LIGHT 3-STARS IN SPARSE PLANE GRAPHS

O.V. BORODIN, A.O. IVANOVA

ABSTRACT. A k-star $S_k(v)$ in a plane graph G consists of a central vertex v and k its neighbor vertices. The height $h(S_k(v))$ and weight $w(S_k(v))$ of $S_k(v)$ is the maximum degree and degree-sum of its vertices, respectively. The height $h_k(G)$ and weight $w_k(G)$ of G is the maximum height and weight of its k-stars.

Lebesgue (1940) proved that every 3-polytope of girth g at least 5 has a 2-star (a path of three vertices) with $h_2 = 3$ and $w_2 = 9$. Madaras (2004) refined this by showing that there is a 3-star with $h_3 = 4$ and $w_3 = 13$, which is tight. In 2015, we gave another tight description of 3-stars for girth $g = 5$ in terms of degree of their vertices and showed that there are only these two tight descriptions of 3-stars.

In 2013, we gave a tight description of 3^--stars in arbitrary plane graphs with minimum degree δ at least 3 and $g \geq 3$, which extends or strengthens several previously known results by Balogh, Jendrol', Harant, Kochol, Madaras, Van den Heuvel, Yu and others and disproves a conjecture by Harant and Jendrol' posed in 2007.

There exist many tight results on the height, weight and structure of 2^--stars when $\delta = 2$. In 2016, Hudák, Maceková, Madaras, and Široczki considered the class of plane graphs with $\delta = 2$ in which no two vertices of degree 2 are adjacent. They proved that $h_3 = w_3 = \infty$ if $g \leq 6$, $h_3 = 5$ if $g = 7$, $h_3 = 3$ if $g \geq 8$, $w_3 = 10$ if $g = 8$ and $w_3 = 3$ if $g \geq 9$. For $g = 7$, Hudák et al. proved $11 \leq w_3 \leq 20$.

The purpose of our paper is to prove that every plane graph with $\delta = 2$, $g = 7$ and no adjacent vertices of degree 2 has $w_3 = 12$.

1. Introduction

The degree of a vertex v or a face f in a plane graph G, that is the number of edges incident with v or f, is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geq k$, etc.

Let $\delta(G)$ be the minimum vertex degree and $g(G)$ be the girth (the length of a shortest cycle) in G. A *k-star* $S_k(v)$ in G consists of a central vertex v and k its neighbor vertices. The height $h(S_k(v))$ and weight $w(S_k(v))$ of $S_k(v)$ is the maximum degree and degree-sum of its vertices, respectively. The height $h_k(G)$ and weight $w_k(G)$ of G is the maximum height and weight of its k-stars. We will often drop the argument when the graph is clear from context.

An edge uv, that is an $S_1(u)$ or $S_1(v)$, is an *(i,j)-edge* if $d(u) \leq i$ and $d(v) \leq j$. More generally, a path $v_1v_2v_3$ (which is an $S_2(v_2)$), is a *path of type (i_1, i_2, i_3)*, or a *(i_1, i_2, i_3)-path* if $d(v_j) \leq i_j$ whenever $1 \leq j \leq 3$. The types of higher stars are defined similarly.

Already in 1904, Wernicke [48] proved that every G with $\delta = 5$ satisfies $w_2 \leq 11$, and Franklin [33] strengthened this to the existence of a $(6, 5, 6)$-path, which description is tight. In [15], we proved that there is another tight description, "a $(5, 6, 6)$-path" and that no other tight descriptions exists.

In [10], we gave a tight description of 3^--stars in arbitrary plane graphs with $\delta \geq 3$ and $g \geq 3$ by proving that there is either a $(3, 10)$-edge, or a $(5, 4, 9)$-path, or a $(6, 4, 8)$-path, or a $(7, 4, 7)$-path, or a $(5; 4, 5, 5)$-star, or a $(5; 5, b, c)$-star with $5 \leq b \leq 6$ and $5 \leq c \leq 7$, or else a $(5; 6, 6, 6)$-star. This extends or strengthens several previously known results by Balogh, Jendrol', Harant, Kochol, Madaras, Van den Heuvel, Yu and others [4, 34, 42, 46, 47] and disproves a conjecture in Harant, Jendrol' [34].

In 1940, Lebesgue [44] gave an approximate description of 5-stars centered at 5-vertices for the case $\delta = 5$ and $g \geq 3$. Recently, we obtained several tight results on the height, weight and structure of such 5-stars assuming the absence of 6^+-vertices from certain degree-sets, see [12, 16, 19, 21, 23, 24, 26, 27, 29–31].

Also, Lebesgue [44] proved that every G with $\delta \geq 3$ and $g = 5$ satisfies $h_2 = 3$ and $w_2 = 9$. In 2004, Madaras [45] refined this by showing that there is a 3-star with $h_3 = 4$ and $w_3 = 13$, which is tight. Recently, we gave [22] another tight description of 3-stars for $g = 5$ in terms of degree of their vertices and showed that there are only these two tight descriptions of 3-stars.

There exist many results on the height, weight and structure of 2^--stars when $\delta = 2$, see, for example, [1–8, 11, 14, 17, 25, 28, 35–40] and also surveys by Jendrol, Voss [43] and Borodin, Ivanova [20].

In 2016, Hudák, Maceková, Madaras and Široczki [35] considered the class of plane graphs with $\delta = 2$ in which no two vertices of degree 2 are adjacent. They proved that $h_3 = w_3 = \infty$ if $g \leq 6$, $h_3 = 5$ if $g = 7$, $h_3 = 3$ if $g \geq 8$, $w_3 = 10$ if $g = 8$ and $w_3 = 9$ if $g \geq 9$. For $g = 7$, Hudák et al. [35] proved $11 \leq w_3 \leq 20$.

The purpose of our paper is to settle the case $g = 7$, as follows.

Theorem 1. *Every plane graph with $\delta = 2$ and $g = 7$ in which no two vertices of degree 2 are adjacent has a 3-star of weight at most 12 centered at a 5^--vertex, where 12 is best possible.*

2. Proof of Theorem 1

In Fig. 1, we see a bit more than a half of a plane graph with the desired properties $\delta = 2$, $g = 7$, and no $(2,2)$-path that confirms the lower bound $w_3 \geq 12$. The invisible "equator" passes through the middles of ten edges joining 3-vertices with 5-vertices in the outside layer. To obtain the whole graph, it suffices to superpose vertices of outer cycle of one half with the vertices of the cycle nearest to it of the other half. Namely, the path of vertices of degrees 5, 3, 2, and 3 of the shaded face are superposed on the path of vertices of degrees 5, 3, 2, and 3 of the white face.
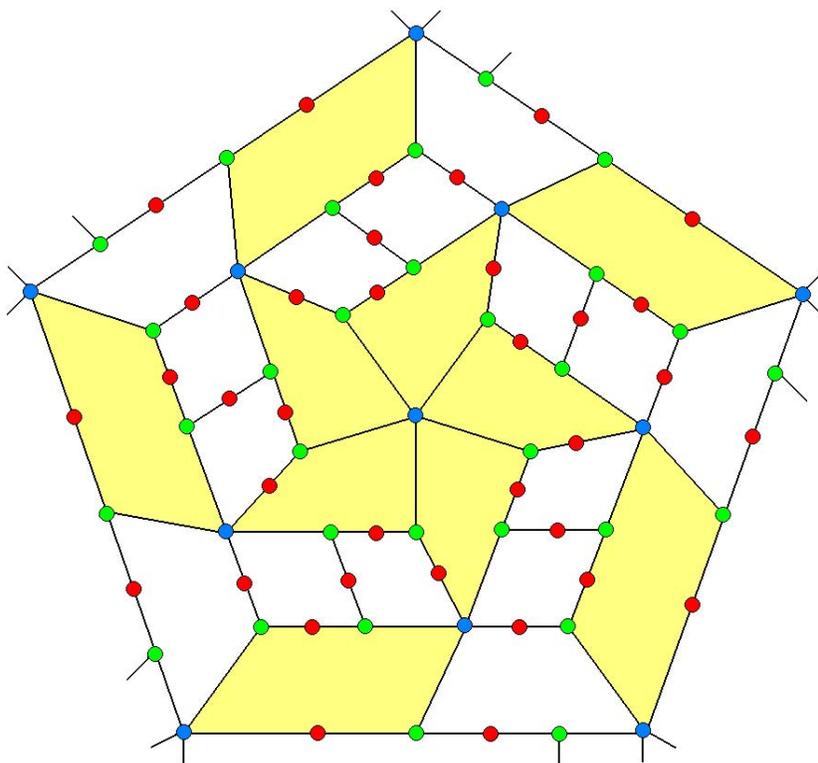
Рис. 1. All 3-stars are of weight at least12.

2.1. **Discharging and its consequences.** Let G be a counterexample to the upper bound 12 in Theorems 1. Without loss of generality, we can assume that G is connected. Let V, E, and F be the sets of vertices, edges and faces of G, respectively. Euler's formula $|V| - |E| + |F| = 2$ for G may be rewritten as

$$(1) \qquad \sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12.$$

Every vertex v contributes the *charge* $\mu(v) = d(v) - 6$ to (1), so only the charges of 5^--vertices are negative. Every face f contributes the non-negative *charge* $\mu(f) = 2d(f) - 6$ to (1). Using the properties of G as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12.

Throughout the paper, we denote the vertices adjacent to a vertex or incident with face x in a cyclic order by $v_1, \ldots, v_{d(x)}$. Let $\partial(f)$ be the boundary of a face f.

We apply the following rules of discharging (see Fig. 2).

R1. *Every face gives* 2 *to each incident 2-vertex.*

R2. *Every face gives* $\frac{1}{3}$ *to each incident vertex* v_2 *such that* $3 \le d(v_2) \le 5$ *and* $d(v_1) = d(v_3) = 2$.

R2. *Every face gives the following charge to each incident 3-vertex* v_2:
(a) $\frac{4}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) \ge 6$,
(b) $\frac{7}{6}$ *if* $d(v_1) \le 3$ *and* $d(v_3) = 5$,
(c) 1 *if* $d(v_3) = 4$,
(d) $\frac{2}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) = 3$,
(e) 1 *if* $d(v_1) = 3$ *while* $d(v_3) = 3$ *or* $d(v_3) \ge 6$, *and*
(f) 1 *if* $d(v_1) \ge 5$ *and* $d(v_3) \ge 5$.

R4. *Every face gives the following charge to each incident 4-vertex* v_2:
(a) $\frac{1}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) = 3$,
(b) $\frac{2}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) \ge 4$, *and*
(c) $\frac{2}{3}$ *if* $d(v_1) \ge 3$ *and* $d(v_3) \ge 3$.

R5. *Every face gives the following charge to each incident 5-vertex* v_2:
(a) $\frac{1}{6}$ *if* $d(v_1) = 2$ *and* $d(v_3) = 3$,
(b) $\frac{1}{3}$ *if* $d(v_1) = 2$ *and* $d(v_3) \ge 4$, *and*
(c) $\frac{1}{3}$ *if* $d(v_1) \ge 3$ *and* $d(v_3) \ge 3$.

2.2. Proving $\mu'(v) \ge 0$ whenever $v \in V$.

CASE 1. $d(v) = 2$. Here, $\mu'(v) = d(v) - 6 + 2 \times 2 = 0$ by R1.

CASE 2. $d(v) = 3$.

SUBCASE 2.1. $d(v_1) = d(v_2) = 2$. Now $d(v_3) \ge 13 - 3 - 2 \times 2 = 6$, so $\mu'(v) = -3 + 2 \times \frac{4}{3} + \frac{1}{3} = 0$ by R2 and R3a.

SUBCASE 2.2. $d(v_1) = 2$ and $d(v_2) = 3$. Now $d(v_3) \ge 5$, so $\mu'(v) = 3 - 6 + 2 \times \frac{7}{6} + \frac{2}{3} = 0$ by R3b, R3d.

SUBCASE 2.3. $d(v_1) = 2$, $d(v_2) \ge 4$, and $d(v_3) \ge 4$. Now v receives at least 1 from each incident face by R3a–R3c, so $\mu'(v) \ge 0$.

SUBCASE 2.4. $d(v_1) \ge 3$, $d(v_2) \ge 3$, and $d(v_3) \ge 4$. Now again v receives at least 1 from each incident face, but this time by R3b, R3c, R3e, and R3f, and we have $\mu'(v) \ge 0$.

CASE 3. $d(v) = 4$. Note that each incident face gives v either $\frac{1}{3}$ by R2 and R4a or $\frac{2}{3}$ by R4b and R4c. We have $\mu'(v) \ge -2 + 2 \times \frac{1}{3} + 2 \times \frac{2}{3} = 0$, unless v receives $\frac{1}{3}$ at

R1

R2

R3a

R3b

R3c

R3d

R3e

R3f

R4a

R4b

R4c

R5a

R5b

R5c
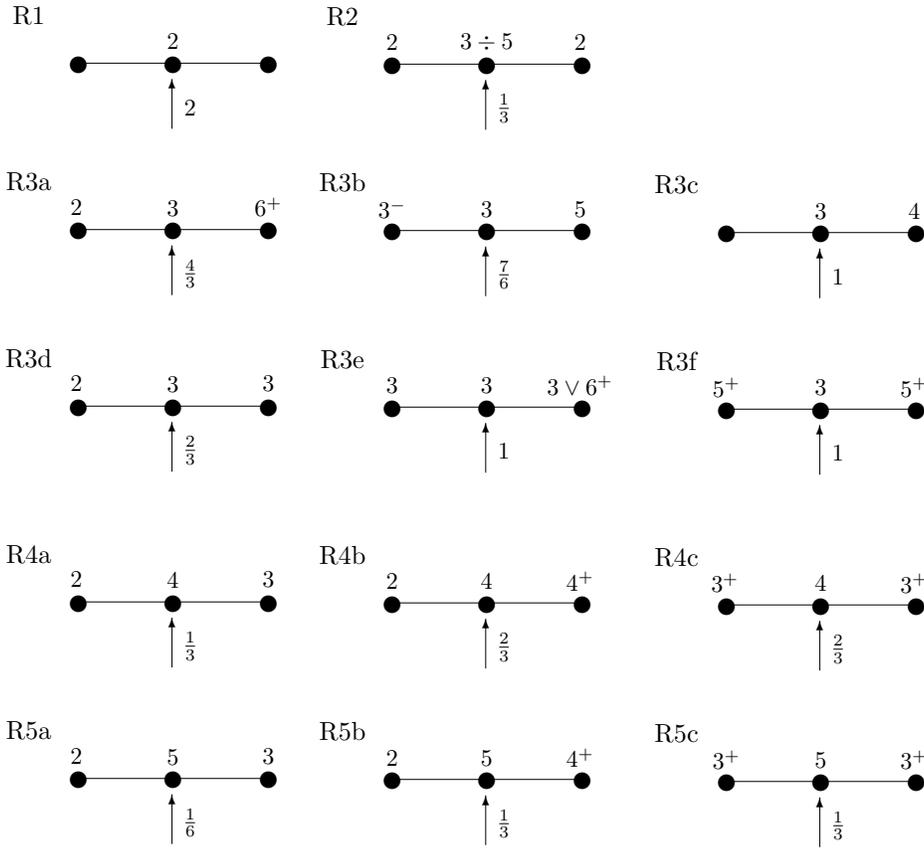
Рис. 2. Rules of discharging

least thrice, in which case v has three 3^--neighbors including a 2-neighbor. However, this implies a 3-star at v of weight at most $4 + 2 \times 3 + 2 < 13$, a contradiction.

CASE 4. $d(v) = 5$. Now each incident face gives v either $\frac{1}{3}$ by R2 and R5b, R5c or $\frac{1}{6}$ by R5a. It suffices to exclude the possibility that v receives $\frac{1}{6}$ five times, since otherwise we have $\mu'(v) \geq -1 + \frac{1}{3} + 4 \times \frac{1}{6} = 0$. This follows easily from the fact that 2-neighbors cannot alternate with 3-neighbors around v.

CASE 5. $d(v) \geq 6$. Since v does not participate in discharging, we have $\mu'(v) = d(v) - 6 \geq 0$.

2.3. **Proving $\mu'(f) \geq 0$ whenever $f \in F$.**

Lemma 1. *Every face f gives the following total charge to strings of incident 3^+-vertices by R2–R5:*
    (a) at most $\frac{1}{3}$ to v_2 if $d(v_1) = d(v_3) = 2$,
    (b) at most $\frac{4}{3}$ to v_2v_3 if $d(v_1) = d(v_4) = 2$,
    (c) at most $\frac{8}{3}$ to $v_2v_3v_4$ if $d(v_1) = d(v_5) = 2$, and
    (d) at most $\frac{11}{6}$ to v_2v_3 if $d(v_1) = 2$ while $d(v_4) \geq 3$,

Proof. (a) This is precisely the rule R2.

(b) If $d(v_2) \geq 6$, then v_2 receives nothing while v_3 receives at most $\frac{4}{3}$ by R3–R5.

Suppose $d(v_2) = 5$. If $d(v_3) = 3$ then v_2 receives $\frac{1}{6}$ by R5a while v_3 receives $\frac{7}{6}$ by R3b, as desired. If $4 \leq d(v_3) \leq 5$ then v_2 receives $\frac{1}{3}$ by R5b while v_3 at most 1 by R5b or R3c.

Next suppose $d(v_2) = 4$. If $d(v_3) = 3$ then v_2 receives $\frac{1}{3}$ by R4a while v_3 receives 1 by R3c. If $d(v_3) = 4$ then each of v_2, v_3 receives $\frac{2}{3}$ by R4b.

Finally, if $d(v_2) = d(v_3) = 3$ then each of v_2, v_3 receives $\frac{2}{3}$ by R3d.

(c) If $d(v_3) \geq 6$, $d(v_3) = 5$, or $d(v_3) = 4$, then $v_2 v_3 v_4$ receives at most $\frac{4}{3} + 0 + \frac{4}{3}$, $\frac{7}{6} + \frac{1}{3} + \frac{7}{6}$, or $1 + \frac{2}{3} + 1$, respectively.

Finally, suppose $d(v_3) = 3$. If $d(v_2) \notin \{3, 5\}$, then $v_2 v_3 v_4$ receives at most $\frac{1}{3} + 1 + \frac{2}{3}$. Now if $d(v_2) = 5$, then $v_2 v_3 v_4$ receives at most $\frac{1}{6} + \frac{7}{6} + \frac{2}{3}$ by R3b, R3d, R4a, R5a. By symmetry, it remains to assume that $d(v_2) = d(v_4) = 3$, in which case $v_2 v_3 v_4$ receives at most $\frac{2}{3} + 1 + \frac{2}{3}$, as desired.

(d) If there is a 5^+-vertex in $v_2 v_3$, then $v_2 v_3$ receives at most $\frac{1}{3} + \frac{4}{3} < \frac{11}{6}$, as desired. Suppose the contrary. Now if v_2 receives at most $\frac{2}{3}$, then $v_2 v_3$ receives at most $\frac{2}{3} + \frac{7}{6} = \frac{11}{6}$ since v_3 never receives as much as $\frac{4}{3}$. Otherwise, v_2 receives 1 by R3c, in which case $d(v_2) = 4$ or $d(v_3) = 4$ and $v_2 v_3$ receives $1 + \frac{2}{3}$ in both cases in view of R4c, and we are done again. ∎

A *k-worm* is a string $v_2 \ldots v_{k+1}$ of 3^+-vertices in the boundary $\partial(f)$ of a face f such that $d(v_1) = d(v_{k+2}) = 2$. In particular, we can have $v_1 = v_{d(v)+1}$, in which case $\partial(f)$ is split into a 2-vertex v_1 and a $(d(v) - 1)$-worm. If f is incident with at least one 2-vertex, then $\partial(f)$ is split into k-worms with variable k.

To estimate the total donation of f by R1–R5, we use the following $\frac{5}{12}$-*averaging*: every 2-vertex v_2 transfers $\frac{5}{12}$ to v_1 and v_3.

The $\frac{5}{12}$-averaged donation of f to a 2-neighbor is $2 - 2 \times \frac{5}{12} = \frac{7}{6}$. Lemma 1 easily implies the maximum donations to k-worms.

Corollary 1. *The $\frac{5}{12}$-averaged total donations of f to k-worms are:*
(a) at most $\frac{1}{3} + 2 \times \frac{5}{12} = \frac{7}{6}$ if $k = 1$,
(b) at most $\frac{4}{3} + \frac{5}{6} = 2 \times \frac{7}{6} - \frac{1}{6}$ if $k = 2$,
(c) at most $\frac{8}{3} + \frac{5}{6} = 3 \times \frac{7}{6}$ if $k = 3$, and
(d) at most $\frac{5}{6} + 2 \times \frac{11}{6} + (k - 4) \times \frac{7}{6} = k \times \frac{7}{6} - \frac{1}{6}$ if $k \geq 4$.

Proof. The only remark possibly needed is that a 3^+-vertex at distance at least 2 from the nearest 2-vertex in $\partial(f)$ receives at most $\frac{7}{6}$ from f both by R2–R5 and after the $\frac{5}{12}$-averaging. ∎

We are now prepared to complete the proof of $\mu'(f) \geq 0$.

CASE 1. $d(f) = 7$. Note that $\mu(f) = 2 \times 7 - 6 = 7 \times \frac{7}{6} - \frac{1}{6}$. This means that we are done by Corollary 1 unless $\partial(f)$ has neither 2-worms nor 4^+-worms.

If f is incident with three 2-vertices, then $\partial(f)$ consists of a 2-worm and two 1-worms, and we are done.

If f is incident with two 2-vertices, then the five incident vertices of f are split into two worms, one of which must be a 2-worm or 4-worm. The presence of precisely one 2-vertex in $\partial(f)$ implies a 6-worm.

Finally, suppose f is not incident with 2-vertices. Now it suffices to observe that there are no three consecutive vertices in $\partial(f)$ each receiving $\frac{7}{6}$ by R2–R5 to see that $\mu'(f) \geq 8 - 6 \times \frac{7}{6} - 1 = 0$.

CASE 2. $d(f) \geq 8$. Since the $\frac{5}{12}$-averaged donation of f to a 2-neighbor is $\frac{7}{6}$, each k-worm receives at most $k \times \frac{7}{6}$ by Corollary 1, and each 3^+-vertex not belonging to a worm receives at most $\frac{7}{6}$, we have $\mu'(f) \geq 2d(f) - 6 - d(f) \times \frac{7}{6} = \frac{5d(f)-36}{6} > 0$.

This completes the proof of Theorem 1.

References

[1] V.A. Aksenov, O.V. Borodin, A.O. Ivanova, *Weight of 3-paths in sparse plane graphs*, Electronic J. Combin., **22**:3 (2015), Paper #P3.28. http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p28 MR3414174

[2] V.A. Aksenov, O.V. Borodin, A.O. Ivanova, *An extension of Kotzig's Theorem*, Discuss. Math. Graph Theory, **36**:4 (2016), 889–897. DOI 10.7151/dmgt.1904 MR3557208

[3] K. Ando, S. Iwasaki, A. Kaneko, *Every 3-connected planar graph has a connected subgraph with small degree sum*, Annual Meeting of Mathematical Society of Japan (1993). (in Japanese)

[4] J. Balogh, M. Kochol, A. Pluhár, X. Yu, *Covering planar graphs with forests*, J. Combin. Theory, Ser. B, **94**:1 (2005), 147–158. DOI 10.1016/j.jctb.2004.12.002 MR2130283

[5] Ts.Ch-D. Batueva, O.V. Borodin, M.A. Bykov, A.O. Ivanova, O.N. Kazak, D.V. Nikiforov, *Refined weight of edges in normal plane maps*, Discrete Math., **340**:11 (2017), 2659–2664. DOI 10.1016/j.disc.2016.10.007 MR3689914

[6] O.V. Borodin, *On the total coloring of planar graphs*, J. Reine Angew. Math., **394** (1989), 180–185. MR0977440

[7] O.V. Borodin, *Joint extension of two Kotzig's theorems on 3-polytopes*, Combinatorica, **13**:1 (1992), 121–125.

[8] O.V. Borodin, *The structure of neighborhoods of an edge in planar graphs and the simultaneous coloring of vertices, edges and faces*, Mat. Zametki, **53**:5 (1993), 35–47. (in Russian) MR1325613

[9] O.V. Borodin, *Minimal vertex degree sum of a 3-path in plane maps*, Discuss. Math. Graph Theory, **17**:2 (1997), 279–284. MR1627959

[10] O.V. Borodin, A.O. Ivanova, *Describing $(d-2)$-stars at d-vertices, $d \leq 5$, in normal plane maps*, Discrete Math., **313**:17 (2013), 1700–1709. DOI 10.1016/j.disc.2013.04.026 MR3061006

[11] O.V. Borodin, A.O. Ivanova, *List injective colorings of planar graphs*, Discrete Math., **311**:2–3 (2011), 154—165. DOI 10.1016/j.disc.2010.10.008 MR2739919

[12] O.V. Borodin, A.O. Ivanova, *Describing 4-stars at 5-vertices in normal plane maps with minimum degree 5*, Discrete Math., **313**:17 (2013), 1710–1714. DOI 10.1016/j.disc.2013.04.025 MR3061007

[13] O.V. Borodin, A.O. Ivanova, *Describing tight descriptions of 3-paths in triangle-free normal plane maps*, Discrete Math., **338**:11 (2015), 1947–1952. DOI 10.1016/j.disc.2015.05.006 MR3357780

[14] O.V. Borodin, A.O. Ivanova, *Low edges in 3-polytopes*, Discrete Math., **338**:12 (2015), 2234–2241. DOI 10.1016/j.disc.2015.05.018 MR3373326

[15] O.V. Borodin, A.O. Ivanova, *An analogue of Franklin's Theorem*, Discrete Math., **339**:10 (2016), 2553–2556. DOI 10.1016/j.disc.2016.04.019 MR3508768

[16] O.V. Borodin, A.O. Ivanova, *Light neighborhoods of 5-vertices in 3-polytopes with minimum degree 5*, Sib. Elektron. Math. Reports, **13** (2016), 584–591. DOI 10.17377/semi.2016.13.045 MR3528631

[17] O.V. Borodin, A.O. Ivanova, *Weight of edges in normal plane maps*, Discrete Math., **339**:5 (2016), 1507–1511. DOI 10.1016/j.disc.2015.12.021 MR3475562

[18] O.V. Borodin, A.O. Ivanova, *Low stars in normal plane maps with minimum degree 4 and no adjacent 4-vertices*, Discrete Math., **339**:2 (2016), 923–930. DOI 10.1016/j.disc.2015.10.018 MR3431405

[19] O.V. Borodin, A.O. Ivanova, *Light and low 5-stars in normal plane maps with minimum degree 5*, Sib. Math. J., **57**:3 (2016), 470–475. DOI 10.1134/S0037446616030071 MR3548785

[20] O.V. Borodin, A.O. Ivanova, *New results about the structure of plane graphs: a survey*, AIP Conference Proceedings, **1907** (2017), 030051. DOI 10.1063/1.5012673

[21] O.V. Borodin, A.O. Ivanova, *On light neighborhoods of 5-vertices in 3-polytopes with minimum degree* 5, Discrete Math., **340**:9 (2017), 2234–2242. DOI 10.1016/j.disc.2017.04.012 MR3665077

[22] O.V. Borodin, A.O. Ivanova, *All tight descriptions of 3-stars in 3-polytopes with girth* 5, Discuss. Math. Graph Theory, **37**:1 (2017), 5–12. DOI 10.7151/dmgt.1905 MR3601028

[23] O.V. Borodin, A.O. Ivanova, *Low 5-stars in normal plane maps with minimum degree* 5, Discrete Math., **340**:2 (2017), 18–22. DOI 10.1016/j.disc.2016.07.013 MR3578794

[24] O.V. Borodin, A.O. Ivanova, T.R. Jensen, *5-stars of low weight in normal plane maps with minimum degree* 5, Discuss. Math. Graph Theory, **34**:3 (2014), 539–546. DOI 10.7151/dmgt.1748 MR3227046

[25] O.V.Borodin, A.O.Ivanova, T.R.Jensen, A.V.Kostochka, M.P.Yancey, *Describing 3-paths in normal plane maps*, Discrete Math., **313:23** (2013), 2702–2711. DOI 10.1016/j.disc.2013.08.018

[26] O.V. Borodin, A.O. Ivanova, O.N. Kazak, *Describing neighborhoods of 5-vertices in 3-polytopes with minimum degree 5 and without vertices of degrees from 7 to 11*, Discuss. Math. Graph Theory, **38**:3 (2018), 615–625. DOI 10.7151/dmgt.2024 MR3811956

[27] O.V. Borodin, A.O. Ivanova, O.N. Kazak, E.I. Vasil'eva, *Heights of minor 5-stars in 3-polytopes with minimum degree 5 and no vertices of degree 6 and 7*, Discrete Math., **341**:3 (2018), 825–829. DOI 10.1016/j.disc.2017.11.021 MR3754396

[28] O.V. Borodin, A.O. Ivanova, A.V. Kostochka, *Tight descriptions of 3-paths in normal plane maps*, J. Graph Theory, **85**:1 (2017), 115–132. DOI 10.1002/jgt.22051 MR3634478

[29] O.V. Borodin, A.O. Ivanova, D.V. Nikiforov, *Low and light minor 5-stars in 3-polytopes with minimum degree 5 and restrictions on the degrees of major vertices*, Sib. Math. J., **58**:4 (2017), 600–605. DOI 10.1134/S003744661704005X MR3762353

[30] O.V. Borodin, A.O. Ivanova, D.V. Nikiforov, *Low minor 5-stars in 3-polytopes with minimum degree 5 and no 6-vertices*, Discrete Math., **340**:7 (2017), 1612–1616. DOI 10.1016/j.disc.2017.03.002 MR3634131

[31] O.V. Borodin, A.O. Ivanova, D.V. Nikiforov, *Describing neighborhoods of 5-vertices in a class of 3-polytopes with minimum degree 5*, Sib. Math. J., **59**:1 (2018), 43–49. DOI 10.1134/S0037446618010056

[32] B. Ferencová, T. Madaras, *Light graphs in families of polyhedral graphs with prescribed minimum degree, face size, edge and dual edge weight*, Discrete Math., **310**:12 (2010), 1661–1675. DOI 10.1016/j.disc.2009.11.027 MR2610267

[33] Ph. Franklin, *The four-color problem*, Amer. J. Math., **44**:3 (1922), 225–236. MR1506473

[34] J. Harant, S. Jendrol', *On the Existence of Specific Stars in Planar Graphs*, Graphs and Combinatorics, **23**:5 (2007), 529–543. MR2346422

[35] P. Hudák, M. Maceková, T. Madaras, P. Široczki, *Light graphs in planar graphs of large girth*, Discuss. Math. Graph Theory, **36**:1 (2016), 227—238. DOI 10.7151/dmgt.1847 MR3449146

[36] S. Jendrol', *A structural property of convex 3-polytopes*, Geom. Dedicata, **68**:1 (1997), 91–99. MR1485387

[37] S. Jendrol', *Paths with restricted degrees of their vertices in planar graphs*, Czechoslovak Math. J., **49(124)**:3 (1999), 481–490. MR1485387

[38] S. Jendrol', M. Maceková, *Describing short paths in plane graphs of girth at least 5*, Discrete Math., **338**:2 (2015), 149–158. DOI 10.1016/j.disc.2014.09.014, MR3279266

[39] S. Jendrol', M. Maceková, M. Montassier, R. Soták, *Optimal unavoidable sets of types of 3-paths for planar graphs of given girth*, Discrete Math., **339**:2 (2016), 780–789. DOI 10.1016/j.disc.2015.10.016, MR3431391

[40] S. Jendrol', M. Maceková, M. Montassier, R. Soták, *3-paths in graphs with bounded average degree*, Discuss. Math. Graph Theory, **36**:2 (2016), 339–353. DOI 10.7151/dmgt.1859, MR3482533

[41] S. Jendrol', M. Maceková and R. Soták, *Note on 3-paths in plane graphs of girth 4*, Discrete Math., **338**:9 (2015), 1643–1648. DOI 10.1016/j.disc.2015.04.011, MR3345601

[42] S. Jendrol', T. Madaras, *On light subgraphs in plane graphs of minimal degree five*, Discuss., Math. Graph Theory, **16**:2 (1996), 207–217. DOI 10.7151/dmgt.1859, MR1446358

[43] S. Jendrol', H.-J. Voss, *Light subgraphs of graphs embedded in the plane — a survey*, Discrete Math., **313**:4 (2013), 406–421. DOI 10.1016/j.disc.2012.11.007, MR3004475

[44] H. Lebesgue, *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl., **19** (1940), 27–43. (in French) MR0001903

[45] T. Madaras, *On the structure of plane graphs of minimum face size* 5, Discuss. Math. Graph Theory, **24**:3 (2004), 403–411. DOI 10.7151/dmgt.1859, MR2120064

[46] T. Madaras, *Two variations of Franklin's theorem*, Tatra Mt. Math. Publ., **36** (2007), 61–70. MR2378741

[47] J. Van den Heuvel, S. McGuinness, *Coloring the square of a planar graph*, J. Graph Theory, **42**:2 (2003), 110–124. DOI 10.1002/jgt.10077, MR1953224

[48] P.Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann., **58** (1904), 413–426. (in German) MR1511242

OLEG VENIAMINOVICH BORODIN
SOBOLEV INSTITUTE OF MATHEMATICS,
PR. KOPTYUGA, 4,
630090, NOVOSIBIRSK, RUSSIA
E-mail address: `brdnoleg@math.nsc.ru`

ANNA OLEGOVNA IVANOVA
AMMOSOV NORTH-EASTERN FEDERAL UNIVERSITY,
STR. KULAKOVSKOGO, 48,
677000, YAKUTSK, RUSSIA
E-mail address: `shmgnanna@mail.ru`