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LIGHT 3-STARS IN SPARSE PLANE GRAPHS

O.V. BORODIN, A.O. IVANOVA

Abstract. A k-star $S_k(v)$ in a plane graph G consists of a central vertex v and k its neighbor vertices. The height $h(S_k(v))$ and weight $w(S_k(v))$ of $S_k(v)$ is the maximum degree and degree-sum of its vertices, respectively. The height $h_k(G)$ and weight $w_k(G)$ of G is the maximum height and weight of its k-stars.

Lebesgue (1940) proved that every 3-polytope of girth g at least 5 has a 2-star (a path of three vertices) with $h_2 = 3$ and $w_2 = 9$. Madaras (2004) refined this by showing that there is a 3-star with $h_3 = 4$ and $w_3 = 13$, which is tight. In 2015, we gave another tight description of 3-stars for girth $g = 5$ in terms of degree of their vertices and showed that there are only these two tight descriptions of 3-stars.

In 2013, we gave a tight description of 3^--stars in arbitrary plane graphs with minimum degree δ at least 3 and $g \geq 3$, which extends or strengthens several previously known results by Balogh, Jendrol', Harant, Kochol, Madaras, Van den Heuvel, Yu and others and disproves a conjecture by Harant and Jendrol' posed in 2007.

There exist many tight results on the height, weight and structure of 2^--stars when $\delta = 2$. In 2016, Hudák, Maceková, Madaras, and Široczki considered the class of plane graphs with $\delta = 2$ in which no two vertices of degree 2 are adjacent. They proved that $h_3 = w_3 = \infty$ if $g \leq 6$, $h_3 = 5$ if $g = 7$, $h_3 = 3$ if $g \geq 8$, $w_3 = 10$ if $g = 8$ and $w_3 = 3$ if $g \geq 9$. For $g = 7$, Hudák et al. proved $11 \leq w_3 \leq 20$.

The purpose of our paper is to prove that every plane graph with $\delta = 2$, $g = 7$ and no adjacent vertices of degree 2 has $w_3 = 12$.

Borodin, O.V., Ivanova, A.O., All tight descriptions of 3-paths in plane graphs with girth at least 9.

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1. Introduction

The degree of a vertex v or a face f in a plane graph G, that is the number of edges incident with v or f, is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc.

Let $\delta(G)$ be the minimum vertex degree and $g(G)$ be the girth (the length of a shortest cycle) in G. A k-star $S_k(v)$ in G consists of a central vertex v and k its neighbor vertices. The height $h(S_k(v))$ and weight $w(S_k(v))$ of $S_k(v)$ is the maximum degree and degree-sum of its vertices, respectively. The height $h_k(G)$ and weight $w_k(G)$ of G is the maximum height and weight of its k-stars. We will often drop the argument when the graph is clear from context.

An edge uv, that is an $S_1(u)$ or $S_1(v)$, is an *(i, j)-edge* if $d(u) \leq i$ and $d(v) \leq j$. More generally, a path $v_1v_2v_3$ (which is an $S_2(v_2)$), is a *path of type* (i_1, i_2, i_3), or a *(i_1, i_2, i_3)-path* if $d(v_j) \leq i_j$ whenever $1 \leq j \leq 3$. The types of higher stars are defined similarly.

Already in 1904, Wernicke [48] proved that every G with $\delta = 5$ satisfies $w_2 \leq 11$, and Franklin [33] strengthened this to the existence of a $(6, 5, 6)$-path, which description is tight. In [15], we proved that there is another tight description, "a $(5, 6, 6)$-path" and that no other tight descriptions exists.

In [10], we gave a tight description of 3^--stars in arbitrary plane graphs with $\delta \geq 3$ and $g \geq 3$ by proving that there is either a $(3, 10)$-edge, or a $(5, 4, 9)$-path, or a $(6, 4, 8)$-path, or a $(7, 4, 7)$-path, or a $(5; 4, 5, 5)$-star, or a $(5; 5, b, c)$-star with $5 \leq b \leq 6$ and $5 \leq c \leq 7$, or else a $(5; 6, 6, 6)$-star. This extends or strengthens several previously known results by Balogh, Jendrol', Harant, Kochol, Madaras, Van den Heuvel, Yu and others [4, 34, 42, 46, 47] and disproves a conjecture in Harant, Jendrol' [34].

In 1940, Lebesgue [44] gave an approximate description of 5-stars centered at 5-vertices for the case $\delta = 5$ and $g \geq 3$. Recently, we obtained several tight results on the height, weight and structure of such 5-stars assuming the absence of 6^+-vertices from certain degree-sets, see [12, 16, 19, 21, 23, 24, 26, 27, 29–31].

Also, Lebesgue [44] proved that every G with $\delta \geq 3$ and $g = 5$ satisfies $h_2 = 3$ and $w_2 = 9$. In 2004, Madaras [45] refined this by showing that there is a 3-star with $h_3 = 4$ and $w_3 = 13$, which is tight. Recently, we gave [22] another tight description of 3-stars for $g = 5$ in terms of degree of their vertices and showed that there are only these two tight descriptions of 3-stars.

There exist many results on the height, weight and structure of 2^--stars when $\delta = 2$, see, for example, [1–8, 11, 14, 17, 25, 28, 35–40] and also surveys by Jendrol, Voss [43] and Borodin, Ivanova [20].

In 2016, Hudák, Maceková, Madaras and Široczki [35] considered the class of plane graphs with $\delta = 2$ in which no two vertices of degree 2 are adjacent. They proved that $h_3 = w_3 = \infty$ if $g \leq 6$, $h_3 = 5$ if $g = 7$, $h_3 = 3$ if $g \geq 8$, $w_3 = 10$ if $g = 8$ and $w_3 = 9$ if $g \geq 9$. For $g = 7$, Hudák et al. [35] proved $11 \leq w_3 \leq 20$.

The purpose of our paper is to settle the case $g = 7$, as follows.

Theorem 1. *Every plane graph with $\delta = 2$ and $g = 7$ in which no two vertices of degree 2 are adjacent has a 3-star of weight at most 12 centered at a 5^--vertex, where 12 is best possible.*

2. Proof of Theorem 1

In Fig. 1, we see a bit more than a half of a plane graph with the desired properties $\delta = 2$, $g = 7$, and no $(2,2)$-path that confirms the lower bound $w_3 \geq 12$. The invisible "equator" passes through the middles of ten edges joining 3-vertices with 5-vertices in the outside layer. To obtain the whole graph, it suffices to superpose vertices of outer cycle of one half with the vertices of the cycle nearest to it of the other half. Namely, the path of vertices of degrees 5, 3, 2, and 3 of the shaded face are superposed on the path of vertices of degrees 5, 3, 2, and 3 of the white face.

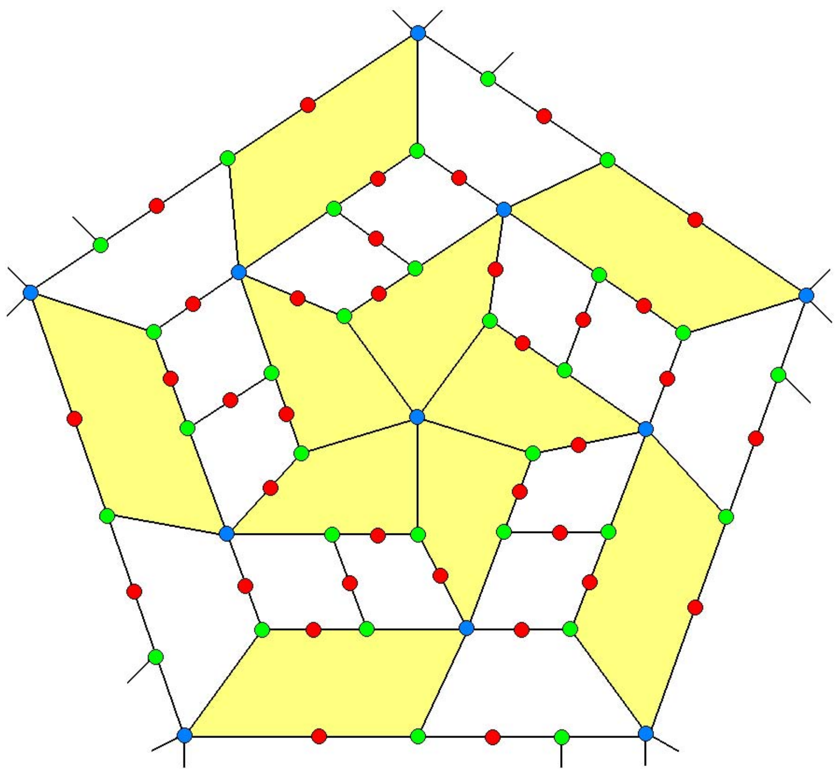

Рис. 1. All 3-stars are of weight at least12.

2.1. **Discharging and its consequences.** Let G be a counterexample to the upper bound 12 in Theorems 1. Without loss of generality, we can assume that G is connected. Let V, E, and F be the sets of vertices, edges and faces of G, respectively. Euler's formula $|V| - |E| + |F| = 2$ for G may be rewritten as

$$\sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2d(f) - 6) = -12. \tag{1}$$

Every vertex v contributes the *charge* $\mu(v) = d(v) - 6$ to (1), so only the charges of 5^--vertices are negative. Every face f contributes the non-negative *charge* $\mu(f) = 2d(f) - 6$ to (1). Using the properties of G as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12.

Throughout the paper, we denote the vertices adjacent to a vertex or incident with face x in a cyclic order by $v_1, \ldots, v_{d(x)}$. Let $\partial(f)$ be the boundary of a face f.

We apply the following rules of discharging (see Fig. 2).

R1. *Every face gives* 2 *to each incident* 2*-vertex.*

R2. *Every face gives* $\frac{1}{3}$ *to each incident vertex* v_2 *such that* $3 \leq d(v_2) \leq 5$ *and* $d(v_1) = d(v_3) = 2$.

R2. *Every face gives the following charge to each incident* 3*-vertex* v_2*:*
(*a*) $\frac{4}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) \geq 6$,
(*b*) $\frac{7}{6}$ *if* $d(v_1) \leq 3$ *and* $d(v_3) = 5$,
(*c*) 1 *if* $d(v_3) = 4$,
(*d*) $\frac{2}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) = 3$,
(*e*) 1 *if* $d(v_1) = 3$ *while* $d(v_3) = 3$ *or* $d(v_3) \geq 6$, *and*
(*f*) 1 *if* $d(v_1) \geq 5$ *and* $d(v_3) \geq 5$.

R4. *Every face gives the following charge to each incident* 4*-vertex* v_2*:*
(*a*) $\frac{1}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) = 3$,
(*b*) $\frac{2}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) \geq 4$, *and*
(*c*) $\frac{2}{3}$ *if* $d(v_1) \geq 3$ *and* $d(v_3) \geq 3$.

R5. *Every face gives the following charge to each incident* 5*-vertex* v_2*:*
(*a*) $\frac{1}{6}$ *if* $d(v_1) = 2$ *and* $d(v_3) = 3$,
(*b*) $\frac{1}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) \geq 4$, *and*
(*c*) $\frac{1}{3}$ *if* $d(v_1) \geq 3$ *and* $d(v_3) \geq 3$.

2.2. Proving $\mu'(v) \geq 0$ whenever $v \in V$.

Case 1. $d(v) = 2$. Here, $\mu'(v) = d(v) - 6 + 2 \times 2 = 0$ by R1.

Case 2. $d(v) = 3$.

Subcase 2.1. $d(v_1) = d(v_2) = 2$. Now $d(v_3) \geq 13 - 3 - 2 \times 2 = 6$, so $\mu'(v) = -3 + 2 \times \frac{4}{3} + \frac{1}{3} = 0$ by R2 and R3a.

Subcase 2.2. $d(v_1) = 2$ and $d(v_2) = 3$. Now $d(v_3) \geq 5$, so $\mu'(v) = 3 - 6 + 2 \times \frac{7}{6} + \frac{2}{3} = 0$ by R3b, R3d.

Subcase 2.3. $d(v_1) = 2$, $d(v_2) \geq 4$, and $d(v_3) \geq 4$. Now v receives at least 1 from each incident face by R3a–R3c, so $\mu'(v) \geq 0$.

Subcase 2.4. $d(v_1) \geq 3$, $d(v_2) \geq 3$, and $d(v_3) \geq 4$. Now again v receives at least 1 from each incident face, but this time by R3b, R3c, R3e, and R3f, and we have $\mu'(v) \geq 0$.

Case 3. $d(v) = 4$. Note that each incident face gives v either $\frac{1}{3}$ by R2 and R4a or $\frac{2}{3}$ by R4b and R4c. We have $\mu'(v) \geq -2 + 2 \times \frac{1}{3} + 2 \times \frac{2}{3} = 0$, unless v receives $\frac{1}{3}$ at

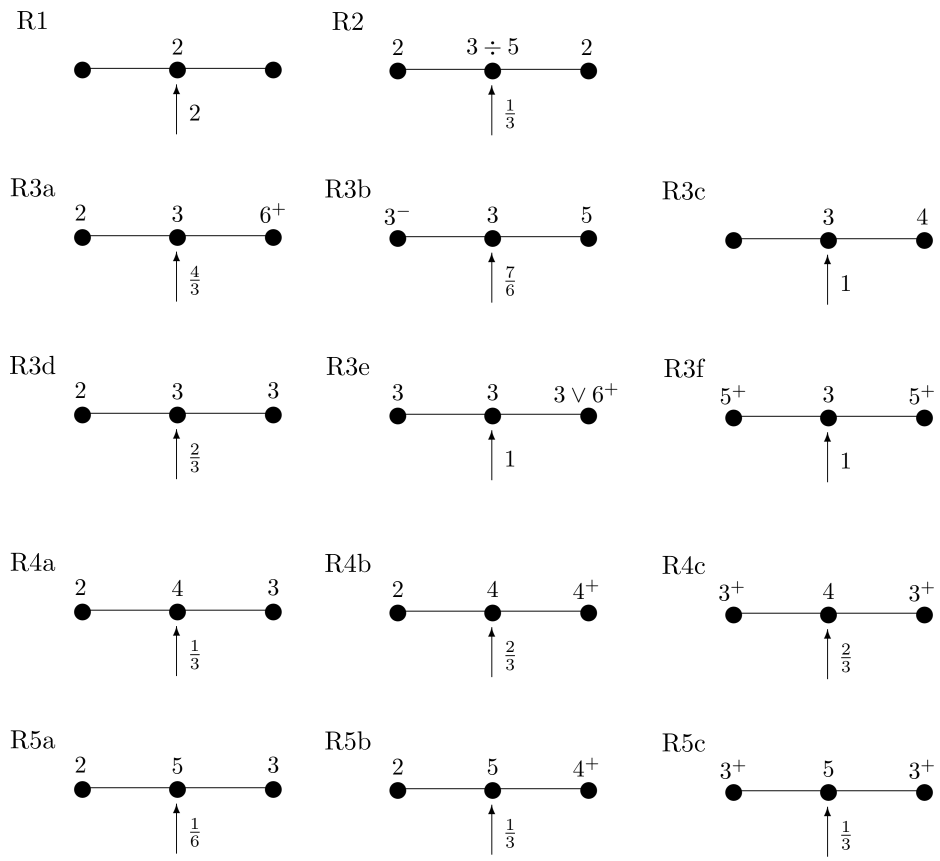

Рис. 2. Rules of discharging

least thrice, in which case v has three 3^--neighbors including a 2-neighbor. However, this implies a 3-star at v of weight at most $4 + 2 \times 3 + 2 < 13$, a contradiction.

Case 4. $d(v) = 5$. Now each incident face gives v either $\frac{1}{3}$ by R2 and R5b, R5c or $\frac{1}{6}$ by R5a. It suffices to exclude the possibility that v receives $\frac{1}{6}$ five times, since otherwise we have $\mu'(v) \geq -1 + \frac{1}{3} + 4 \times \frac{1}{6} = 0$. This follows easily from the fact that 2-neighbors cannot alternate with 3-neighbors around v.

Case 5. $d(v) \geq 6$. Since v does not participate in discharging, we have $\mu'(v) = d(v) - 6 \geq 0$.

2.3. Proving $\mu'(f) \geq 0$ whenever $f \in F$.

Lemma 1. *Every face f gives the following total charge to strings of incident 3^+-vertices by R2–R5:*

(*a*) *at most $\frac{1}{3}$ to v_2 if $d(v_1) = d(v_3) = 2$,*
(*b*) *at most $\frac{4}{3}$ to v_2v_3 if $d(v_1) = d(v_4) = 2$,*
(*c*) *at most $\frac{8}{3}$ to $v_2v_3v_4$ if $d(v_1) = d(v_5) = 2$, and*
(*d*) *at most $\frac{11}{6}$ to v_2v_3 if $d(v_1) = 2$ while $d(v_4) \geq 3$,*

Proof. (a) This is precisely the rule R2.

(b) If $d(v_2) \geq 6$, then v_2 receives nothing while v_3 receives at most $\frac{4}{3}$ by R3–R5.

Suppose $d(v_2) = 5$. If $d(v_3) = 3$ then v_2 receives $\frac{1}{6}$ by R5a while v_3 receives $\frac{7}{6}$ by R3b, as desired. If $4 \leq d(v_3) \leq 5$ then v_2 receives $\frac{1}{3}$ by R5b while v_3 at most 1 by R5b or R3c.

Next suppose $d(v_2) = 4$. If $d(v_3) = 3$ then v_2 receives $\frac{1}{3}$ by R4a while v_3 receives 1 by R3c. If $d(v_3) = 4$ then each of v_2, v_3 receives $\frac{2}{3}$ by R4b.

Finally, if $d(v_2) = d(v_3) = 3$ then each of v_2, v_3 receives $\frac{2}{3}$ by R3d.

(c) If $d(v_3) \geq 6$, $d(v_3) = 5$, or $d(v_3) = 4$, then $v_2v_3v_4$ receives at most $\frac{4}{3} + 0 + \frac{4}{3}$, $\frac{7}{6} + \frac{1}{3} + \frac{7}{6}$, or $1 + \frac{2}{3} + 1$, respectively.

Finally, suppose $d(v_3) = 3$. If $d(v_2) \notin \{3, 5\}$, then $v_2v_3v_4$ receives at most $\frac{1}{3}+1+\frac{2}{3}$. Now if $d(v_2) = 5$, then $v_2v_3v_4$ receives at most $\frac{1}{6} + \frac{7}{6} + \frac{2}{3}$ by R3b, R3d, R4a, R5a. By symmetry, it remains to assume that $d(v_2) = d(v_4) = 3$, in which case $v_2v_3v_4$ receives at most $\frac{2}{3} + 1 + \frac{2}{3}$, as desired.

(d) If there is a 5^+-vertex in v_2v_3, then v_2v_3 receives at most $\frac{1}{3} + \frac{4}{3} < \frac{11}{6}$, as desired. Suppose the contrary. Now if v_2 receives at most $\frac{2}{3}$, then v_2v_3 receives at most $\frac{2}{3} + \frac{7}{6} = \frac{11}{6}$ since v_3 never receives as much as $\frac{4}{3}$. Otherwise, v_2 receives 1 by R3c, in which case $d(v_2) = 4$ or $d(v_3) = 4$ and v_2v_3 receives $1 + \frac{2}{3}$ in both cases in view of R4c, and we are done again. ■

A *k-worm* is a string $v_2 \ldots v_{k+1}$ of 3^+-vertices in the boundary $\partial(f)$ of a face f such that $d(v_1) = d(v_{k+2}) = 2$. In particular, we can have $v_1 = v_{d(v)+1}$, in which case $\partial(f)$ is split into a 2-vertex v_1 and a $(d(v) - 1)$-worm. If f is incident with at least one 2-vertex, then $\partial(f)$ is split into k-worms with variable k.

To estimate the total donation of f by R1–R5, we use the following $\frac{5}{12}$-*averaging*: every 2-vertex v_2 transfers $\frac{5}{12}$ to v_1 and v_3.

The $\frac{5}{12}$-averaged donation of f to a 2-neighbor is $2 - 2 \times \frac{5}{12} = \frac{7}{6}$. Lemma 1 easily implies the maximum donations to k-worms.

Corollary 1. *The $\frac{5}{12}$-averaged total donations of f to k-worms are:*
(*a*) *at most* $\frac{1}{3} + 2 \times \frac{5}{12} = \frac{7}{6}$ *if* $k = 1$,
(*b*) *at most* $\frac{4}{3} + \frac{5}{6} = 2 \times \frac{7}{6} - \frac{1}{6}$ *if* $k = 2$,
(*c*) *at most* $\frac{8}{3} + \frac{5}{6} = 3 \times \frac{7}{6}$ *if* $k = 3$, *and*
(*d*) *at most* $\frac{5}{6} + 2 \times \frac{11}{6} + (k - 4) \times \frac{7}{6} = k \times \frac{7}{6} - \frac{1}{6}$ *if* $k \geq 4$.

Proof. The only remark possibly needed is that a 3^+-vertex at distance at least 2 from the nearest 2-vertex in $\partial(f)$ receives at most $\frac{7}{6}$ from f both by R2–R5 and after the $\frac{5}{12}$-averaging. ■

We are now prepared to complete the proof of $\mu'(f) \geq 0$.

Case 1. $d(f) = 7$. Note that $\mu(f) = 2 \times 7 - 6 = 7 \times \frac{7}{6} - \frac{1}{6}$. This means that we are done by Corollary 1 unless $\partial(f)$ has neither 2-worms nor 4^+-worms.

If f is incident with three 2-vertices, then $\partial(f)$ consists of a 2-worm and two 1-worms, and we are done.

If f is incident with two 2-vertices, then the five incident vertices of f are split into two worms, one of which must be a 2-worm or 4-worm. The presence of precisely one 2-vertex in $\partial(f)$ implies a 6-worm.

Finally, suppose f is not incident with 2-vertices. Now it suffices to observe that there are no three consecutive vertices in $\partial(f)$ each receiving $\frac{7}{6}$ by R2–R5 to see that $\mu'(f) \geq 8 - 6 \times \frac{7}{6} - 1 = 0$.

Case 2. $d(f) \geq 8$. Since the $\frac{5}{12}$-averaged donation of f to a 2-neighbor is $\frac{7}{6}$, each k-worm receives at most $k \times \frac{7}{6}$ by Corollary 1, and each 3^+-vertex not belonging to a worm receives at most $\frac{7}{6}$, we have $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{7}{6} = \frac{5d(f)-36}{6} > 0$.

This completes the proof of Theorem 1.

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