Phi(v) \le \Phi(\tilde v) \quad \textit{for all} \quad \tilde v \in H_0^1(\Omega).$$

This function has the following properties:

- (i) $\|\nabla v\| \le d(\Omega)\,\|w - u_0\|$, *where $d(\Omega)$ is the diameter of the domain Ω;*
- (ii) $v(x) \le \gamma$ *for almost all $x \in \Omega$;*
- (iii) $\|\varphi(v)\,v\| \le \|w - u_0\|$;
- (iv) $\|\Delta v\| \le 2\,\|w - u_0\|$;
- (v) $\|\varphi(v)\| \le C$, *where the constant C depends on $\|w - u_0\|$ and Ω.*

$\triangleright$ The functional Φ is coercive and strictly convex on $H_0^1(\Omega)$. The strict convexity gives us the uniqueness of the solution of the minimization problem. In order to prove the existence, we have to verify the weak lower semi-continuity of Φ in $H_0^1(\Omega)$ (see, e.g., [6]). This will be true if the level set

$$E_a = \{v \in H_0^1(\Omega) \mid \int_\Omega F(v)\,dx \le a\}$$

is weakly closed in $H_0^1(\Omega)$ for every $a \in \mathbb{R}$. Let us fix an arbitrary $a \in \mathbb{R}$ and take a sequence $\{v_k\}_{k\in\mathbb{N}}$ such that $v_k \in E_a$ for all $k \in \mathbb{N}$ and $v_k \to v$ weakly in $H_0^1(\Omega)$ as $k \to \infty$. There exists a subsequence $v_{k'}$ that converges to v almost everywhere in Ω. Since F is a continuous function on $[0,\gamma)$, $F(v_{k'}(x)) \to F(v(x))$ for almost all $x \in \Omega$. Due to the nonnegativity of F and the Fatou lemma, we have that

$$\int_\Omega F(v)\,dx \le \liminf_{k'\to\infty} \int_\Omega F(v_{k'})\,dx \le a$$

which means that $v \in E_a$.

Let us establish the properties of the solution of the variational problem (6).

(i) If we take $\tilde v \equiv 0$ in (6), then we obtain that

$$\int_\Omega \Big(\frac{1}{2}\,|\nabla v|^2 + F(v) + (w - u_0)\,v\Big)\,dx \le 0.$$

This estimate and the Poincaré inequality imply that

(7) $$\frac{1}{2}\,\|\nabla v\|^2 + \int_\Omega F(v)\,dx \le \|w - u_0\|\,\|v\| \le \frac{d(\Omega)}{2}\,\|w - u_0\|\,\|\nabla v\|.$$

The nonnegativity of F implies the first property of v.

(ii) As it follows from estimate (7), $\int_\Omega F(v)\,dx < \infty$. Since $F(s) = +\infty$ for $s \geq \gamma$, we conclude that $v(x) \leq \gamma$ for almost all $x \in \Omega$. Notice that, generally speaking, v can take negative values.

(iii) For brevity, denote the function $s \mapsto \varphi(s)s$ by η. The multiplication of (4) by $\eta(v)$ and the integration by parts lead to the following equality:

$$\int_\Omega \big(\eta'(v)\,|\nabla v|^2 + \eta^2(v) + (w-u_0)\,\eta(v)\big)\,dx = 0.$$

Since $\eta' \geq 0$, this equality together with the Hölder inequality yields the third property of v.

(iv) This property is a direct consequence of the previous property and (4).

(v) If $A = \{x \in \Omega \mid v(x) \geq \gamma/2\}$, then

$$\int_A \varphi^2(v)\,dx \leq \Big(\frac{2}{\gamma}\Big)^2 \int_\Omega \varphi^2(v)\,v^2\,dx.$$

Since the function φ is increasing, we have

$$\int_{\Omega\setminus A} \varphi^2(v)\,dx \leq \varphi^2(\gamma/2)\,\mu(\Omega),$$

where $\mu(\Omega)$ is the Lebesgue measure of Ω. These inequalities together with the third property imply the required estimate with $C^2 = (2/\gamma)^2\,\|w-u_0\|^2 + \varphi^2(\gamma/2)\,\mu(\Omega)$.

The lemma is proven. $\lhd$

Denote by V the mapping from $L^2(\Omega)$ into $H_0^1(\Omega)$ such that $v = V(w)$ is the solution of problem (6).

Lemma 2. *Let $\{w_k\}$ be a sequence in $L^2(\Omega)$ that converges weakly in this space to w. If $v_k = V(w_k)$ and $v = V(w)$, then*

(i) *$v_k \to v$ in $H_0^1(\Omega)$ as $k \to \infty$;*
(ii) *$\Delta v_k \to \Delta v$ weakly in $L^2(\Omega)$ as $k \to \infty$;*
(iii) *$\varphi(v_k) \to \varphi(v)$ weakly in $L^2(\Omega)$ as $k \to \infty$;*

$\rhd$ Since v_k and v are solutions of problem (4),

(8) $$-\Delta(v_k - v) + \varphi(v_k)\,v_k - \varphi(v)\,v + w_k - w = 0, \quad (v_k - v)|_{\partial\Omega} = 0.$$

Due to the monotonicity of the function $s \mapsto \varphi(s)s$, we obtain that

$$\int_\Omega |\nabla(v_k - v)|^2\,dx \leq \Big|\int_\Omega (w_k - w)(v_k - v)\,dx\Big| \leq \|w_k - w\|_{H^{-1}(\Omega)}\|v_k - v\|_{H_0^1(\Omega)}.$$

The first assertion of the lemma follows from the fact that $\|w_k - w\|_{H^{-1}(\Omega)} \to 0$ as $k \to \infty$.

The fact just proven and the integration by parts immediately imply that

$$\int_\Omega \big(\varphi(v_k)\,v_k - \varphi(v)\,v\big)\,\psi\,dx \to 0 \quad \text{as} \quad k \to \infty$$

for an arbitrary $\psi \in H_0^1(\Omega)$. Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ and the sequence $\{\varphi(v_k)\,v_k\}$ is bounded in $L^2(\Omega)$, we obtain that

(9) $$\varphi(v_k)\,v_k \to \varphi(v)\,v \quad \text{weakly in} \quad L^2(\Omega) \quad \text{as} \quad k \to \infty.$$

This relation together with (8) yields the second assertion of the lemma.

Since $v_k \to v$ in $L^2(\Omega)$ and the sequence $\{\varphi(v_k)\}$ is bounded in $L^2(\Omega)$, we have that

$$\varphi(v_k)\, v \to \varphi(v)\, v \quad \text{weakly in} \quad L^2(\Omega) \quad \text{as} \quad k \to \infty. \tag{10}$$

Really, for every $\psi \in L^\infty(\Omega)$,

$$\begin{aligned}\Big|\int_\Omega \big(\varphi(v_k) - \varphi(v)\big)\, v\, \psi\, dx\Big| \\ \le \Big|\int_\Omega \big(\varphi(v_k)\, v_k - \varphi(v)\, v\big)\, \psi\, dx\Big| + \Big|\int_\Omega \varphi(v_k)(v_k - v)\, \psi\, dx\Big| \\ \le \Big|\int_\Omega \big(\varphi(v_k)\, v_k - \varphi(v)\, v\big)\, \psi\, dx\Big| + \|\varphi(v_k)\|\, \|v_k - v\|\, \|\psi\|_{L^\infty(\Omega)}.\end{aligned}$$

The density of $L^\infty(\Omega)$ in $L^2(\Omega)$ together with (9) implies (10).

Relation (10), generally speaking, does not yield the third assertion of the lemma. The function v can be equal to zero on a set of positive measure. Let us fix an arbitrary $\varepsilon > 0$ and introduce the set $A_\varepsilon = \{x \in \Omega \mid |v(x)| < \varepsilon\}$. As it follows from (10),

$$\int_{\Omega \setminus A_\varepsilon} \big(\varphi(v_k) - \varphi(v)\big)\, \psi\, dx \to 0 \quad \text{as} \quad k \to \infty$$

for every $\psi \in L^2(\Omega)$. In order to prove the third assertion of the lemma, we have to establish that

$$\int_{A_\varepsilon} \big(\varphi(v_k) - \varphi(v)\big)\, \psi\, dx \to 0 \quad \text{as} \quad k \to \infty$$

for every $\psi \in L^2(\Omega)$. As before, it is enough to prove this relation for $\psi \in L^\infty(\Omega)$. Since $v_k \to v$ in $L^2(\Omega)$ as $k \to \infty$, the sequence $\{v_k\}$ converges to v in measure and, in particular,

$$\lim_{k\to\infty} \mu(A_\varepsilon^k) = 0,$$

where μ is the Lebesgue measure and $A_\varepsilon^k = \{x \in A_\varepsilon \mid |v_k(x) - v(x)| \ge \varepsilon\}$. For every $k \in \mathbb{N}$, we have

$$\Big|\int_{A_\varepsilon^k} \big(\varphi(v_k) - \varphi(v)\big)\, \psi\, dx\Big| \le \|\varphi(v_k) - \varphi(v)\|\, \|\psi\|_{L^\infty(\Omega)}\, \mu(A_\varepsilon^k)^{1/2}.$$

Since $\|\varphi(v_k) - \varphi(v)\|$ are uniformly bounded, the right-hand side of this inequality tends to zero as $k \to \infty$. On the other hand, $|v_k| < 2\varepsilon$ almost everywhere on the set $A_\varepsilon \setminus A_\varepsilon^k$. Therefore,

$$\Big|\int_{A_\varepsilon \setminus A_\varepsilon^k} \big(\varphi(v_k) - \varphi(v)\big)\, \psi\, dx\Big| \le \big(\varphi(2\varepsilon) + \varphi(\varepsilon)\big)\, \|\psi\|_{L^\infty(\Omega)}\, \mu(A_\varepsilon).$$

Thus,

$$\begin{aligned}\limsup_{k\to\infty} \Big|\int_{A_\varepsilon} \big(\varphi(v_k) - \varphi(v)\big)\, \psi\, dx\Big| \\ \le \lim_{k\to\infty} \Big|\int_{A_\varepsilon^k} \big(\varphi(v_k) - \varphi(v)\big)\, \psi\, dx\Big| + \limsup_{k\to\infty} \Big|\int_{A_\varepsilon \setminus A_\varepsilon^k} \big(\varphi(v_k) - \varphi(v)\big)\, \psi\, dx\Big| \\ \le \big(\varphi(2\varepsilon) + \varphi(\varepsilon)\big)\, \|\psi\|_{L^\infty(\Omega)}\, \mu(A_\varepsilon).\end{aligned}$$

Since ε is arbitrary small and $\varphi(s) \to 0$ as $s \to 0$,

$$\lim_{k\to\infty} \Big| \int_{A_\varepsilon} \big(\varphi(v_k) - \varphi(v)\big)\, \psi\, dx \Big| = 0.$$

The lemma is proven. $\lhd$

The advantage of the lemma just proven is that we have established the convergence results not for a subsequence but for the entire sequence $\{V(w_k)\}$. These results will be used for the proof of the complete continuity of the mapping Ψ.

Lemma 3. *If $\varphi(v)$ and u_0 are in $L^2(\Omega)$, then problem (5) has a unique weak solution $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ such that $\sqrt{\varphi(v)}\, u \in L^2(\Omega_T)$, where $\Omega_T = \Omega \times [0,T]$. Moreover, if, in addition, $u_0 \in L^\infty(\Omega)$ and $u_0 \geq 0$, then $u \in L^\infty(\Omega_T)$, $u \geq 0$, $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$, and $u \in C(0,T;L^2(\Omega))$.*

$\rhd$ Since $\varphi(v)$ is a nonnegative function, the linear problem (5) is standard and we omit the proof of its unique weak solvability as well as various justifications (see, e.g., [7]). We restrict ourselves to the derivation of the estimates for the function u. The multiplication of (5) by u and the integration over $\Omega \times [0,s]$ for almost all $s \in (0,T]$, give

$$\frac{1}{2}\, \|u(\cdot,s)\|^2 + \int_0^s \|\nabla u\|^2\, dxdt + \int_0^s \int_\Omega \varphi(v)\, u^2\, dxdt \leq \frac{1}{2}\, \|u_0\|^2. \tag{11}$$

The inequality instead of the equality is obtained after the passage to the limit in the corresponding equalities for the approximate solutions. This estimate ensures that u is in the function classes encountered in the first part of the lemma.

Let us consider the second part of the assertion of the lemma, where it is assumed that u_0 is a bounded nonnegative function. Denote by γ_0 a positive number such that $u_0 \leq \gamma_0$ almost everywhere in Ω. Let $f : \mathbb{R} \to [0,\infty)$ be a smooth convex function such that $f(s) = 0$ for $s \in [0,\gamma_0]$ and $f(s) > 0$ for all other s. The multiplication of (5) by $f'(u)$, where f' is the derivative of f, leads to the following estimate:

$$\int_\Omega f\big(u(x,s)\big)\, dx \leq \int_\Omega f\big(u_0(x)\big)\, dx = 0$$

for almost all $s \in [0,T]$. We have used that $f''(u)$ and $uf'(u)$ are nonnegative. Since $f(u_0) = 0$ almost everywhere in Ω, this estimate implies that $f(u) = 0$ and, as a consequence, that $0 \leq u \leq \gamma_0$ almost everywhere in Ω_T.

Finally, in order to prove the continuity of u in $L^2(\Omega)$, we multiply (5) by a smooth function $\psi = \psi(x,t)$ which is equal to zero on $\partial\Omega$, for $t = 0$, and for $t = T$. As a result, we find that $\partial_t u$ is bounded in $L^2(0,T;H^{-1}(\Omega))$, which is a direct consequence of the following estimate:

$$\Big| \int_0^T \int_\Omega \varphi\big(v(x)\big)\, u(x,t)\, \psi(x,t)\, dxdt \Big| \leq \|\varphi(v)\|\, \|u\|_{L^2(0,T;L^\infty(\Omega))}\, \|\psi\|_{L^2(\Omega_T)}.$$

Since $u \in L^2(0,T;H_0^1(\Omega))$, we conclude that $u \in C(0,T;L^2(\Omega))$ (see, e.g., [7]). $\lhd$

Due to the continuity of u in $L^2(\Omega)$, the function $u_T = u(\cdot,T)$ is well defined as an element of $L^2(\Omega)$. This fact enables us to construct the mapping $\Psi : L^2(\Omega) \to L^2(\Omega)$ as it was described in the beginning of the proof of the theorem. We have to find a closed convex bounded set $E \subset L^2(\Omega)$ such that $\Psi(E) \subset E$ and to prove the complete continuity of Ψ on E.

We take $E = \{w \in L^2(\Omega) \,|\, \|w\| \leq \|u_0\|\}$. Due to (11), $\|u_T\| \leq \|u_0\|$ independently of v, therefore, $\Psi(w) \in E$ for all $w \in L^2(\Omega)$. In particular, $\Psi(E) \subset E$.

Let us investigate the complete continuity of Ψ on E. Suppose that $\{w_k\}$ is an arbitrary sequence in E that converges weakly in $L^2(\Omega)$ to $w \in E$. Our goal is to prove that $\Psi(w_k) \to \Psi(w)$ strongly in $L^2(\Omega)$ as $k \to \infty$. Due to Lemma 2, $\varphi(v_k) \to \varphi(v)$ weakly in $L^2(\Omega)$ as $k \to \infty$, where $v_k = V(w_k)$ and $v = V(w)$. Denote by u_k and u the corresponding solutions of problem (5). It remains to prove that $u_{kT} \to u_T$ in $L^2(\Omega)$ as $k \to \infty$.

At first, we note that, due to Lemma 3,

$$\partial_t(u_k - u) - \Delta(u_k - u) + \varphi(v_k)\, u_k - \varphi(v)\, u = 0$$

in $L^2(0,T;H^{-1}(\Omega))$. The multiplication of this equation by $u_k - u$ and the integration over Ω yield:

$$\frac{1}{2}\frac{d}{dt}\|u_k - u\|^2 + \|\nabla(u_k - u)\|^2 + \int_\Omega \varphi(v_k)\,(u_k - u)^2\,dx \\ + \int_\Omega u\,\big(\varphi(v_k) - \varphi(v)\big)\,(u_k - u)\,dx = 0,$$

where the time derivative is understood in the distributional sense. Since the second and the third terms on the left-hand side of this equality are nonnegative and $u_k - u = 0$ for $t = 0$, we have the following estimate:

$$\frac{1}{2}\,\|u_{kT} - u_T\|^2 \le \|\varphi(v_k) - \varphi(v)\|_{H^{-1}(\Omega)} \int_0^T \|u\,(u_k - u)\|_{H_0^1(\Omega)} dt.$$

Since u and u_k are uniformly bounded in $L^\infty(\Omega_T) \cap L^2(0,T;H_0^1(\Omega))$, the sequence $\{u(u_k - u)\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$. Therefore, there exists an independent of k constant C such that

$$\|u_{kT} - u_T\|^2 \le C\,\|\varphi(v_k) - \varphi(v)\|_{H^{-1}(\Omega)}.$$

As the embedding $L^2(\Omega)$ into $H^{-1}(\Omega)$ is compact, the right-hand side of this inequality tends to 0 as $k \to \infty$. Thus, $u_{kT} \to u_T$ and, as a consequence, $\Psi(w_k) \to \Psi(w)$ in $L^2(\Omega)$ as $k \to \infty$. The solvability of problem (1)–(3) is proven. □

References

[1] V.N. Starovoitov, B.N. Starovoitova, *Modeling the dynamics of polymer chains in water solution. Application to sensor design*, Journal of Physics: Conference series, **894** (2017), 012088.

[2] C.V. Pao, *Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions*, Journal of Mathematical Analysis and Applications, **195**:3 (1995), 702–718. MR1356638

[3] V.V. Shelukhin, *A problem with time-averaged data for nonlinear parabolic equations*, Siberian Mathematical Journal, **32**:2 (1991), 309-–320. MR1138450

[4] V.V. Shelukhin, *A variational principle for linear evolution problems nonlocal in time.* Siberian Mathematical Journal, **34**:2 (1993), 369–384. MR1223770

[5] A.Sh. Lyubanova, *On nonlocal problems for systems of parabolic equations*, J. Math. Anal. Appl., **421**:2 (2015), 1767-–1778. MR3258348

[6] I. Ekeland, R. Témam, *Convex analysis and variational inequalities*, Amsterdam: North-Holland, 1976.

[7] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen.* Mathematische Lehrbücher und Monographien, II. Abteilung, Mathematische Monographien, B. 38, Akademie-Verlag, Berlin, 1974. MR0636412

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