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A NOTE ON REGULAR SUBGROUPS OF THE AUTOMORPHISM GROUP OF THE LINEAR HADAMARD CODE

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Abstract. We consider the subgroups of the automorphism group of the linear Hadamard code that act regularly on the codewords of the code. These subgroups correspond to the regular subgroups of the general affine group $\mathrm{GA}(r,2)$ with respect to the action on the vectors of F_2^r, where $n = 2^r - 1$ is the length of the Hamadard code. We show that the dihedral group $D_{2^{r-1}}$ is a regular subgroup of $\mathrm{GA}(r,2)$ only when $r = 3$. Following the approach of [13] we study the regular subgroups of the automorphism group of the Hamming code obtained from the regular subgroups of the automorphism group of the Hadamard code of length 15.

Keywords: error-correcting code, automorphism group, regular action, affine group.

1. Introduction

Let F_2^n be n-dimensional vector space over the field F_2 of two elements. The *Hamming distance* $d(x, y)$ between vectors x and y is defined to be the number of coordinate positions at which two vectors differ. A *code* of length n is an arbitrary set of vectors of F_2^n which are called codewords of C. A code is called *linear* if it is a linear subspace of F_2^n. The *minimum distance* of a code is the minimum of the Hamming distances between distinct codewords. For any $r, r \geq 2$, $n = 2^r - 1$ *the Hamming code* $\mathcal{H}_n$ is the solution space of the system $H_r x^T = 0$, where H_r is $r \times 2^r - 1$ matrix, whose columns are all nonzero vectors of F_2^r. Equivalently,

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the Hamming code $\mathcal{H}_n$ is the linear perfect code (i.e. attaining the well-known Hamming bound) with minimum distance 3. The *Hadamard code* $\mathcal{A}_n$ is the linear code spanned by rows of H_r. There are other definitions of the Hadamard code, which are not necessarily linear [12]. It is well-known that any linear code of length $n = 2^r - 1$, dimension r and minimum distance 2^{r-1} is $\mathcal{A}_n$ up to a permutation of the coordinate positions.

Let x be a binary vector of F_2^n, π be a permutation of the coordinate positions of x. Consider the transformation (x, π) acting on a binary vector y by the following rule:

$$(x, \pi)(y) = x + \pi(y),$$

where $\pi(y) = (y_{\pi^{-1}(1)}, \dots, y_{\pi^{-1}(n)})$. The composition of two transformations (x, π), (y, π') is defined as follows

$$(x, \pi) \cdot (y, \pi') = (x + \pi(y), \pi \circ \pi'),$$

where $\circ$ is the composition of permutations π and π'.

The *automorphism group* $\mathrm{Aut}(C)$ of a code C is the group of all transformations (x, π) such that $x + \pi(C) = C$ with respect to composition. The *symmetry group* (also known as the permutation automorphism group) of C is defined as $\mathrm{Sym}(C) = \{\pi \in S_n : \pi(C) = C\}$.

A group G acting on a set M is called *regular* if the action is transitive and the order of G coincides with the size of M. Throughout the paper a subgroup of the automorphism group of a code is called regular if it acts regularly w.r.t. on the codewords of C. A code C is *propelinear* [14] if Aut(C) contains a regular subgroup.

Propelinear codes provide a general view on additive and several classes of optimal codes. The concept is specially important in cases where there are many nonisomorphic codes with the same parameters, separating the codes that are "close" to linear. In particular, among propelinear codes there are Z_2Z_4-linear codes that could be defined via Gray map. Generally, the Hadamard codes are those that could be obtained from Hadamard matrices of order n [12] and can have length n or $n-1$. Z_2Z_4-linear perfect codes were classified in [3], while Z_2Z_4-linear Hadamard codes were classified in [5], [10], along with the description of their automorphism groups in [9]. In work [15] $Z_2Z_4Q_8$-Hadamard codes are discussed. Another point of study is finding a proper generalization of the Gray map, and its further implementation for construction of codes, see [11] for a research on Z_{2^k}-linear Hadamard codes.

The *general linear group* $\mathrm{GL}(r, q)$ is the group of invertible $r \times r$ matrices over the field F_q of q elements. The *general affine group* $\mathrm{GA}(r, q)$ is the group of affine transformations of F_q^r, i.e. transformations (a, A), $a \in F_q^r$, $A \in \mathrm{GL}(r, q)$ acting on a vector $x \in F_q^r$ in the following way: $(a, A)(x) = a + Ax$. Throughout the paper by a regular subgroup of the general affine group $\mathrm{GA}(r, q)$ we mean a group which is regular w.r.t. its natural action on the vectors from F_q^r. Consider the following embedding of $\mathrm{GA}(r, q)$ to $\mathrm{GL}(r+1, q)$: $(a, A) \to \begin{pmatrix} A & a \\ 0^r & 1 \end{pmatrix}$. It is easy to see that the regular subgroups of $\mathrm{GA}(r, q)$ correspond to the subgroups of the matrix representation in $\mathrm{GL}(r+1, q)$ that act regularly on the set of vectors $\{(x, 1), x \in F_q^r\}$ [7].

In Section 2 of the current paper we give auxiliary statements. In particular, we show that the regular subgroups of $\mathrm{Aut}(\mathcal{A}_n)$ correspond to those of the general affine group $\mathrm{GA}(\log(n+1), 2)$.

There are few references on regular subgroups of the affine group from strictly algebraic point of view. A regular subgroup of $\mathrm{GA}(r, q)$ without nontrivial translations was constructed in [8]. In works [6], [7] it was shown that the abelian regular subgroups of $\mathrm{GA}(r, q)$ correspond to certain algebraic structures on the vector space F_q^r. The example below is from [7]. Consider the matrix representation of $\mathrm{G}A(r, q)$ in $\mathrm{G}L(r+1, q)$ described above. Let G be its subgroup which is the centralizer of the Jordan block of size $r+1$ in the group of upper triangular matrices. The group G acts regularly on elements $\{(x, 1),\ x \in F_q^r\}$, thus it gives rise to an example of a regular subgroup of $\mathrm{G}A(r, q)$.

One of the main problems arising in the theory of propelinear codes is a construction of codes with regular subgroups of their automorphism group that are abelian or "close" to them in a sense, such as for example Z_4^l, cyclic or dihedral groups. The same question could be asked for the regular subgroups of the affine group. In Section 3, we see that the dihedral group is a regular subgroup of the affine group if and only if $r = 3$. The nontrivial case of the proof is $r = 4$, when there is a dihedral subgroup of the affine group, but there is no regular dihedral subgroup. The maximum order of the elements in the regular subgroups of $\mathrm{G}A(r, 2)$ is $2^{\lfloor \log_2 r \rfloor + 1}$ which follows from the work [2]. This implies that the dihedral group of order 2^r is not a subgroup of $\mathrm{G}A(r, 2)$ for $r \geq 4$.

The Hamming code $\mathcal{H}_n$ is known to have the largest order of the automorphism group in the class of perfect binary codes of length n [17] and it would be natural to suggest that it has the maximum number of the regular subgroups of its automorphism group among propelinear perfect codes. However, the fact that

$$|\mathrm{Aut}(\mathcal{H}_n)| = |\mathrm{GL}(\log(n+1), 2)| 2^{n - \log(n+1)}$$

makes attempts of even partial classification of regular subgroups impossible for ordinary calculational machinery starting with length $n = 15$.

Regular subgroups of the automorphism group of the Hamming code could be constructed from the regular subgroups of the automorphism group of its subcodes whose automorphism groups are embedded into that of the Hamming code in a certain way. In work [13], this idea was implemented for the Nordstrom-Robinson code in case of extended length $n = 16$. In Section 4 we apply this approach to regular subgroups of the automorphism group of the Hadamard code of length 15.

2. Preliminaries

We begin with the following two well-known facts, e.g. see [12].

Proposition 1. *Let C be a linear code of length n. Then*

$$\mathrm{Aut}(C) = F_2^n \leftthreetimes \mathrm{Sym}(C) = \{(x, \pi) : x \in C, \pi \in \mathrm{Sym}(C)\}.$$

The Hadamard code is known to be the dual code of the Hamming code of length $n = 2^r - 1$, which implies that their symmetry groups coincide and are isomorphic to the general linear group of F_2^r.

Proposition 2. *Let $\mathcal{A}_n$ and $\mathcal{H}_n$ be respectively the Hadamard and the Hamming codes of length $n = 2^r - 1$. Then*

$$\mathrm{Sym}(\mathcal{H}_n) = \mathrm{Sym}(\mathcal{A}_n) \cong \mathrm{GL}(r,2).$$

As far as the automorphism groups are concerned, the following fact holds.

Proposition 3. *Let $\mathcal{A}_n$ be the Hadamard code of length $n = 2^r - 1$. Then $\mathrm{Aut}(\mathcal{A}_n) \cong \mathrm{GA}(r,2)$ and the action of $\mathrm{Aut}(\mathcal{A}_n)$ on the codewords of $\mathcal{A}_n$ is equivalent to the natural action of $\mathrm{GA}(r,2)$ on the vectors of F_2^r. In particular, the regular subgroups of $\mathrm{Aut}(\mathcal{A}_n)$ correspond to the regular subgroups of $\mathrm{GA}(r,2)$.*

Proof. We use a well-known representation of the Hadamard code, see e.g. [12]. For a vector $a \in F_2^r$ consider the vector c_a of values of the function $\sum_{i=1,\dots,n} x_i a_i$ of variable x from $F_2^r \setminus 0^r$ to F_2. It is easy to see that the code $\mathcal{A}_n = \{c_a : a \in F_2^r\}$ is linear of length $n = 2^r - 1$, dimension r and minimum distance $(n+1)/2$, i.e. $\mathcal{A}_n$ is the Hadamard code. By Propositions 1 and 2 any automorphism of $\mathrm{Aut}(\mathcal{A}_n)$ is (c_a, π_A) for a vector $a \in F_2^r$ and $A \in \mathrm{GL}(r,2)$, therefore the mapping $(c_a, \pi_A) \to (a, A)$ is an isomorphism from $\mathrm{Aut}(\mathcal{A}_n)$ to $\mathrm{GA}(r,2)$.

□

In [2] the maximum orders of elements of $\mathrm{GL}(r,q)$ were described. In particular, the following was shown:

Proposition 4. *The maximum of orders of elements of $\mathrm{GL}(r,2)$ of type 2^l is $2^{1+\lfloor \log_2(r-1) \rfloor}$.*

Corollary 1. *The maximum of orders of the elements of $\mathrm{GA}(r,2)$ of type 2^l is $2^{1+\lfloor \log_2 r \rfloor}$.*

Proof. The representation for $GA(r,2)$ in $GL(r+1,2)$: $(a,A) \to \begin{pmatrix} A & a \\ 0^r & 1 \end{pmatrix}$ and Proposition 4 implies that the orders of elements of type 2^l in $GA(r,2)$ are upper bounded by $2^{\lfloor \log_2 r \rfloor} + 1$. On the other hand it is easy to see that the Jordan block of size $r+1$, which belongs to the matrix representation of $GL(r,2)$ is of order $2^{\lfloor \log_2 r \rfloor + 1}$. □

We finish the section by giving a version of the direct product construction for regular subgroups of the general affine group.

Proposition 5. *Let G and G' be regular subgroups of $\mathrm{GA}(r,2)$ and $\mathrm{GA}(r',2)$ repectively. Then there is a regular subgroup of $\mathrm{GA}(r+r',2)$ isomorphic to $G \times G'$.*

Proof. Given elements $\alpha = (a,A)$ of $\mathrm{GA}(r,2)$ and $\beta = (b,B)$ of $\mathrm{GA}(r',2)$, define $\alpha \cdot \beta$ to be $((a|b), \begin{pmatrix} A & 0^{rr'} \\ 0^{r'r} & B \end{pmatrix})$, where $(a|b)$ is the concatenation of vectors a and b. Obviously, the elements $\{\alpha \cdot \beta : \alpha \in G, \beta \in G'\}$ form a regular subgroup of $\mathrm{GA}(r+r',2)$, isomorphic to $G \times G'$.

□

3. Dihedral regular subgroups of $\mathrm{GA}(r,2)$

The cyclic group Z_{2^r} is not a regular subgroup of $\mathrm{GA}(r,2)$ for any r, $r \geq 3$ as we see from the bound in Corollary 1. Therefore, we address the question of being a regular subgroup of the affine group to other groups, that are close to cyclic. The dihedral group, which we denote by D_n, is the group composed by all $2n$ symmetries of the n-sided polygon. It is well-known that any group generated by an element α of order n and an involution β satisfying $\beta\alpha\beta = \alpha^{-1}$ is isomorphic to D_n. By Corollary 1 we see that there are no elements of order 2^{r-1} in $\mathrm{G}A(r,2)$. Therefore $D_{2^{r-1}}$ is not a subgroup of $\mathrm{G}A(r,2)$ for $r \geq 5$.

Theorem 1. *$D_{2^{r-1}}$ is a regular subgroup of* $\mathrm{GA}(r,2)$ *if and only if* $r = 3$.

Proof. Consider the subgroup G of $\mathrm{GA}(3,2)$ generated by $(a,A),(b,I)$, where $a = (101)^T$, $b = (011)^T$, $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Here and below I denotes the identity matrix. The orbit of $(000)^T$ under the action of the subgroup generated by (a,A) consists of vectors $a = (101)^T, a+Aa = (100)^T, Aa = (001)^T, (000)^T$. Since the orbit is a subspace that does not contain the vector b, G acts transitively on the elements of F_2^3. Moreover, $(b,I)(a,A)(b,I) = (Ab+a+b, A) = ((001)^T, A) = (a,A)^{-1}$, so G is D_4 and it is regular.

Suppose there is a regular subgroup of $\mathrm{GA}(4,2)$, generated by an element (a,A) of order 8 and an element (b,B) of order 2, satisfying relation

$$(b,B)(a,A)(b,B) = (a,A)^{-1}. \tag{1}$$

Note that the order of A is 4 by Proposition 4.

Since $(b,B)^2 = (0^4, I)$, we have that

$$b = Bb. \tag{2}$$

We have the following:

$$(a,A)^{-1} = (a,A)^7 = (\sum_{j=0,\ldots,6} A^j a, A^3) = (A^3 a, A^3) =$$
$$(b,B)(a,A)(b,B) = (Ba + BAb + b, BAB),$$

Using equality (2), $BAB = A^3$ and $B^2 = I$ we have that $BAb = A^3 b$, so we obtain

$$Ba = A^3 b + A^3 a + b. \tag{3}$$

The matrix A is similar to the Jordan block of size 4 with the eigenvalue 1. Since $(a,A)^4 = ((I+A)^3 a, I)$ and the transformation (a,A) is of order 8, then the vector $(I+A)^3 a$ is nonzero. Moreover, since A has order 4, the vector $(I+A)^3 a = (I+A+A^2+A^3)a$ is the unique eigenvector of A. The Jordan chain (the basis for which A is the Jordan block) containing $(I+A)^3 a$ are vectors $a, (I+A)a, (I+A)^2 a, (I+A)^3 a$, which implies that $a, Aa, A^2 a, A^3 a$ is a basis of F_2^4, so

$$b = c_0 a + c_1 Aa + c_2 A^2 a + c_3 A^3 a, \tag{4}$$

for some c_i in F_2, $i \in \{0,\ldots,3\}$.

Substituting the expression (4) for b into the equality (3), we obtain the following expression for Ba:

$$Ba = (c_0 + c_1)a + (c_1 + c_2)Aa + (c_2 + c_3)A^2 a + (c_0 + c_3 + 1)A^3 a. \tag{5}$$

Putting the expression (4) for b into (2) and using the equality $BAB = A^3$, we obtain

$$c_0Ba + c_1A^3Ba + c_2A^2Ba + c_3ABa + c_0a + c_1Aa + c_2A^2a + c_3A^3a = 0^4.$$

Substituting the expression (5) for Ba in the previous equality, we obtain that

$$(c_0c_3 + c_0c_1 + c_1c_2 + c_2c_3 + c_2 + c_3)(a + Aa + A^2a + A^3a) = 0^4.$$

Finally, we see that the only binary vectors $(c_0c_1c_2c_3)$ satisfying

$$c_0c_3 + c_0c_1 + c_1c_2 + c_2c_3 + c_2 + c_1 = 0$$

are exactly

$$(0000), (1000), (1100), (1110), (1111), (0111), (0011), (0001),$$

which are, in turn, the coefficients of the linear combinations expressing elements $\sum_{j=0,\dots,i} A^ja$, $0 \leq i \leq 7$ in the basis a, Aa, A^2a, A^3a. Therefore for some i the elements $(b, B) = (\sum_{j=0,\dots,i} A^ja, B)$ and $(a, A)^{i+1} = (\sum_{j=0,\dots,i} A^ja, A^{i+1})$ are distinct elements of the dihedral subgroup, sending 0^4 to b. We conclude that the considered group is not regular.

□

Remark 1. The subgroup of GA(4, 2) generated by $((0001)^T, A)$ and $((0000)^T, B)$ of GA(4, 2), where $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is an irregular subgroup of GA(4, 2) isomorphic to D_8.

Remark 2. Consider the elements $((0001)^T, A)$ and $((0100)^T, B)$, where A is the same as in Remark 1, $B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. It is easy to see that the element $((0100)^T, B)$ is an involution, $((0001)^T, A)$ is of order 8. The elements commute and moreover they generate a regular subgroup isomorphic to Z_2Z_8. Actually, the group is isomorphic to the abelian regular subgroup, arising from the centralizer of the Jordan block of size 5 in the group of upper triangular 5×5 matrices, described in [7].

4. Extending to regular subgroups of the automorphism group of the Hamming code of length 15

Regular subgroups of the automorphism group of the Hamming code could be constructed from regular subgroups of its subcodes whose automorphism groups are embedded into that of the Hamming code in a certain way. In work [13], narrow-sense extensions of the regular subgroups of the automorphism group of Nordstrom-Robinson code to those of the automorphism group of the extended Hamming code were considered. In the section we extend the regular subgroups of the automorphism group of the Hadamard code to regular subgroups of the automorphism group of the Hamming code.

Given a subgroup G of $\mathrm{Aut}(F_2^n) = \{(x, \pi) : x \in F_2^n, \pi \in S_n\}$, denote by Π_G the subgroup of S_n whose elements are $\{\pi : (x, \pi) \in G\}$. Let H and G be subgroups of $\mathrm{Aut}(F_2^n)$. Following [13] we say that a group H is *narrow-sense extended* to a subgroup G if $H \leq G$ and $\Pi_G = \Pi_H$.

It is well-known that the Hadamard code $\mathcal{A}_{15}$ and the punctured Nordstrom-Robinson, which we denote by $\mathcal{N}$, are subcodes of the code $\mathcal{H}_{15}$ [12], [16]. Obviously, $\Pi_{\mathrm{Aut}(C)} = \mathrm{Sym}(C)$ if C is linear. So, $\mathrm{Aut}(\mathcal{A}_{15})$ is narrow-sense extended to that of the Hamming code $\mathcal{H}_{15}$, because their symmetry groups coincide (see Proposition 2). The linear span of the punctured Nordstrom-Robinson code is the Hamming code [16], thus its automorphism group is a subgroup of that of the Hamming code. Moreover, the extension is in the narrow sense. We recall a description of the symmetry group of $\mathcal{N}$ from [1].

Proposition 6. $\mathrm{Sym}(\mathcal{N}) \cong A_7 < \mathrm{Sym}(\mathcal{H}_{15}) \cong \mathrm{GL}(4,2) \cong A_8$.

Corollary 2. $\mathrm{Aut}(\mathcal{N})$ *is narrow-sense extended to* $\mathrm{Aut}(\mathcal{H}_{15})$.

Proof. The punctured Nordstrom-Robinson code is propelinear. Then it is not hard to see that $|\Pi_{\mathrm{Aut}(\mathcal{N})}| = |\mathrm{Sym}(\mathcal{N})||\mathcal{N}|/|Ker(\mathcal{N})|$, where $Ker(\mathcal{N}) = \{x \in \mathcal{N} : x + \mathcal{N} = \mathcal{N}\}$ see e.g. [4], Proposition 4.3. Then, since $Ker(\mathcal{N})$ is $\mathcal{A}_{15}$ augmented by the all-ones vector, see [16] and the size of $\mathcal{N}$ is 2^8, we see that

$$|\Pi_{\mathrm{Aut}(\mathcal{N})}| = 8|\mathrm{Sym}(N)| = |\mathrm{Sym}(\mathcal{H}_{15})|. \tag{6}$$

Let π be an element of $\Pi_{\mathrm{Aut}(\mathcal{N})}$, i.e. $x + \pi(\mathcal{N}) = \mathcal{N}$. The linear span of $\mathcal{N}$ is $\mathcal{H}_{15}$ [16], therefore π is a symmetry of $\mathcal{H}_{15}$. Taking into account the equality (6), we obtain that $\Pi_{\mathrm{Aut}(\mathcal{N})} = \mathrm{Sym}(\mathcal{H}_{15}) = \Pi_{\mathrm{Aut}(\mathcal{H}_{15})}$.

□

Now we find the regular subgroups of the automorphism group of $\mathrm{Aut}(\mathcal{A}_{15})$ (regular subgroups of $\mathrm{GA}(4,2)$). The results below were obtained using PC.

Theorem 2. *There are* 39 *conjugacy classes of regular subgroups of* $\mathrm{Aut}(\mathcal{A}_{15})$*, which fall into 11 isomorphism classes.*

Remark 3. Four of 11 isomorphism classes of regular subgroups are abelian and are isomorphic to the groups Z_2^4, Z_2Z_8, $Z_2^2Z_4$ and Z_4^2. A group isomorphic to Z_2Z_8 is given in Remark 2. It is not hard to see that there is a regular subgroup of $\mathrm{GA}(2,2)$, isomorphic to Z_4. Then the regular subgroups isomorphic to $Z_2^2Z_4$ and Z_4^2 could be constructed using the direct product construction (see Proposition 5).

The narrow-sense extensions into regular subgroups of the automorphism group of the Hamming code were found. The bound on the number of isomorphism classes below was obtain by comparing the orders of the centralizers of elements of groups.

Theorem 3. *The regular subgroups of* $\mathrm{Aut}(\mathcal{A}_{15})$ *are narrow-sense extended to at least 1207 conjugacy classes of regular subgroups of* $\mathrm{Aut}(\mathcal{H}_{15})$*, which fall into at least 48 isomorphism classes.*

The result is somewhat disappointing, as extensions of Nordstrom-Robinson code in Hamming code gave significantly better bound for isomorphism classes.

Theorem 4. [13] *There are* 73 *conjugacy classes of regular subgroups of* $\mathrm{Aut}(\mathcal{N})$ *that fall into* 45 *isomorphism classes. The regular subgroups of* $\mathrm{Aut}(\mathcal{N})$ *are narrow-sense extended to exactly* 605 *conjugacy classes of regular subgroups of* $\mathrm{Aut}(\mathcal{H}_{15})$*, which fall into at least* 219 *isomorphism classes.*

One might suggest a tighter interconnection of regular subgroups of $\mathrm{Aut}(\mathcal{A}_{15})$ and that of $\mathrm{Aut}(\mathcal{N})$. However, despite that $\mathcal{A}_{15} \subset \mathcal{N}$, $\mathrm{Aut}(\mathcal{A}_{15})$ is not extended to that of $\mathrm{Aut}(N)$ in narrow sense, which in turn, follows, for example, from a proper containment of $\mathrm{Sym}(\mathcal{N})$ in $\mathrm{Sym}(\mathcal{A}_{15})$, see Proposition 6. Moreover, only 6 of 39 conjugacy classes of regular subgroups of $\mathrm{Aut}(\mathcal{A}_{15})$ are subgroups of $\mathrm{Aut}(\mathcal{N})$.

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