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FINITE ALMOST SIMPLE GROUPS WHOSE GRUENBERG-KEGEL GRAPHS AS ABSTRACT GRAPHS ARE ISOMORPHIC TO SUBGRAPHS OF THE GRUENBERG-KEGEL GRAPH OF THE ALTERNATING GROUP A_{10}

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Abstract. We consider the problem of describing finite groups whose the Gruenberg-Kegel graphs as abstract graphs are isomorphic to the Gruenberg-Kegel graph of the alternating group A_{10}. In the given paper, we prove that if such group is non-solvable then its quotient group by solvable radical is almost simple and classify all finite almost simple groups whose the Gruenberg-Kegel graphs as abstract graphs are isomorphic to subgraphs of the Gruenberg-Kegel graph of A_{10}.

Keywords: finite group, almost simple group, 4-primary group, Gruenberg-Kegel graph.

1. Introduction

Let G be a finite group. Denote by $\pi(G)$ the set of prime divisors of the order of G. The Gruenberg-Kegel graph (prime graph) $\Gamma(G)$ of G is a graph with the vertex set $\pi(G)$, in which two different vertices p and q are adjacent if and only if there exists an element of order pq in G. If $|\pi(G)| = n$ then the group G is called n-primary.

In 2012–2013, the first author described finite groups with the Gruenberg-Kegel graphs as for groups $Aut(J_2)$ (see [7]) and A_{10} (see [8]), respectively. The Gruenberg-Kegel graphs of these groups as abstract graphs are isomorphic to the graph Γ of the form

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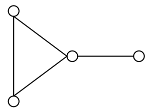


We put a more general problem: to describe finite groups whose Gruenberg-Kegel graphs as abstract graphs are isomorphic to the graph Γ.

In the frame of this problem, we prove in this paper the following two theorems.

Theorem 1. *Let G be a finite non-solvable group and the graph $\Gamma(G)$ as abstract graph is isomorphic to the graph $\Gamma(A_{10})$. Then quotient group $G/S(G)$ is almost simple.*

Theorem 2. *Let G be a finite almost simple group. Then the graph $\Gamma(G)$ as abstract graph is isomorphic to a subgraph of the graph $\Gamma(A_{10})$ if and only if one of the following statements holds:*

(*a*) *the graph $\Gamma(G)$ is disconnected and the group G is isomorphic to one of the following groups:*

(1) *A_n for $5 \le n \le 9$, S_n for $5 \le n \le 8$, M_{10}, $L_2(q)$ for $q \in \{7, 8, 16, 17, 25, 49, 81\}$, $PGL_2(q)$ for $q \in \{7, 9, 17\}$, $L_2(q).2$ for $q \in \{16, 25, 49, 81\}$, $Aut(L_2(16))$, $L_2(27).3$, $L_2(81).4_1$, $L_2(81).4_2$, $L_3(q)$ for $q \in \{3, 4, 5, 7, 8, 17\}$, $Aut(L_3(q))$ for $q \in \{3, 5, 8, 17\}$, $L_3(q).2$ for $q \in \{2, 7, 8\}$, $L_3(8).3$, $L_4(3)$, $L_4(3).2_2$, $L_4(3).2_3$, $S_4(q)$ for $q \in \{3, 4, 5, 7, 9\}$, $Aut(S_4(q))$ for $q \in \{3, 4\}$, $S_4(4).2$, $S_4(9).2_1$, $S_4(9).2_3$, $S_6(2)$, $Aut(S_6(2))$, $U_3(q)$ for $q \in \{3, 4, 5, 7, 8, 9\}$, $Aut(U_3(q))$ for $q \in \{4, 5, 7, 9\}$, $U_3(q).2$ for $q \in \{5, 8, 9\}$, $U_3(8).3_1$, $U_3(8).3_3$, $U_3(8).6$, $U_4(3)$, $U_4(3).2_2$, $U_4(3).2_3$, $U_5(2)$, $Aut(U_5(2))$, $Sz(8)$, $Sz(32)$, $Aut(Sz(32))$, ${}^3D_4(2)$, $Aut({}^3D_4(2))$, ${}^2F_4(2)'$, ${}^2F_4(2)$, M_{11}, M_{12}, $Aut(M_{12})$, J_2;*

(2) *$L_2(r)$ or $PGL_2(r)$, where r is a prime, $17 \ne r \ge 11$, $r^2 - 1 = 2^a 3^b s^c$, $s > 3$ is a prime, $a, b \in \mathbb{N}$ and c equals to either 1 or 2 for $r \in \{97, 577\}$;*

(3) *$L_2(2^m)$, where m, $2^m - 1$ and $(2^m + 1)/3$ are primes greater than 3;*

(4) *$L_2(3^m)$ or $PGL_2(3^m)$, where m and $(3^m - 1)/2$ are odd primes and $(3^m + 1)/4$ equals to either a prime or 11^2 (for $m = 5$);*

(*b*) *the graph $\Gamma(G)$ is connected and the group G is isomorphic to one of the following groups: $Aut(A_6)$, S_9, A_{10}, $Aut(L_2(q))$ for $q \in \{25, 27, 49, 81\}$, $L_2(81).2^2$, $PGL_3(4)$, $L_3(4).6$, $L_3(4).2^2$, $PGL_3(4).2_2$, $PGL_3(4).2_3$, $Aut(L_3(4))$, $PGL_3(7)$, $Aut(L_3(7))$, $L_4(3).2_1$, $Aut(L_4(3))$, $Aut(S_4(q))$ for $q \in \{5, 7, 9\}$, $S_4(9).2_2$, $Aut(U_3(q))$ for $q \in \{5, 8\}$, $U_3(5).3$, $U_3(8).3_2$, $U_3(8).3^2$, $U_3(8).S_3$, $U_4(3).2_1$, $U_4(3).2^2$, $U_4(3).4$, $Aut(U_4(3))$, $O_8^+(2).2$, $O_8^+(2).3$, $Aut(J_2)$.*

Theorem 2 and [2, 4] imply

Corollary 1. *Let G be a finite almost simple group whose Gruenberg-Kegel graph as abstract graph is isomorphic to $\Gamma(A_{10})$. Then G is isomorphic one of the following groups: S_9, A_{10}, $Aut(L_3(4))$, $PGL_3(7)$, $Aut(L_3(7))$, $Aut(S_4(5))$, $Aut(S_4(7))$, $S_4(9).2_2$, $Aut(S_4(9))$, $U_3(5).3$, $Aut(U_3(5))$, $U_3(8).3_2$, $U_3(8).3^2$, $U_3(8).S_3$, $Aut(U_3(8))$, $O_8^+(2).2$, $O_8^+(2).3$, $Aut(J_2)$.*

For the proving of Theorems, we use the results from [9, 10].

2. Notation and auxiliary results

We use mainly standard notation and terminology (see [2, 4]). The solvable radical (the largest solvable normal subgroup) of a finite group G is denoted by $S(G)$.

For the proving of Theorems, we need the following two lemmas.

Lemma 1 ([5]). *If G be a finite simple 3-primary group, then G is isomorphic to one of following groups: A_5, $L_2(7)$, A_6, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.*

Lemma 2 ([1, 6, 11]). *Let G be a finite simple 4-primary group. Then G is isomorphic to one of the following groups:*

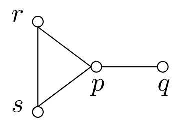
(1) A_n for $7 \leq n \leq 10$, $L_2(q)$ for $q \in \{16, 25, 49, 81\}$, $L_3(q)$ for $q \in \{4, 5, 7, 8, 17\}$, $L_4(3)$, $S_4(q)$ for $q \in \{4, 5, 7, 9\}$, $S_6(2)$, $U_3(q)$ for $q \in \{4, 5, 7, 8, 9\}$, $U_4(3)$, $U_5(2)$, $O_8^+(2)$, $G_2(3)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, M_{11}, M_{12}, J_2;

(2) $L_2(r)$, where r is a prime, $17 \neq r \geq 11$, $r^2 - 1 = 2^a 3^b s^c$, $s > 3$ is a prime, $a, b \in \mathbb{N}$, and c equals to either 1 or 2 for $r \in \{97, 577\}$;

(3) $L_2(2^m)$, where m, $2^m - 1$ and $(2^m + 1)/3$ are primes greater than 3;

(4) $L_2(3^m)$, where m and $(3^m - 1)/2$ are odd primes, and $(3^m + 1)/4$ equals to either a prime or 11^2 (for $m = 5$).

Further we shall assume that the Gruenberg-Kegel graph of a investigated group has the form



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3. Proof of theorem 1

Let G be a group satisfying the conditions of Theorem 1 and $\overline{G} = G/S(G)$. Show that $\overline{G}$ is almost simple. Let M be a minimum normal subgroup in $\overline{G}$. Then $M = M_1 \times \cdots \times M_n$, where $M_1, \ldots, M_n$ are isomorphic non-abelian simple groups.

Suppose that $n > 1$. Then any vertex of the graph $\Gamma(M)$ is adjacent at least two other its vertices, therefore $\pi(M_1) = \{r, s, p\}$. By Lemma 1, $\{2, 3\} \in \pi(M_1)$ and $Out(M_1)$ is a 2-group.

Suppose that $q \in \pi(\overline{G})$. Then there is an element x of order q in $\overline{G}$. The subgroup $\langle x \rangle$ acts (by the conjugation) on the set $\{M_1, \ldots, M_n\}$ without fixed points. Indeed, suppose that the element x normalizes M_1. Since $N_{\overline{G}}(M_1)/M_1 C_{\overline{G}}(M_1)$ is a 2-group, the element x centralizes M_1. Hence x centralizes elements of order 2 and 3 in M_1, a contradiction with the form of the graph $\Gamma(G)$. Thus, subgroup $K = \langle M_1, x \rangle$ is isomorphic to the wreath product $M_1 \wr \mathbb{Z}_q$. Hence, $C_K(x) \cong M_1$, and therefore x centralizes some elements of order 2 and 3 in K. This is impossible.

So, $q \notin \pi(\overline{G})$. Hence $q \in \pi(S(G))$. Let $Q \in Syl_q(S(G))$. By Frattini lemma $G = S(G)N_G(Q)$, and hence $N_G(Q)/N_{S(G)}(Q) \cong \overline{G}$. Since $n > 1$, a Sylow 2-subgroup from $N_G(Q)$ contains a four-subgroup, hence $p = 2$. Therefore the Sylow r- and s-subgroups in $N_G(Q)$, and hence in $\overline{G}$, are cyclic, a contradiction with the supposition $n > 1$.

So, $n = 1$, i.e., the subgroup M is simple. If $\overline{G}$ contains a different from M minimal normal subgroups N, then, as it is proved above, N is a simple and centralize M. Then, arguing as above, we come to a contradiction.

Theorem 1 is proved.

4. Proof of theorem 2

Let G be a group satisfying the conditions of Theorem 2, and let L be its socle.

Lemma 3. *If* $|\pi(L)| = 3$*, then* G *either satisfies the statement* $(a1)$ *of Theorem* 2*, or is isomorphic to* $Aut(A_6)$*.*

Proof. The lemma follows from Lemma 1 and [9, Table]. □

In view of Lemma 3, further we can assume that $|\pi(L)| = 4$.

Lemma 4. *If the graph* $\Gamma(G)$ *is disconnected, then* G *satisfies the statement* (a) *of Theorem* 2*.*

Proof. The lemma follows from Theorem 1 and [10, Table 1]. □

In view of Lemma 4, further we can assume that the graph $\Gamma(G)$ is connected.

Lemma 5. *If* L *is isomorphic to a group from the item* (1) *of Lemma* 2*, then* G *satisfies the statement* (b) *of Theorem* 2*.*

Proof. The lemma follows from Theorem 1 and [10, Table 1]. □

Lemma 6. L *is not isomorphic to a group from the item* (2) *of Lemma* 2*.*

Proof. Suppose the contrary. Then $G \cong Aut(L)$ and in view of [10, Table 1] the graph $\Gamma(G)$ disconnected, a contradiction. □

Lemma 7. L *is not isomorphic to a group from the item* (3) *of Lemma* 2*..*

Proof. Suppose the contrary. Then $L \cong L_2(2^m)$, where $m, u = 2^m - 1$ and $t = (2^m + 1)/3$ are primes greater than 3. Since the graph $\Gamma(L)$ is disconnected, we have $L < G$. Hence $G \cong Aut(L) \cong L_2(2^m) : \mathbb{Z}_m$. Since $\pi(G) = \{2, 3, u, t\}$, we have $m \in \{u, t\}$.

Let $m = u$. Then $m = 2^m - 1$, i.e., $2^m = m + 1$. Show by induction on m that $2^m > m + 1$ for $m \geq 2$. For $m = 2$, we have $2^2 = 4 > 2 + 1 = 3$, so the base of induction is satisfied. Suppose that $m \geq 2$ and $2^m > m + 1$. Then $2^{m+1} > 2m + 2 = m + (m + 2) > m + 2$, so the induction step is satisfied too. Thus, $m \neq u$.

So, $m = t = (2^m + 1)/3$. Then $2^m = 3m - 1$. Show by induction on m that $2^m > 3m - 1$ for $m > 3$. For $m = 4$, we have $2^4 = 16 > 3 \cdot 4 - 1 = 11$, so the base of induction is satisfied. Suppose that $m > 3$ and $2^m > 3m - 1$. Then $2^{m+1} > 6m - 2 = 3(m + 1) - 1 + (3m - 4) > 3(m + 1) - 1$, so the induction step is satisfied too. Thus, $m \neq t$.

The obtained contradiction proves the lemma. □

Lemma 8. *If* L *is isomorphic to a group from the item* (4) *of Lemma* 2*, then* $G \cong Aut(L_2(27))$*.*

Proof. Suppose that $L \cong L_2(3^m)$, where m and $u = (3^m - 1)/2$ are odd primes, and $(3^m + 1)/4$ equals to either a prime or 11^2 for $m = 5$. Then $\pi((3^m + 1)/4) = \{t\}$ for some prime t. In view of [10, Table 1], the graphs $\Gamma(L_2(3^m))$ and $\Gamma(PGL_2(3^m))$ are disconnected.

Since $|Out(L)| = 2m$ (see [2, Table 5]) and the graph $\Gamma(G)$ is connected, the group G is isomorphic to either $L : \mathbb{Z}_m$ or $Aut(L)$. Hence $m \in \pi(G) = \pi(L) = \{2, 3, u, t\}$.

Suppose that $m \in \{u, t\}$. Then $m > 3$, and hence $m \in \pi(L)$. But then a field automorphism φ of order m of the group L centralizes an element of order m in L. We have $C_L(\varphi) \cong L_2(3) \cong A_4$ (see [4, 4.9.1]); a contradiction. Thus, $m = 3$. In view of [10, Table 1] the graph $\Gamma(L_2(3^3).\mathbb{Z}_3)$ is disconnected, hence $G \cong Aut(L_2(3^3))$. □

The statement of the necessity of Theorem 2 follows from Lemmas 3–8.

The statement of the sufficiency of Theorem 2 follows from [9, 10, 2, 4].

Theorem 2 is proved.

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