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DISTORTION THEOREM FOR COMPLEX POLYNOMIALS

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Abstract. For any complex polynomial P of degree $n \geq 2$ and any complex number z, we consider a sharp inequality involving of the absolute values of $P(z)$, $P'(z)$, leading coefficient of P and an upper bound of the moduli of the critical values of P. All cases of an equality in this inequality are established.

Keywords: distortion theorems, complex polynomials, inequalities, critical values.

1. Introduction

Let P be a complex polynomial of degree $n \geq 2$ and let $0 < c < \infty$. The set

$$L_P(c) = \{z : \ |P(z)| = c\}$$

is called a *lemniscate* of the polynomial P. The motivation of our study goes back to the well-known Erdős problem of finding the maximum of the absolute value of the derivative of a monomial P on the connected lemniscate $L_P(1)$ [1, 2]. The Erdős problem was solved by Eremenko and Lempert [3] (also see [4]). In the papers [5, 6] general distortion theorems for polynomials were obtained. These theorems imply, in particular, a sharp inequality for the absolute value of the derivative of a polynomial at any point in the plane (not necessarily belonging to the connected lemniscate). In this paper, we give a direct proof of a stronger version of the above-mentioned inequality. In addition, we find all cases of equality in this inequality. Note that a lemniscates $L_P(c)$ is connected if and only if

$$c \geq M(P) := \max\{|P(z)| : \ P'(z) = 0\}.$$

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This fact will be used repeatedly in this paper, so that restrictions will be imposed on $M(P)$ instead of the connectivity of the lemniscate.

The following assertion is valid.

Theorem. *Let $P(z) = c_0 + c_1 z + ... + c_n z^n$ be a polynomial of degree $n \geq 2$, and let c be arbitrary real number, $c \geq M(P)$. Then, for any point z, the inequality*

$$|P'(z)| \leq \frac{c}{2}\left(\frac{2|c_n|}{c}\right)^{\frac{1}{n}} T_n'\left(T_n^{-1}\left(\left|\frac{P(z)}{c}\right|\right)\right) \tag{1}$$

holds, where $T_n(z) = 2^{n-1}z^n + ...$ is the Chebyshev polynomial of the first kind, and the value $T_n^{-1}(|P(z)/c|)$ is taken on the ray $[\cos(\pi/(2n)), +\infty)$.

Equality in (1) is attained at a point $z = z_0$ only for polynomials of the form

$$P(z) = ce^{i\theta}T_n(az + b), \tag{2}$$

where θ is any real number and a and b are any complex numbers satisfying the conditions $a \neq 0,\ az_0 + b \in [\cos(\pi/(2n)), +\infty)$.

The inequality (1) is proved by using a symmetrization method [7]. The direct proof of (1) (for $c = 1$) was announced in [8]. It was discussed at some conferences but has not yet been published. The study of the equality case in (1) goes back to [9] but the result is presented here for the first time.

Directly from the Theorem we obtain the following statement.

Corollary (see [8, Theorem 3]). *Let $P(z) = c_0 + c_1 z + ... + c_n z^n$ be a polynomial of degree $n \geq 2$, and let z_0 be arbitrary zero of this polynomial. Then*

$$M(P) \geq 2|c_n|^{\frac{1}{1-n}}\left(\frac{|P'(z_0)|}{n}\sin\frac{\pi}{2n}\right)^{\frac{n}{n-1}}. \tag{3}$$

Equality in (3) is attained only for polynomials of the form (2), where θ is any real number and a and b are any complex numbers satisfying the conditions $a \neq 0,\ az_0 + b = \cos(\pi/(2n))$.

2. Preliminaries

Let $w = P(z)$ be a polynomial of degree $n \geq 2$ with $M(P) \leq 1$. Denote by $\mathscr{R}(P)$ the Riemann surface of the function inverse to P. The surface $\mathscr{R}(P)$ lies over the complex w-sphere and satisfies the following condition: for any given $1 \leq \rho < \infty$, each closed Jordan curve on $\mathscr{R}(P)$ lying over the circle $\gamma(\rho) := \{w : |w| = \rho\}$ which does not pass through ramification points of $\mathscr{R}(P)$ covers this circle with multiplicity n. So, the surface $\mathscr{R}(P)$ belongs to class $\mathfrak{R}_n$ defined in [7]. It is important for us to consider a special surface $\mathscr{R}(T_n)$ of the inverse function of the Chebyshev polynomial T_n. One of the representations of this Riemann surface is as follows. Let D_1 be the w-plane cut along the ray $L^- := [-\infty, -1]$, and let $D_2, ..., D_{n-1}$ be copies of the w-plane cut along the rays L^- and $L^+ := [1, +\infty]$; finally, let D_n be w-plane cut along the ray L^- if n is even or along L^+ if n is odd. The Riemann surface $\mathscr{R}(T_n)$ can be obtained by gluing the domains $D_k,\ k = 1, ..., n$, together as follows. The domain D_1 is attached crosswise to D_2 along the banks of the cuts L^-, D_2 is attached to D_3 along the banks of L^+, and so on. The domain D_{n-1} is attached to D_n along the banks of the cuts L^- in the case of even n and along the banks of L^+ in the case of odd n. The domains D_k regarded as subsets of the surface $\mathscr{R}(T_n)$ will be denoted by $\mathscr{D}_k,\ k = 1, ..., n$. Let $\mathscr{L}$ denote the “ray” lying on the sheet $\mathscr{D}_1$ over the ray $[0,+\infty]$. We treat the polynomial T_n as a map of the sphere $\overline{\mathbb{C}}_z$ onto the Riemann surface $\mathscr{R}(T_n)$ under which $T_n([1,+\infty])\subset\mathscr{L}$.

We now take an arbitrary surface $\mathscr{R}=\mathscr{R}(P)$ and proceed to the definition of the circular symmetrization Sym of sets and condensers on $\mathscr{R}$ [7].

Let $\mathscr{B}$ be an open set in $\mathscr{R}$. Then symmetrization Sym transforms $\mathscr{B}$ into a subset $\mathrm{Sym}\,\mathscr{B}$ of $\mathscr{R}(T_n)$ with the following properties. Fix some $\rho,\ 0\le\rho\le\infty$. If no points in $\mathscr{B}$ lie over the circle $\gamma(\rho)$, then no points in $\mathrm{Sym}\,\mathscr{B}$ lie over it either. If $\mathscr{B}$ covers $\gamma(\rho),\ 1\le\rho\le\infty$, with multiplicity n, then $\mathrm{Sym}\,\mathscr{B}$ also covers $\gamma(\rho)$ with multiplicity n. If $\mathscr{B}$ covers $\gamma(\rho),\ 0\le\rho<1$, with multiplicity $l\le n$, then the part of $\mathrm{Sym}\,\mathscr{B}$ over $\gamma(\rho)$ consists of l circles lying on the sheets $\mathscr{D}_1,...,\mathscr{D}_l$. In the other cases, for $1\le\rho<\infty$ the part of $\mathrm{Sym}\,\mathscr{B}$ lying over $\gamma(\rho)$ is an open arc[1] on $\mathscr{R}(T_n)$ with midpoint on the ray $\mathscr{L}$ and with linear measure equal to the measure of $\mathscr{B}(\rho):=\{W\in\mathscr{B}:\ |\mathrm{pr}W|=\rho\}$. For $0<\rho<1$ the part of $\mathrm{Sym}\,\mathscr{B}$ over $\gamma(\rho)$ is a union of m circles $\Gamma_1,...,\Gamma_m,\ 0\le m\le n-1$, and an open arc Γ_{m+1} such that $\Gamma_k=\Gamma_k(\mathscr{B},\rho)\subset\mathscr{D}_k,\ k=1,...,m+1$; the total linear measure of these curves is equal to the measure of $\mathscr{B}(\rho)$, and the midpoint of Γ_{m+1} lies over $(-1)^m\rho$. Here the number m of the circles depends on the measure of $\mathscr{B}(\rho)$. If this measure is less that $2\pi\rho$, then necessarily $m=0$, and there are no full circles.

The result $\mathrm{Sym}\,\mathscr{E}$ of the symmetrization of a closed set $\mathscr{E}\subset\mathscr{R}$ also lies on $\mathscr{R}(T_p)$ and is defined as follows. Fix some $\rho,\ 0\le\rho\le\infty$. If no points in the set $\mathscr{E}$ lie over $\gamma(\rho)$, then $\mathrm{Sym}\,\mathscr{E}$ contains no points over this circle either. If $\mathscr{E}$ covers $\gamma(\rho),\ 1\le\rho\le\infty$, with multiplicity n, then $\mathrm{Sym}\,\mathscr{E}$ also covers $\gamma(\rho)$ with multiplicity n. If $\mathscr{E}$ covers $\gamma(\rho),\ 0\le\rho<1$, with multiplicity $l\le n$, then the part of $\mathrm{Sym}\,\mathscr{E}$ over $\gamma(\rho)$ consists of l circles in the sheets $\mathscr{D}_1,...,\mathscr{D}_l$. Otherwise, the part of $\mathrm{Sym}\,\mathscr{E}$ lying over $\gamma(\rho),\ 1\le\rho<\infty$, is a closed arc segment (that is, an arc with its endpoints) on $\mathscr{R}(T_n)$, with midpoint on the ray $\mathscr{L}$ and with linear measure equal to the measure of $\mathscr{E}(\rho):=\{W\in\mathscr{E}:\ |\mathrm{pr}W|=\rho\}$ (if the latter is equal to zero, then the corresponding arc segment is a point on $\mathscr{L}$). The part of $\mathrm{Sym}\,\mathscr{E}$ over $\gamma(\rho),\ 0<\rho<1$, is the union of m circles $\Gamma_1,...,\Gamma_m,\ 0\le m\le n-1$, and a closed arc segment Γ_{m+1} such that $\Gamma_k\subset\mathscr{D}_k,\ k=1,..,m+1$, the total linear measure of these curves is equal to that of $\mathscr{E}(\rho)$ and the midpoint of Γ_{m+1} lies over $(-1)^m\rho$ (if the measure in question is $2\pi\rho m$, where m is a nonnegative integer, then Γ_{m+1} reduce to a point).

A *condenser on the surface* $\mathscr{R}$ is an ordered pair of sets $\mathscr{C}=(\mathscr{B},\mathscr{E})$, where $\mathscr{B}$ is an open subset of $\mathscr{R}$ and $\mathscr{E}$ is a compact subset of $\mathscr{B}$. We call $\mathscr{B}\setminus\mathscr{E}$ the *field of the condenser* $\mathscr{C}$. Now we set

$$\mathrm{Sym}\,\mathscr{C}=(\mathrm{Sym}\,\mathscr{B},\ \mathrm{Sym}\,\mathscr{E}).$$

The capacity $\mathrm{cap}\mathscr{C}$ of the condenser $\mathscr{C}=(\mathscr{B},\mathscr{E})$ is defined by

$$\mathrm{cap}\mathscr{C}=\inf\int_{\mathscr{B}}|\bigtriangledown\mathscr{V}|^2d\sigma,$$

where the infimum is taken over all *admissible* functions $\mathscr{V}$: real-valued functions $\mathscr{V}$ which have compact support in $\mathscr{B}$, are equal to 1 on $\mathscr{E}$ and are locally Lipschitz in $\mathscr{B}$. If there exists a function $\mathscr{P}$ which is continuous in $\overline{\mathscr{B}}$, equal to zero on $\partial\mathscr{B}$, to 1 on $\mathscr{E}$, and harmonic in the field $\mathscr{B}\setminus\mathscr{E}$, then it is called the *potential function*

[1]For $\rho>1$, this is an open Jordan arc, but for $\rho=1$ it can contain self-tangency points.

of the condenser $\mathscr{C}$. Then by Dirichlet's principle

$$\operatorname{cap}\mathscr{C} = \int_{\mathscr{B}\setminus\mathscr{E}} |\bigtriangledown \mathscr{P}|^2 d\sigma.$$

Let $\mathscr{G}$ be a subdomain of $\mathscr{R}$ and $\tilde{\mathscr{G}}$ be a subdomain of another Riemann surface $\tilde{\mathscr{R}}$. A bijective map $\Phi : \mathscr{G} \to \tilde{\mathscr{G}}$ is called a *motion* if it preserves distances, that is, for any points $W,\ W' \in \mathscr{G}$ we have

$$d_{\mathscr{G}}(W, W') = d_{\tilde{\mathscr{G}}}(\Phi(W), \Phi(W')),$$

where $d_{\mathscr{G}}(W, W')$ is the distance between W and W' within $\mathscr{G}$.

A *rotation* about the origin through the angle θ (where θ is a real number) taking $\mathscr{G}$ to $\tilde{\mathscr{G}}$ is a motion $\Phi_\theta :\ \mathscr{G} \to \tilde{\mathscr{G}}$ such that for any $W \in \mathscr{G}$,

$$\operatorname{pr}\Phi_\theta(W) = e^{i\theta}\operatorname{pr} W.$$

The following is a special case of the central result of [7].

Lemma (see [7, Theorem 1.1]). *For each condenser $\mathscr{C}$ on a surface $\mathscr{R}$*

$$\operatorname{cap}\mathscr{C} \geq \operatorname{cap}\operatorname{Sym}\mathscr{C}. \tag{4}$$

In addition, if $\mathscr{C} = (\mathscr{B}, \mathscr{E})$ has a connected field and the potential function of $\mathscr{C}$ exists, then equality in (4) *holds only in the following cases*:

(i) *the field of $\mathscr{C}$ coincides with the field of* $\operatorname{Sym}\mathscr{C}$ *up to rotations about the origin;*

(ii) *there exist numbers s, t and l, $0 < s < t < \infty$, $1 < l \leq n$, for which the field of the condenser $\mathscr{C}$ covers the circular annulus $s < |w| < \tau$ with multiplicity*[2] *l in such a way that, over each boundary circle of this annulus, there are either only boundary points of $\mathscr{B}$ or only boundary points of $\mathscr{E}$.*

3. Proof of Theorem

Fix any point z_0 such that $P'(z_0) \neq 0$. Given any $0 < r < r_1 < r_2 < 1/r$, consider the four condensers

$$C_1(r, r_1) = (U(r_1), \overline{U(r)}), \quad C_2(r_1, r_2) = (U(r_2), \overline{U(r_1)}),$$
$$C_3(r_2, r) = (U(1/r)), \overline{U(r_2)}), \quad C(r) = (U(1/r), \overline{U(r)})$$

in the closed ζ-plane, where $U(\rho) := \{\zeta :\ |\zeta| < \rho\}$. Let $\tilde{C}_1(r, r_1),\ \tilde{C}_2(r_1, r_2),\ \tilde{C}_3(r_2, r)$ and $\tilde{C}(r)$ denote, respectively, the results of the transformations of these condensers consisting in successive application of the following maps:

$$z = \varphi(z) := \zeta + z_0, \ \ W = Q(z) := \frac{P(z)}{c}, \ \ W \in \mathscr{R}(Q),$$

the symmetrization map Sym from Section 2, the map $z = T_n^{-1}(W)$, where $W \in \mathscr{R}(T_n)$, and the map

$$\omega = \psi(z) := z - T_n^{-1}(|Q(z_0)|),$$

where $T_n^{-1}(|Q(z_0)|) \in [\cos(\pi/(2n), +\infty)$. We treat the polynomial Q as a map of the sphere $\overline{\mathbb{C}}_z$ onto the Riemann surface $\mathscr{R}(Q)$, $M(Q) \leq 1$. Taking into account the conformal invariance of condenser capacity and Lemma from Section 2, we see that

$$\operatorname{cap} C_1(r, r_1) \geq \operatorname{cap}\tilde{C}_1(r, r_1), \quad \operatorname{cap} C_2(r_1, r_2) \geq \operatorname{cap}\tilde{C}_2(r_1, r_2),$$

[2] This means that precisely l points (counting multiplicities) in the field lie over each point of the annulus.

$$\operatorname{cap} C_3(r_2, r) \geq \operatorname{cap} \tilde{C}_3(r_2, r), \quad \operatorname{cap} C(r) \geq \operatorname{cap} \tilde{C}(r).$$

According to Grötzsch's lemma, the capacities of the condensers satisfy the relations

$$\frac{1}{\operatorname{cap} C(r)} = \frac{1}{\operatorname{cap} C_1(r, r_1)} + \frac{1}{\operatorname{cap} C_2(r_1, r_2)} + \frac{1}{\operatorname{cap} C_3(r_2, r)},$$

$$\frac{1}{\operatorname{cap} \tilde{C}(r)} \geq \frac{1}{\operatorname{cap} \tilde{C}_1(r, r_1)} + \frac{1}{\operatorname{cap} \tilde{C}_2(r_1, r_2)} + \frac{1}{\operatorname{cap} \tilde{C}_3(r_2, r)}$$

(see, e.g., [10, Theorem 1.14], where a somewhat different notation for condensers was used).

Hence,

$$\frac{1}{\operatorname{cap} \tilde{C}(r)} - \frac{1}{\operatorname{cap} C(r)} \geq \frac{1}{\operatorname{cap} \tilde{C}_2(r_1, r_2)} - \frac{1}{\operatorname{cap} C_2(r_1, r_2)} \geq 0. \tag{5}$$

We define the sets $B(r)$ and $E(r)$ by the equality

$$\tilde{C}(r) = (B(r), E(r)).$$

It follows from the definition of the symmetrization Sym and the elementary properties of the conformal maps that the boundary of $E(r)$ is almost a circle of radius

$$\frac{r(1 + o(1))|Q'(z_0)|}{|T_n'(T_n^{-1}(|Q(z_0)|))|}$$

centred at the origin; on the boundary of $B(r)$, we have

$$|\omega| = \frac{1}{2r(1 + o(1))} \left(\frac{2|c_n|}{c} \right)^{\frac{1}{n}}, \quad r \to 0.$$

Therefore, inequality (5) gives

$$\frac{1}{2\pi} \log \left[\frac{|T_n'(T_n^{-1}(|Q(z_0)|))|}{2|Q'(z_0)|} \left(\frac{2|c_n|}{c} \right)^{\frac{1}{n}} \right] \geq \frac{1}{\operatorname{cap} \tilde{C}_2(r_1, r_2)} - \frac{1}{\operatorname{cap} C_2(r_1, r_2)} \geq 0. \tag{6}$$

This implies inequality (1) for $z = z_0$ under the condition $P'(z_0) \neq 0$. In the case when $P'(z_0) = 0$ this inequality is a clear.

Now, suppose that (1) turns into equality for $z = z_0$. Then $P'(z_0) \neq 0$ and (6) implies that

$$\operatorname{cap} \tilde{C}_2(r_1, r_2) = \operatorname{cap} C_2(r_1, r_2)$$

for all r_1 and r_2, $0 < r_1 < r_2 < \infty$. It follows from the equality conditions (i) and (ii) in the Lemma that the condenser $Q(\varphi(C_2(r_1, r_2)))$ is obtained from $\operatorname{Sym} Q(\varphi(C_2(r_1, r_2)))$ by a rotation through an angle θ about the origin. As the ramification points are mapped to the ramification points under rotation, the angle θ is the same for all small r_1 and large r_2. Therefore, the superposition

$$\psi \circ T_n^{-1} \circ \lambda_\theta \circ Q \circ \varphi$$

($\lambda_\theta(W) = e^{-i\theta} W$) is a univalent conformal map of the ζ-plane onto ω-plane which takes 0 to 0 and ∞ to ∞. This yields the representation (2). Conversely, it is seen from a direct calculation that for any polynomial of the form (2), (1) becomes an equality. This completes the proof of the theorem.

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