DISTORTION THEOREM FOR COMPLEX POLYNOMIALS

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Abstract. For any complex polynomial $P$ of degree $n \geq 2$ and any complex number $z$, we consider a sharp inequality involving of the absolute values of $P(z)$, $P'(z)$, leading coefficient of $P$ and an upper bound of the moduli of the critical values of $P$. All cases of an equality in this inequality are established.

Keywords: distortion theorems, complex polynomials, inequalities, critical values.

1. Introduction

Let $P$ be a complex polynomial of degree $n \geq 2$ and let $0 < c < \infty$. The set

$$L_P(c) = \{z : |P(z)| = c\}$$

is called a lemniscate of the polynomial $P$. The motivation of our study goes back to the well-known Erdős problem of finding the maximum of the absolute value of the derivative of a monomial $P$ on the connected lemniscate $L_P(1)$ [1, 2]. The Erdős problem was solved by Eremenko and Lempert [3] (also see [4]). In the papers [5, 6] general distortion theorems for polynomials were obtained. These theorems imply, in particular, a sharp inequality for the absolute value of the derivative of a polynomial at any point in the plane (not necessarily belonging to the connected lemniscate). In this paper, we give a direct proof of a stronger version of the abovementioned inequality. In addition, we find all cases of equality in this inequality. Note that a lemniscates $L_P(c)$ is connected if and only if

$$c \geq M(P) := \max\{|P(z)| : P'(z) = 0\}.$$
This fact will be used repeatedly in this paper, so that restrictions will be imposed on \(M(P)\) instead of the connectivity of the lemniscate.

The following assertion is valid.

**Theorem.** Let \(P(z) = c_0 + c_1 z + ... + c_n z^n\) be a polynomial of degree \(n \geq 2\), and let \(c\) be arbitrary real number, \(c \geq M(P)\). Then, for any point \(z\), the inequality

\[
|P'(z)| \leq \frac{c}{2} \left(\frac{2|c_n|}{c}\right)^{\frac{1}{n}} T_n^{\frac{1}{n}} \left(\frac{|P(z)|}{c}\right)
\]

holds, where \(T_n(z) = 2^{n-1} z^n + ...\) is the Chebyshev polynomial of the first kind, and the value \(T_n^{-1}(|P(z)/c|)\) is taken on the ray \(\cos(\pi/(2n)), +\infty)\).

Equality in (1) is attained at a point \(z = z_0\) only for polynomials of the form

\[
P(z) = c e^{i\theta} T_n(az + b),
\]

where \(\theta\) is any real number and \(a\) and \(b\) are any complex numbers satisfying the conditions \(a \neq 0, az_0 + b \in \cos(\pi/(2n)), +\infty)\).

The inequality (1) is proved by using a symmetrization method [7]. The direct proof of (1) (for \(c = 1\)) was announced in [8]. It was discussed at some conferences but has not yet been published. The study of the equality case in (1) goes back to [9] but the result is presented here for the first time.

Directly from the Theorem we obtain the following statement.

**Corollary** (see [8, Theorem 3]). Let \(P(z) = c_0 + c_1 z + ... + c_n z^n\) be a polynomial of degree \(n \geq 2\), and let \(z_0\) be arbitrary zero of this polynomial. Then

\[
M(P) \geq 2|c_n| \left(\frac{|P'(z_0)|}{n} \sin \frac{\pi}{2n}\right)^{\frac{n}{2n}}.
\]

Equality in (3) is attained only for polynomials of the form (2), where \(\theta\) is any real number and \(a\) and \(b\) are any complex numbers satisfying the conditions \(a \neq 0, az_0 + b = \cos(\pi/(2n))\).

2. Preliminaries

Let \(w = P(z)\) be a polynomial of degree \(n \geq 2\) with \(M(P) \leq 1\). Denote by \(\mathcal{R}(P)\) the Riemann surface of the function inverse to \(P\). The surface \(\mathcal{R}(P)\) lies over the complex \(w\)-sphere and satisfies the following condition: for any given \(1 \leq \rho < \infty\), each closed Jordan curve on \(\mathcal{R}(P)\) lying over the circle \(\gamma(\rho) := \{w : |w| = \rho\}\) which does not pass through ramification points of \(\mathcal{R}(P)\) covers this circle with multiplicity \(n\). So, the surface \(\mathcal{R}(P)\) belongs to class \(N_n\) defined in [7]. It is important for us to consider a special surface \(\mathcal{R}(T_n)\) of the inverse function of the Chebyshev polynomial \(T_n\). One of the representations of this Riemann surface is as follows. Let \(D_1\) be the \(w\)-plane cut along the ray \(L^- := [-\infty, -1]\), and let \(D_2, ..., D_{n-1}\) be copies of the \(w\)-plane cut along the rays \(L^- \text{ and } L^+ := [1, +\infty]\); finally, let \(D_n\) be \(w\)-plane cut along the ray \(L^+\) if \(n\) is even or along \(L^-\) if \(n\) is odd. The Riemann surface \(\mathcal{R}(T_n)\) can be obtained by gluing the domains \(D_k, k = 1, ..., n\), together as follows. The domain \(D_1\) is attached crosswise to \(D_2\) along the banks of the cuts \(L^-\), \(D_2\) is attached to \(D_3\) along the banks of \(L^+\), and so on. The domain \(D_{n-1}\) is attached to \(D_n\) along the banks of the cuts \(L^-\) in the case of even \(n\) and along the banks of \(L^+\) in the case of odd \(n\). The domains \(D_k\) regarded as subsets of the surface \(\mathcal{R}(T_n)\) will be denoted by \(\mathcal{R}_k, k = 1, ..., n\). Let \(\mathcal{L}\) denote the “ray”
lying on the sheet $D_1$ over the ray $[0, +\infty]$. We treat the polynomial $T_n$ as a map of the sphere $\overline{\mathbb{C}}$, onto the Riemann surface $\mathcal{R}(T_n)$ under which $T_n([1, +\infty]) \subset \mathcal{L}$.

We now take an arbitrary surface $\mathcal{B} = \mathcal{B}(P)$ and proceed to the definition of the circular symmetrization $\text{Sym}$ of sets and condensers on $\mathcal{B}$ [7].

Let $\mathcal{B}$ be an open set in $\mathcal{R}$. Then symmetrization $\text{Sym}$ transforms $\mathcal{B}$ into a subset $\text{Sym} \mathcal{B}$ of $\mathcal{R}(T_n)$ with the following properties. Fix some $\rho$, $0 \leq \rho \leq \infty$. If no points in $\mathcal{B}$ lie over the circle $\gamma(\rho)$, then no points in $\text{Sym} \mathcal{B}$ lie over it either. If $\mathcal{B}$ covers $\gamma(\rho)$, $1 \leq \rho \leq \infty$, with multiplicity $n$, then $\text{Sym} \mathcal{B}$ also covers $\gamma(\rho)$ with multiplicity $n$. If $\mathcal{B}$ covers $\gamma(\rho)$, $0 \leq \rho < 1$, with multiplicity $l \leq n$, then the part of $\text{Sym} \mathcal{B}$ over $\gamma(\rho)$ consists of $l$ circles lying on the sheets $D_1, \ldots, D_l$. In the other cases, for $1 \leq \rho < \infty$ the part of $\text{Sym} \mathcal{B}$ lying over $\gamma(\rho)$ is an open arc on $\mathcal{R}(T_n)$ with midpoint on the ray $\mathcal{L}$ and with linear measure equal to the measure of $\mathcal{B}(\rho) := \{W \in \mathcal{R} : |\rho W| = \rho\}$. For $0 < \rho < 1$ the part of $\text{Sym} \mathcal{B}$ over $\gamma(\rho)$ is a union of $m$ circles $\Gamma_1, \ldots, \Gamma_m$, $0 \leq m \leq n-1$, and an open arc $\Gamma_{m+1}$ such that $\Gamma_k = \Gamma_k(\mathcal{B}, \rho) \subset D_k$, $k = 1, \ldots, m+1$; the total linear measure of these curves is equal to the measure of $\mathcal{B}(\rho)$, and the midpoint of $\Gamma_{m+1}$ lies over $(-1)^m \rho$. Here the number $m$ of the circles depends on the measure of $\mathcal{B}(\rho)$. If this measure is less than $2\pi \rho$, then necessarily $m = 0$, and there are no full circles.

The result $\text{Sym} \mathcal{E}$ of the symmetrization of a closed set $\mathcal{E} \subset \mathcal{B}$ also lies on $\mathcal{R}(T_n)$ and is defined as follows. Fix some $\rho$, $0 \leq \rho \leq \infty$. If no points in the set $\mathcal{E}$ lie over $\gamma(\rho)$, then $\text{Sym} \mathcal{E}$ contains no points over this circle either. If $\mathcal{E}$ covers $\gamma(\rho)$, $1 \leq \rho \leq \infty$, with multiplicity $n$, then $\text{Sym} \mathcal{E}$ also covers $\gamma(\rho)$ with multiplicity $n$. If $\mathcal{E}$ covers $\gamma(\rho)$, $0 \leq \rho < 1$, with multiplicity $l \leq n$, then the part of $\text{Sym} \mathcal{E}$ over $\gamma(\rho)$ consists of $l$ circles in the sheets $D_1, \ldots, D_l$. Otherwise, the part of $\text{Sym} \mathcal{E}$ lying over $\gamma(\rho)$, $1 \leq \rho < \infty$, is a closed arc segment (that is, an arc with its endpoints) on $\mathcal{R}(T_n)$, with midpoint on the ray $\mathcal{L}$ and with linear measure equal to the measure of $\mathcal{E}(\rho) := \{W \in \mathcal{E} : |\rho W| = \rho\}$ (if the latter is equal to zero, then the corresponding arc segment is a point on $\mathcal{L}$). The part of $\text{Sym} \mathcal{E}$ over $\gamma(\rho)$, $0 < \rho < 1$, is the union of $m$ circles $\Gamma_1, \ldots, \Gamma_m$, $0 \leq m \leq n-1$, and a closed arc segment $\Gamma_{m+1}$ such that $\Gamma_k \subset D_k$, $k = 1, \ldots, m+1$; the total linear measure of these curves is equal to that of $\mathcal{E}(\rho)$ and the midpoint of $\Gamma_{m+1}$ lies over $(-1)^m \rho$ (if the measure in question is $2\pi \rho m$, where $m$ is a nonnegative integer, then $\Gamma_{m+1}$ reduce to a point).

A condenser on the surface $\mathcal{B}$ is an ordered pair of sets $\mathcal{C} = (\mathcal{B}, \mathcal{E})$, where $\mathcal{B}$ is an open subset of $\mathcal{B}$ and $\mathcal{E}$ is a compact subset of $\mathcal{B}$. We call $\mathcal{B} \setminus \mathcal{E}$ the field of the condenser $\mathcal{C}$. Now we set

$$\text{Sym} \mathcal{C} = (\text{Sym} \mathcal{B}, \text{Sym} \mathcal{E}).$$

The capacity $\text{cap} \mathcal{C}$ of the condenser $\mathcal{C} = (\mathcal{B}, \mathcal{E})$ is defined by

$$\text{cap} \mathcal{C} = \inf \int_{\mathcal{E}} |\nabla \psi|^2 d\sigma,$$

where the infimum is taken over all admissible functions $\psi$: real-valued functions $\psi$ which have compact support in $\mathcal{B}$, are equal to 1 on $\mathcal{E}$ and are locally Lipschitz in $\mathcal{B}$. If there exists a function $\mathcal{P}$ which is continuous in $\overline{\mathcal{B}}$, equal to zero on $\partial \mathcal{B}$, to 1 on $\mathcal{E}$, and harmonic in the field $\mathcal{B} \setminus \mathcal{E}$, then it is called the potential function

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1For $\rho > 1$, this is an open Jordan arc, but for $\rho = 1$ it can contain self-tangency points.
of the condenser $\mathcal{C}$. Then by Dirichlet’s principle
\[ \operatorname{cap} \mathcal{C} = \int_{\partial S \setminus \mathcal{C}} |\nabla \hat{\mathcal{P}}|^2 d\sigma. \]

Let $\mathcal{B}$ be a subdomain of $\hat{\mathcal{R}}$ and $\mathcal{B}$ be a subdomain of another Riemann surface $\hat{\mathcal{R}}$. A bijective map $\Phi : \mathcal{B} \to \mathcal{B}$ is called a motion if it preserves distances, that is, for any points $W, W' \in \mathcal{B}$ we have
\[ d_\mathcal{B}(W, W') = d_\mathcal{B}(\Phi(W), \Phi(W')), \]
where $d_\mathcal{B}(W, W')$ is the distance between $W$ and $W'$ within $\mathcal{B}$.

A rotation about the origin through the angle $\theta$ (where $\theta$ is a real number) taking $\mathcal{B}$ to $\mathcal{B}$ is a motion $\Phi_\theta : \mathcal{B} \to \mathcal{B}$ such that for any $W \in \mathcal{B}$,
\[ \operatorname{pr} \Phi_\theta(W) = e^{i\theta} \operatorname{pr} W. \]

The following is a special case of the central result of [7].

**Lemma** (see [7, Theorem 1.1]). For each condenser $\mathcal{C}$ on a surface $\hat{\mathcal{R}}$ (4)
\[ \operatorname{cap} \mathcal{C} \geq \operatorname{cap} \operatorname{Sym} \mathcal{C}. \]

In addition, if $\mathcal{C} = (\hat{\mathcal{R}}, \mathcal{E})$ has a connected field and the potential function of $\mathcal{C}$ exists, then equality in (4) holds only in the following cases:

(i) the field of $\mathcal{C}$ coincides with the field of $\operatorname{Sym} \mathcal{C}$ up to rotations about the origin;

(ii) there exist numbers $s$, $t$ and $l$, $0 < s < t < \infty$, $1 < l \leq n$, for which the field of the condenser $\mathcal{C}$ covers the circular annulus $s < |w| < t$ with multiplicity $l$ in such a way that, over each boundary circle of this annulus, there are either only boundary points of $\hat{\mathcal{R}}$ or only boundary points of $\mathcal{E}$.

3. **Proof of Theorem**

Fix any point $z_0$ such that $P'(z_0) \neq 0$. Given any $0 < r < r_1 < r_2 < 1/r$, consider the four condensers
\[ C_1(r, r_1) = (U(r_1), U(r)), \quad C_2(r_1, r_2) = (U(r_2), U(r_1)), \]
\[ C_3(r, r_2) = (U(1/r), U(r_2)), \quad C(r) = (U(1/r), U(r)) \]
in the closed $\zeta$-plane, where $U(\rho) := \{ \zeta : |\zeta| < \rho \}$. Let $\hat{C}_1(r, r_1)$, $\hat{C}_2(r_1, r_2)$, $\hat{C}_3(r_2, r)$ and $\hat{C}(r)$ denote, respectively, the results of the transformations of these condensers consisting in successive application of the following maps:
\[ z = \varphi(z) := \zeta + z_0, \quad W = Q(z) := \frac{P(z)}{e}, \quad W \in \hat{\mathcal{R}}(Q), \]
the symmetrization map $\operatorname{Sym}$ from Section 2, the map $z = T_n^{-1}(W)$, where $W \in \hat{\mathcal{R}}(T_n)$, and the map
\[ \omega = \psi(z) := z - T_n^{-1}(\{Q(z_0)\}), \]
where $T_n^{-1}(\{Q(z_0)\}) \in \{\cos(\pi/(2n)), +\infty\}$. We treat the polynomial $Q$ as a map of the sphere $\overline{\mathbb{C}}$ onto the Riemann surface $\hat{\mathcal{R}}(Q)$, $M(Q) \leq 1$. Taking into account the conformal invariance of condenser capacity and Lemma from Section 2, we see that
\[ \operatorname{cap} C_1(r, r_1) \geq \operatorname{cap} \hat{C}_1(r, r_1), \quad \operatorname{cap} C_2(r_1, r_2) \geq \operatorname{cap} \hat{C}_2(r_1, r_2), \]
\[ \text{This means that precisely } l \text{ points (counting multiplicities) in the field lie over each point of the annulus.} \]
\[
\text{cap} C_3(r_2, r) \geq \text{cap} \tilde{C}_3(r_2, r), \quad \text{cap} C(r) \geq \text{cap} \tilde{C}(r).
\]

According to Grötzsch’s lemma, the capacities of the condensers satisfy the relations
\[
\frac{1}{\text{cap} C(r)} = \frac{1}{\text{cap} C_1(r, r_1)} + \frac{1}{\text{cap} C_2(r_1, r_2)} + \frac{1}{\text{cap} C_3(r_2, r)},
\]
\[
\frac{1}{\text{cap} C(r)} \geq \frac{1}{\text{cap} C_1(r, r_1)} + \frac{1}{\text{cap} C_2(r_1, r_2)} + \frac{1}{\text{cap} C_3(r_2, r)}
\]
(see, e.g., [10, Theorem 1.14], where a somewhat different notation for condensers was used).

Hence,
\[
\text{(5)} \quad \frac{1}{\text{cap} C(r)} - \frac{1}{\text{cap} C(r)} \geq \frac{1}{\text{cap} C_2(r_1, r_2)} - \frac{1}{\text{cap} C_2(r_1, r_2)} \geq 0.
\]

We define the sets \( B(r) \) and \( E(r) \) by the equality
\[
\tilde{C}(r) = (B(r), E(r)).
\]

It follows from the definition of the symmetrization \( \text{Sym} \) and the elementary properties of the conformal maps that the boundary of \( E(r) \) is almost a circle of radius
\[
\frac{r(1 + o(1))|Q'(z_0)|}{|T_n^r(T_n^{-1}(|Q(z_0)|))|}
\]
centred at the origin; on the boundary of \( B(r) \), we have
\[
|\omega| = \frac{1}{2r(1 + o(1))} \left( \frac{2|c_n|}{c} \right)^{\frac{1}{2}}, \quad r \to 0.
\]

Therefore, inequality (5) gives
\[
\text{(6)} \quad \frac{1}{2\pi} \log \left[ \frac{|T_n^r(T_n^{-1}(|Q(z_0)|))|}{2|Q'(z_0)|} \left( \frac{2|c_n|}{c} \right)^{\frac{1}{2}} \right] \geq \frac{1}{\text{cap} C_2(r_1, r_2)} - \frac{1}{\text{cap} C_2(r_1, r_2)} \geq 0.
\]

This implies inequality (1) for \( z = z_0 \) under the condition \( P'(z_0) \neq 0 \). In the case when \( P'(z_0) = 0 \) this inequality is a clear.

Now, suppose that (1) turns into equality for \( z = z_0 \). Then \( P'(z_0) \neq 0 \) and (6) implies that
\[
\text{cap} \tilde{C}_2(r_1, r_2) = \text{cap} C_2(r_1, r_2)
\]
for all \( r_1 \) and \( r_2, 0 < r_1 < r_2 < \infty \). It follows from the equality conditions (i) and (ii) in the Lemma that the condenser \( Q(\varphi(C_2(r_1, r_2))) \) is obtained from \( \text{Sym} Q(\varphi(C_2(r_1, r_2))) \) by a rotation through an angle \( \theta \) about the origin. As the ramification points are mapped to the ramification points under rotation, the angle \( \theta \) is the same for all small \( r_1 \) and large \( r_2 \). Therefore, the superposition
\[
\psi \circ T_n^{-1} \circ \lambda_\theta \circ Q \circ \varphi
\]
(\( \lambda_\theta(W) = e^{-i\theta}W \)) is a univalent conformal map of the \( \zeta \)-plane onto \( \omega \)-plane which takes 0 to 0 and \( \infty \) to \( \infty \). This yields the representation (2). Conversely, it is seen from a direct calculation that for any polynomial of the form (2), (1) becomes an equality. This completes the proof of the theorem.
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