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ON MODELING ELASTIC BODIES WITH DEFECTS

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Abstract. The paper concerns a mathematical analysis of equilibrium problems for 2D elastic bodies with thin defects. The defects are characterized with a damage parameter. A presence of defects implies that the problems are formulated in a nonsmooth domain with a cut. Nonlinear boundary conditions at the cut faces are considered to prevent a mutual penetration between the faces. Weak and strong formulations of the problems are analyzed. The paper provides an asymptotic analysis with respect to the damage parameter. We obtain invariant integrals over curves surrounding the defect tip. An optimal control problem is investigated with a cost functional equal to the derivative of the energy functional with respect to the defect length, and the damage parameter being a control function.

Keywords: Defect, damage parameter, non-penetration boundary conditions, variational inequality, optimal control, derivative of energy functional.

1. Introduction

Progress in the development of many areas of modern engineering, aerospace engineering, civil engineering and other specialties is associated with an increase in the share of the use of complex materials. This is due to the fact that their use increases the strength properties and bearing capacity of the elements structures, while the weight of products decreases simultaneously. We also note that materials containing different inclusions are now widely used, since they have high strength and rigidity. Due to the lack of adhesive ability of inclusions, they can be detached from the elastic matrix. As a result, cracks and other defects may appear in the structures. In this case, the presence of defects reduces the rigidity and strength of

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structures. Thus, the study of the properties of heterogeneous materials containing thin inclusions in the presence of defects is important from the point of view of constructing complex materials and optimizing their characteristics. The study of the corresponding models of continuous media considered in non-smooth domains is very important for describing complex materials and is of considerable theoretical interest due to the presence of solution singularities. The problem considered in the paper provides a mathematical analysis of equilibrium problems for elastic bodies with defects. In the frame of this paper, a defect means a manifold located inside of the elastic body whose dimension is less compared to the dimension of the body. The defect may have different properties depending on a suitable parameter (damage parameter). In particular, as the parameter goes to infinity and to zero we obtain well known crack models and elasticity models without any singularities, respectively. The last years, a lot of papers and books were published related to the crack models with non-linear boundary conditions at the crack faces excluding a mutual penetration between the faces [5, 6, 7, 11, 13, 16, 18, 23]. This analysis included, in particular, a shape sensitivity of solutions, see also [3, 4]. Moreover, equilibrium problems for elastic bodies with thin delaminated inclusions and non-linear boundary conditions were investigated [9, 10, 24]. We also may mention different approaches for describing thin inclusions in elastic bodies [20, 28]. General results related to elastic composite structures can be found in [15, 19].

In the paper, we provide differential and variational formulations of the equilibrium problem for elastic bodies with defects. Asymptotic analysis is fulfilled provided that the damage parameter goes to zero and to infinity. We derive a formula for the derivative of the energy functional with respect to the defect length which can be used provided that the Griffith rapture criterion is applied for describing the defect propagation. We prove a solution existence of an optimal control problem with a cost functional related to the Griffith criterion, and the damage parameter being a control function.

As for optimal control problems related to elastic structures with cracks and thin inclusions in the frame of models with zero friction and non-penetration boundary conditions, we refer the reader to [8, 9, 17, 25, 26, 27]. Different crack models with boundary conditions providing a connection between the crack faces can be found in [1, 2, 12, 14, 21, 22].

2. Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ. Denote $\gamma_0 = (0,1)\times\{0\}$ assuming $\bar{\gamma}_0 \subset \Omega$, see Fig. 1. Let $\nu = (0,1) = (\nu_1, \nu_2)$ be a unit normal vector to γ_0; $\tau = (1,0) = (\tau_1, \tau_2)$ is a tangential vector, $\Omega_0 = \Omega \setminus \bar{\gamma}_0$.

The domain Ω_0 represents a region occupied with an elastic material, and γ_0 fits to a defect in the material. The damage parameter $\delta > 0$ characterizes the defect properties. Inequality type boundary conditions will be imposed at the defect faces $\gamma_0^{\pm}$ to prevent a mutual penetration. In this section, we plan to analyze a problem formulation for a fixed δ, and then to investigate passages to limits as $\delta \to 0, \infty$.

By $A = \{a_{ijkl}\}, i,j,k,l = 1,2$, we denote a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$a_{ijkl} = a_{jikl} = a_{klij},\ \ i,j,k,l = 1,2,\ a_{ijkl} \in L^\infty(\Omega),$$
$$a_{ijkl}\xi_{ij}\xi_{kl} \ge c_0|\xi|^2 \quad \forall \xi_{ij},\ c_0 = const > 0.$$

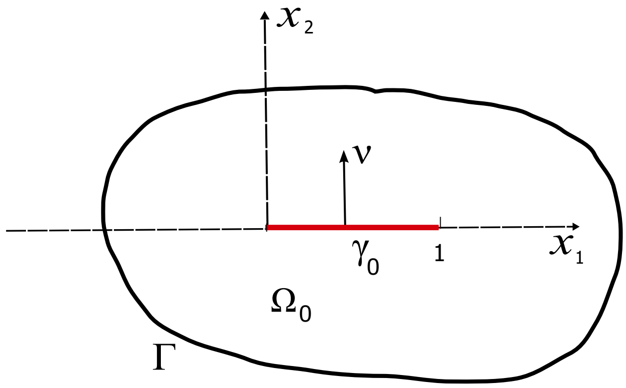

FIG. 1. Geometry of the problem

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices.

An equilibrium problem for the body Ω_0 and the defect γ_0 is formulated as follows. For given external forces $f = (f_1, f_2) \in L^2(\Omega)^2$ acting on the body we want to find a displacement field $u = (u_1, u_2)$ and a stress tensor $\sigma = \{\sigma_{ij}\}, i, j = 1, 2,$ defined in Ω_0 such that

$$-\text{div}\ \sigma = f,\ \sigma - A\varepsilon(u) = 0 \ \text{ in } \Omega_0, \tag{1}$$

$$u = 0 \text{ on } \Gamma, \tag{2}$$

$$[u_\nu] \geq 0, \quad [\sigma_\nu] = 0,\ [\sigma_\tau] = 0 \text{ on } \gamma_0, \tag{3}$$

$$-\sigma_\nu + \frac{1}{\delta}[u_\nu] \geq 0,\ -\sigma_\tau + \frac{1}{\delta}[u_\tau] = 0 \text{ on } \gamma_0, \tag{4}$$

$$[u_\nu](-\sigma_\nu + \frac{1}{\delta}[u_\nu]) = 0 \text{ on } \ \gamma_0. \tag{5}$$

Here $[\phi] = \phi^+ - \phi^-$ is a jump of a function ϕ on γ_0, where $\phi^\pm$ are the traces of ϕ on the defect faces $\gamma_0^\pm$. The signs $\pm$ correspond to positive and negative directions of ν; $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the strain tensor, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ i, j = 1, 2;\ \sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j),\ \sigma_\nu = \sigma_{ij}\nu_j\nu_i,\ \sigma_\tau = \sigma\nu \cdot \tau,\ u_\nu = u\nu.$

Relations (1) are the equilibrium equations for the elastic body and Hooke's law. The first inequality of (3) provides a non-penetration between the defect faces. If there is no contact at a given point $x_0 \in \gamma_0$, i.e. $[u_\nu(x_0)] > 0$, from (5) we get the following relation: $(-\sigma_\nu + \frac{1}{\delta}[u_\nu])(x_0) = 0$. On the other hand, if $(-\sigma_\nu + \frac{1}{\delta}[u_\nu])(x_0) > 0$ we obtain a contact condition $[u_\nu(x_0)] = 0$.

The problem (1)-(5) admits a variational formulation. To this end, introduce the Sobolev space

$$H^1_\Gamma(\Omega_0) = \{v \in H^1(\Omega_0) \mid v = 0 \text{ on } \Gamma\}$$

and consider the energy functional

$$\Pi(v) = \frac{1}{2}\int\limits_{\Omega_0} \sigma(v)\varepsilon(v) - \int\limits_{\Omega_0} fv + \frac{1}{2\delta}\int\limits_{\gamma_0} [v]^2.$$

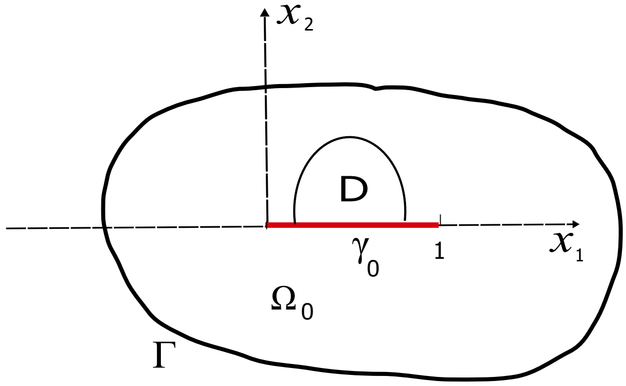

FIG. 2. Neighborhood D

Here $\sigma(v)$ is defined from the second relation of (1), i.e., $\sigma(v) = A\varepsilon(v)$, and for simplicity we write $\sigma(v)\varepsilon(v) = \sigma_{ij}(v)\varepsilon_{ij}(v)$, $fv = f_i v_i$.

Let K_0 be a convex set of kinematically admissible displacements,

$$K_0 = \{v \mid v \in H^1_\Gamma(\Omega_0)^2,\ [v_\nu] \geq 0 \text{ on } \gamma_0\}.$$

Then the minimization problem

$$\inf_{v \in K_0} \Pi(v)$$

has a unique solution satisfying the variational inequality

$$u \in K_0, \int\limits_{\Omega_0} \sigma(u)\varepsilon(v-u) - \int\limits_{\Omega_0} f(v-u) + \frac{1}{\delta}\int\limits_{\gamma_0} [u][v-u] \geq 0 \quad \forall\, v \in K_0. \tag{6}$$

To check a solvability of the problem (6) it suffices to establish a coercivity of the functional Π on the set K_0 since its weak lower semicontinuity in the space $H^1_\Gamma(\Omega_0)^2$ is obvious. We do not go in details, see a close situation concerning a solvability in [6], pp. 47-48.

In what follows we check an equivalence of (1)-(5) and (6) for smooth solutions.

Theorem 1 *Problem formulations* (1)-(5) *and* (6) *are equivalent provided that* $u \in H^2(\Omega_0)^2$.

Proof Let (6) be fulfilled. We take test functions of the form $v = u \pm \varphi$, $\varphi \in C_0^\infty(\Omega_0)^2$. It gives the equilibrium equation fulfilled in the sense of distributions, see (1). Next, take a test function of the form $v = u + \varphi$ in (6), $\varphi_\nu \geq 0$ on γ_0^+; *supp* $\varphi \subset \bar{D}$, see Fig 2. It gives

$$\int\limits_{\Omega_0} \sigma(u)\varepsilon(\varphi) - \int\limits_{\Omega_0} f\varphi + \frac{1}{\delta}\int\limits_{\gamma_0} [u][\varphi] \geq 0.$$

Integrating here by parts, we find

$$-\int\limits_{\gamma_0} (\sigma\nu)^+ \cdot \varphi^+ + \frac{1}{\delta}\int\limits_{\gamma_0} [u][\varphi] \geq 0$$

or

$$-\int_{\gamma_0} \sigma_\nu^+ \varphi_\nu^+ - \int_{\gamma_0} \sigma_\tau^+ \varphi_\tau^+ + \frac{1}{\delta}\int_{\gamma_0} [u_\nu]\varphi_\nu^+ + \frac{1}{\delta}\int_{\gamma_0} [u_\tau]\varphi_\tau^+ \geq 0.$$

From here, boundary conditions follow

$$-\sigma_\nu^+ + \frac{1}{\delta}[u_\nu] \geq 0, \ -\sigma_\tau^+ + \frac{1}{\delta}[u_\tau] = 0 \text{ on } \gamma_0. \tag{7}$$

Similarly, we obtain the following boundary conditions at γ_0 :

$$-\sigma_\nu^- + \frac{1}{\delta}[u_\nu] \geq 0, \ -\sigma_\tau^- + \frac{1}{\delta}[u_\tau] = 0 \text{ on } \gamma_0, \tag{8}$$

consequently we derive (4).

From the second relations of (7), (8), we obtain $[\sigma_\tau] = 0$ on γ_0.

Now, we substitute $v = u \pm \varphi$ in (6), $[\varphi_\nu] = 0$ on γ_0. It gives

$$\int_{\Omega_0} \sigma(u)\varepsilon(\varphi) - \int_{\Omega_0} f\varphi + \frac{1}{\delta}\int_{\gamma_0} [u][\varphi] = 0$$

or

$$-\int_{\gamma_0} [\sigma\nu \cdot \varphi] + \frac{1}{\delta}\int_{\gamma_0} [u][\varphi] = 0.$$

Hence

$$-\int_{\gamma_0} [\sigma_\nu]\varphi_\nu - \int_{\gamma_0} \sigma_\tau[\varphi_\tau] + \frac{1}{\delta}\int_{\gamma_0} [u_\nu][\varphi_\nu] + \frac{1}{\delta}\int_{\gamma_0} [u_\tau][\varphi_\tau] = 0.$$

The third term here is equal to zero by the choice of φ. The sum of the second and the forth term is zero by (7), (8). Since φ_ν is arbitrary on γ_0, it provides

$$[\sigma_\nu] = 0 \text{ on } \gamma_0.$$

To conclude the arguments, we substitute test functions $v = 0, v = 2u$ in (6). It implies

$$\int_{\Omega_0} \sigma(u)\varepsilon(u) - \int_{\Omega_0} fu + \frac{1}{\delta}\int_{\gamma_0} [u]^2 = 0.$$

Thus

$$-\int_{\gamma_0} \sigma_\nu[u_\nu] - \int_{\gamma_0} \sigma_\tau[u_\tau] + \frac{1}{\delta}\int_{\gamma_0} [u_\nu]^2 + \frac{1}{\delta}\int_{\gamma_0} [u_\tau]^2 = 0$$

or

$$\int_{\gamma_0} [u_\nu](-\sigma_\nu + \frac{1}{\delta}[u_\nu]) + \int_{\gamma_0} [u_\tau](-\sigma_\tau + \frac{1}{\delta}[u_\tau]) = 0.$$

The second term here is equal to zero. Taking into account nonnegativity of the terms in the first integral we arrive at boundary condition (5).

Conversely, let (1)-(5) be fulfilled. Take $v \in K_0$ and multiply the first equation of (1) by $v - u$. We get

$$\int_{\Omega_0} (-\text{div } \sigma - f)(v - u) = 0.$$

Hence

$$\int\limits_{\Omega_0} \sigma(u)\varepsilon(v-u) - \int\limits_{\Omega_0} f(v-u) + \int\limits_{\gamma_0} [\sigma\nu(v-u)] + \pm\frac{1}{\delta}\int\limits_{\gamma_0} [u][v-u] = 0. \tag{9}$$

Let us check the inequality

$$I \equiv \int\limits_{\gamma_0} [\sigma\nu(v-u)] - \frac{1}{\delta}\int\limits_{\gamma_0} [u][v-u] \le 0.$$

If this is the case, from (9) it follows (6). We have

$$I = \int\limits_{\gamma_0} \sigma_\nu[v_\nu - u_\nu] + \int\limits_{\gamma_0} \sigma_\tau[v_\tau - u_\tau] -\frac{1}{\delta}\int\limits_{\gamma_0} [u_\nu][v_\nu - u_\nu] - \frac{1}{\delta}\int\limits_{\gamma_0} [u_\tau][v_\tau - u_\tau]. \tag{10}$$

We can rewrite (10) in the equivalent form

$$I = \int\limits_{\gamma_0} (\sigma_\nu - \frac{1}{\delta}[u_\nu])[v_\nu - u_\nu] + \int\limits_{\gamma_0} (\sigma_\tau - \frac{1}{\delta}[u_\tau])[v_\tau - u_\tau],$$

and the inequality $I \le 0$ is clear by (3)-(5). Thus, the variational inequality (6) is fulfilled.

Theorem 1 is proved.

3. Passage to the limit in (6) as $\delta \to \infty$

This section is concerned with a justification of a passage to the limit as $\delta \to \infty$ in the problem (6). To this end, we denote by u^δ the solution of the problem (6). Thus, for any fixed $\delta > 0$, there exists a solution of the problem

$$u^\delta \in K_0, \tag{11}$$

$$\int\limits_{\Omega_0} \sigma(u^\delta)\varepsilon(v-u^\delta) - \int\limits_{\Omega_0} f(v-u^\delta) +\frac{1}{\delta}\int\limits_{\gamma_0} [u^\delta][v-u^\delta] \ge 0 \quad \forall\, v \in K_0. \tag{12}$$

From (11)-(12), it follows

$$\int\limits_{\Omega_0} \sigma(u^\delta)\varepsilon(u^\delta) - \int\limits_{\Omega_0} fu^\delta + \frac{1}{\delta}\int\limits_{\gamma_0} [u^\delta]^2 = 0.$$

Thus, by Korn's inequality, the following a priori estimate holds being uniform in δ,

$$\|u^\delta\|_{H^1_\Gamma(\Omega_0)^2} \le c.$$

Consequently

$$\int_{\gamma_0} [u^\delta]^2 \le c.$$

By these estimates, choosing a sequence, if necessary, we assume that as $\delta \to \infty$

$$u^\delta \to u^\infty \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ [u^\delta] \to [u^\infty] \text{ weakly in } L^2(\gamma_0). \tag{13}$$

Passing to the limit in (11)-(12) as $\delta \to \infty$ we obtain

$$u^\infty \in K_0,\ \int_{\Omega_0} \sigma(u^\infty)\varepsilon(v-u^\infty) - \int_{\Omega_0} f(v-u^\infty) \ge 0 \quad \forall\, v \in K_0. \tag{14}$$

We can provide an equivalent differential formulation of the problem (14). Namely, find a displacement field $u^\infty = (u_1^\infty, u_2^\infty)$ and a stress tensor $\sigma(u^\infty) = \sigma = \{\sigma_{ij}\}, i,j = 1,2$, defined in Ω_0 such that

$$-\text{div } \sigma = f,\ \sigma - A\varepsilon(u^\infty) = 0 \text{ in } \Omega_0, \tag{15}$$

$$u^\infty = 0 \text{ on } \Gamma, \tag{16}$$

$$[u_\nu^\infty] \ge 0,\quad [\sigma_\nu] = 0,\ \sigma_\nu \le 0,\ \sigma_\tau^\pm = 0,\ \sigma_\nu [u_\nu^\infty] = 0 \text{ on } \gamma_0. \tag{17}$$

The problem (14) or (15)-(17) describes an equilibrium state of the elastic body with the crack γ_0. This model is analyzed in many papers and books, see [5, 6].

Thus, we have proved the following statement.

Theorem 2 Solutions of the problems (11)-(12) converge in the sense (13) to the solution of the problem (14) as $\delta \to \infty$.

4. Passage to the limit in (6) as $\delta \to 0$

In this section, we analyze a passage to the limit as $\delta \to 0$ in the problem (6). As in the previous section, denote by u^δ the solution of the problem (6). For any fixed $\delta > 0$, consider the problem

$$u^\delta \in K_0, \tag{18}$$

$$\int_{\Omega_0} \sigma(u^\delta)\varepsilon(v-u^\delta) - \int_{\Omega_0} f(v-u^\delta) + \frac{1}{\delta}\int_{\gamma_0} [u^\delta][v-u^\delta] \ge 0 \quad \forall\, v \in K_0. \tag{19}$$

The solutions of this problem satisfy the equality

$$\int_{\Omega_0} \sigma(u^\delta)\varepsilon(u^\delta) - \int_{\Omega_0} f u^\delta + \frac{1}{\delta}\int_{\gamma_0} [u^\delta]^2 = 0. \tag{20}$$

From (20) it follows that estimates hold

$$\|u^\delta\|_{H^1_\Gamma(\Omega_0)^2} \le c,\ \int_{\gamma_0} [u^\delta]^2 \le c\delta$$

being uniform in δ. By these estimates, we can assume that as $\delta \to 0$,

$$u^\delta \to u^0 \text{ weakly in } H^1_\Gamma(\Omega_0)^2,\ [u^\delta] \to [u^0] = 0 \text{ in } L^2(\gamma_0). \tag{21}$$

It is clear that $u^0 \in H^1_0(\Omega)^2$, where

$$H^1_0(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma\}.$$

Let $\bar{v} \in H^1_0(\Omega)^2$. We substitute a test function $v = u^\delta \pm \bar{v}$ in (19). It provides

$$\int_{\Omega_0} \sigma(u^\delta)\varepsilon(\bar{v}) - \int_{\Omega_0} f\bar{v} + \frac{1}{\delta}\int_{\gamma_0} [u^\delta][\bar{v}] = 0. \tag{22}$$

Since $[\bar{v}] = 0$ on γ_0 it is possible to pass to the limit in (22) as $\delta \to 0$. It gives

$$\int_{\Omega_0} \sigma(u^0)\varepsilon(\bar{v}) - \int_{\Omega_0} f\bar{v} = 0 \quad \forall \bar{v} \in H^1_0(\Omega)^2,$$

i.e., by (21),

$$\int_{\Omega} \sigma(u^0)\varepsilon(\bar{v}) - \int_{\Omega} f\bar{v} = 0 \quad \forall \bar{v} \in H^1_0(\Omega)^2. \tag{23}$$

Thus, the limit problem can be written in the form (23). Its equivalent differential formulation is as follows. We have to find functions $u^0 = (u^0_1, u^0_2), \sigma(u^0) = \{\sigma_{ij}\}, i, j = 1, 2$, defined in Ω such that

$$-\text{div}\ \sigma(u^0) = f,\ \sigma(u^0) - A\varepsilon(u^0) = 0 \ \text{ in } \Omega, \tag{24}$$

$$u^0 = 0 \text{ on } \Gamma. \tag{25}$$

We see that the limit problem (24)-(25) corresponds to the elastic body without any defects and cracks.

To conclude, we formulate the result of this section.

Theorem 3 Solutions of the problems (18)-(19) converge in the sense (21) to the solution of the problem (23) as $\delta \to 0$.

5. Derivative of the energy functional

This section concerns an analysis of formulae for the derivatives of the energy functional. The derivatives are defined with to respect of the defect length. Suitable formulae will be used in the next section for the analysis of an optimal control problem. For simplicity, in this section, we consider the homogeneous elastic body, i.e. $a_{ijkl} = \mathit{const}$. Moreover, $f \in C^1(\bar{\Omega})^2$. First, we introduce a problem being perturbed to (6). Denote $\gamma_\lambda = (0, 1+\lambda) \times \{0\}$, $\lambda \ge 0$ is a parameter; $\Omega_\lambda = \Omega \setminus \bar{\gamma}_\lambda$. Introduce a set of admissible displacements

$$K_\lambda = \{v \mid v \in H^1_\Gamma(\Omega_\lambda)^2,\ [v_\nu] \ge 0 \text{ on } \gamma_\lambda\}.$$

There exists a solution of the problem

$$u^\lambda \in K_\lambda,\ \int_{\Omega_\lambda} \sigma(u^\lambda)\varepsilon(v - u^\lambda) - \int_{\Omega_\lambda} f(v - u^\lambda) + \frac{1}{\delta}\int_{\gamma_\lambda} [u^\lambda][v - u^\lambda] \ge 0 \quad \forall\, v \in K_\lambda. \tag{26}$$

The problem (26) corresponds to a minimization of the energy functional

$$\Pi_\delta(\Omega_\lambda; v) = \frac{1}{2}\int_{\Omega_\lambda} \sigma(v)\varepsilon(v) - \int_{\Omega_\lambda} fv + \frac{1}{2\delta}\int_{\gamma_\lambda} [v]^2 \tag{27}$$

over the set K_λ. In the definition of the energy functional (27), we indicate a dependence of the functional on Ω_λ and δ since in the future, Ω_λ and δ are subject to change.

It can be proved that the derivative of the energy functional $\Pi_\delta(\Omega_\lambda; u^\lambda)$ with respect to the defect length exists, i.e. there exists

$$\frac{d}{d\lambda}\Pi_\delta(\Omega_\lambda; u^\lambda)|_{\lambda=0+} = J_\delta(\Omega_0; u^\delta), \tag{28}$$

where u^δ is the solution of the problem (6), i.e. of the problem (26) corresponding to $\lambda = 0$. We refer the reader to [6] for details concerning this statement, Section 3.1.2. Compared to [6], the energy functional (27) includes an additional term $\frac{1}{2\delta}\int\limits_{\gamma_\lambda}[v]^2$, but this term does not provide any additional difficulties in finding the derivative. The formula (28) for the derivative of the energy functional takes the form, compare with [6]

$$J_\delta(\Omega_0; u^\delta) = \frac{1}{2}\int\limits_{\Omega_0}\sigma_{ij}(u^\delta)\varepsilon_{ij}(u^\delta)\theta_{,1} - \int\limits_{\Omega_0}\sigma_{ij}(u^\delta)u^\delta_{i,1}\theta_{,j} - \int\limits_{\Omega_0}(\theta f_i)_{,1}u_i^\delta + \frac{1}{2\delta}\int\limits_{\gamma_0}[u^\delta]^2\theta_{,1}. \tag{29}$$

Here θ is a smooth function equal to 1 near the point $(1,0)$ with a support in a small neighborhood of this point. We should remark that the derivative (29) does not depend on θ.

The suitable formula for the derivative can be derived and for the limiting case $\delta = \infty$ as well, i.e. for the problem (14). Namely, see [6], Section 3.1.2,

$$J_\infty(\Omega_0; u^\infty) = \frac{1}{2}\int\limits_{\Omega_0}\sigma_{ij}(u^\infty)\varepsilon_{ij}(u^\infty)\theta_{,1} - \int\limits_{\Omega_0}\sigma_{ij}(u^\infty)u^\infty_{i,1}\theta_{,j} - \int\limits_{\Omega_0}(\theta f_i)_{,1}u_i^\infty, \tag{30}$$

where u^∞ is a unique solution of the problem (14), and the function θ is taken like in (29). Again, there is no dependence of $J_\infty(\Omega_0; u^\infty)$ on the function θ.

6. Invariant integrals

It is turned out that the formula (29) for the derivative of the energy functional can be written in the form of an invariant integral over the curve surrounding the defect tip $(1,0)$ provided that the solution is quite smooth, for example, provided $u \in H^2(\Omega_0)^2$. This form does not depend on any function θ. The invariance means that the integral does not depend on a choice of the curve. Also, to write the invariant integral, we assume that $f \equiv 0$ in a neighborhood of the point $(1,0)$. Denote $\gamma = (a,b)\times\{0\}$, $a < b$; $a, b \in (0,1)$. Assume that inside of a closed smooth curve C_2, a smooth function θ is equal to 1, and $\theta \equiv 0$ outside of a closed smooth curve C_1, see Fig. 3. In this case, in (29), we have to integrate over the domain Ω_1 located between C_1 and C_2. Let $n = (n_1, n_2)$ be a unit normal vector to C_2. Integrating by parts in (29), we write $J_\delta(\Omega_0; u^\delta)$ in the form ($\sigma_{ij} = \sigma_{ij}(u^\delta)$, $\varepsilon_{ij} =$

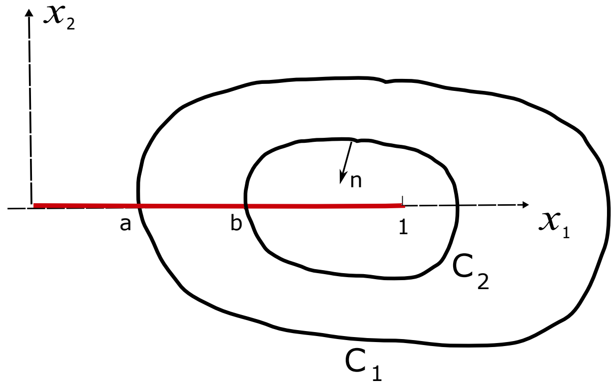

FIG. 3. Curves C_1, C_2

$\varepsilon_{ij}(u^\delta))$

$$J_\delta(\Omega_0; u^\delta) = -\frac{1}{2}\int\limits_{\Omega_1}(\sigma_{ij}\varepsilon_{ij})_{,1}\theta + \int\limits_{\Omega_1}(\sigma_{ij}u^\delta_{i,1})_{,j}\theta - \frac{1}{2\delta}\int\limits_\gamma [u^\delta]^2_{,1}\theta + \frac{1}{2}\int\limits_{C_2}\sigma_{ij}\varepsilon_{ij}n_1 - \int\limits_{C_2}\sigma_{ij}u^\delta_{i,1}n_j + \int\limits_\gamma[\sigma_{i2}u^\delta_{i,1}]\theta + \frac{1}{2\delta}[u^\delta]^2(B), \tag{31}$$

where $B = (b, 0)$. It is easy to see that

$$-\frac{1}{2}\int\limits_{\Omega_1}(\sigma_{ij}\varepsilon_{ij})_{,1}\theta + \int\limits_{\Omega_1}(\sigma_{ij}u^\delta_{i,1})_{,j}\theta = 0. \tag{32}$$

Moreover, we can check that

$$M \equiv -\frac{1}{2\delta}\int\limits_\gamma [u^\delta]^2_{,1}\theta + \int\limits_\gamma[\sigma_{i2}u^\delta_{i,1}]\theta = 0. \tag{33}$$

Indeed,

$$M = -\frac{1}{\delta}\int\limits_\gamma([u^\delta_1][u^\delta_{1,1}] + [u^\delta_2][u^\delta_{2,1}])\theta + \int\limits_\gamma \sigma_\nu[u^\delta_{2,1}]\theta + \int\limits_\gamma \sigma_\tau[u^\delta_{1,1}]\theta.$$

By the second formula (4), we have

$$M = -\frac{1}{\delta}\int\limits_\gamma [u^\delta_2][u^\delta_{2,1}]\theta + \int\limits_\gamma \sigma_\nu[u^\delta_{2,1}]\theta = \int\limits_\gamma(\sigma_\nu - \frac{1}{\delta}[u^\delta_\nu])[u^\delta_{\nu,1}]\theta.$$

Remark that in addition to (5), the following relation holds

$$(\sigma_\nu - \frac{1}{\delta}[u^\delta_\nu])[u^\delta_{\nu,1}] = 0 \text{ on } \gamma_0. \tag{34}$$

To prove this, we denote

$$N = \{x \in \gamma_0 \mid [u^\delta_\nu(x)] > 0\}.$$

If $x \in N$, then (34) takes place at the point x in view of (5). On the other hand, if $x \in \gamma_0 \setminus N$, we have $[u^\delta_\nu(x)] = 0$, and thus $[u^\delta_{\nu,1}](x) = 0$. Consequently (34) is true for $x \in \gamma_0 \setminus N$. As a result of the previous arguments, $M = 0$, what is needed.

In this case, by (32) and (33), the formula (31) can be written in the form of invariant integral

$$J_\delta(\Omega_0; u^\delta) = \frac{1}{2}\int\limits_{C_2} \sigma_{ij}(u^\delta)\varepsilon_{ij}(u^\delta)n_1 - \int\limits_{C_2} \sigma_{ij}(u^\delta)u^\delta_{i,1}n_j + \frac{1}{2\delta}[u^\delta]^2(B). \tag{35}$$

The formula (35) does not depend on a choice of the curve C_2 (and consequently, does not depend on the point B).

7. Optimal control problem

The well known Griffith criterion says that a defect propagates in an elastic body provided that a derivative of the energy functional with respect to the defect length reaches a critical value. This section concerns an optimal control problem for a cost functional being equal to the derivative of the energy functional with respect to the defect length. The damage parameter δ serves as a control function. For any $\delta > 0$ we can find the derivative of the energy functional by the formula (29). As for $\delta = \infty$, the formula for the derivative is defined by (30).

It is easy to see that $J_\delta(\Omega_0; u^\delta) \le 0$ for all $\delta \in (0, \infty]$. Indeed, by increasing a length of the defect, we increase a set of admissible functions. Since the solution is found by minimizing the energy functional over the set of admissible functions, the value of the energy functional (calculated for the solutions) is less for a bigger defect length. Thus, the inequality $J_\delta(\Omega_0; u^\delta) \le 0$ follows. A maximization of $J_\delta(\Omega_0; u^\delta)$ with respect to the defect length provides the most safe case from the standpoint of the Griffith criterion.

Let $\delta_0 > 0$ be given. Consider an optimal control problem

$$\sup_{\delta \in [\delta_0, \infty]} J_\delta(\Omega_0; u^\delta). \tag{36}$$

We prove the following statement.

Theorem 2 There exists a solution of the problem (36).

Proof Let $\delta^n \in [\delta_0, \infty]$ be a maximizing sequence. We assume that δ^n is a converging sequence. There are two cases may appear:

1. $\delta^n \in \mathbb{R},\ \delta^n \to \delta^*,\ n \to \infty,\ \delta^* \in \mathbb{R}$.
2. $\delta^n \in \mathbb{R},\ \delta^n \to \infty,\ n \to \infty$.

If $\delta^n = \infty,\ n \ge n_0$, we easy obtain a solution existence of the problem (36).

Case 1. $\delta^n \in \mathbb{R},\ \delta^n \to \delta^*,\ n \to \infty,\ \delta^* \in \mathbb{R}$.

For any n there exists a solution of the problem

$$u^n \in K_0,\ \int\limits_{\Omega_0} \sigma(u^n)\varepsilon(v - u^n) - \int\limits_{\Omega_0} f(v - u^n) + \frac{1}{\delta^n}\int\limits_{\gamma_0} [u^n][v - u^n] \ge 0 \quad \forall\, v \in K_0. \tag{37}$$

A priori estimates for the solutions u^n can be obtained as that in Section 3. Hence, uniformly in n

$$\|u^n\|_{H^1_\Gamma(\Omega_0)^2} \le c.$$

Consequently, it can be assumed that as $n \to \infty$,

$$u^n \to u^* \text{ weakly in } H^1_\Gamma(\Omega_0)^2. \tag{38}$$

Moreover, u^* is the solution of the problem

$$u^* \in K_0, \int\limits_{\Omega_0} \sigma(u^*)\varepsilon(v-u^*) - \int\limits_{\Omega_0} f(v-u^*) + \frac{1}{\delta^*}\int\limits_{\gamma_0} [u^*][v-u^*] \ge 0 \quad \forall v \in K_0, \tag{39}$$

i.e. $u^* = u^{\delta^*}$. We can prove a strong convergence of the sequence u^n. According to (37), (38), (39), we have, as $n \to \infty$,

$$\int\limits_{\Omega_0} \sigma(u^n)\varepsilon(u^n) + \frac{1}{\delta^n}\int\limits_{\gamma_0} [u^n]^2 = \int\limits_{\Omega_0} fu^n \to \int\limits_{\Omega_0} fu^* = \int\limits_{\Omega_0} \sigma(u^*)\varepsilon(u^*) + \frac{1}{\delta^*}\int\limits_{\gamma_0} [u^*]^2.$$

In particular, this means

$$\|u^n\|^2_{H^1_\Gamma(\Omega_0)^2} \to \|u^*\|^2_{H^1_\Gamma(\Omega_0)^2}. \tag{40}$$

By (38), (40), we obtain the strong convergence,

$$u^n \to u^* \text{ strongly in } H^1_\Gamma(\Omega_0)^2.$$

Hence, taking into account the formula for the derivative (29), we conclude

$$J_{\delta^n}(\Omega_0; u^n) \to J_{\delta^*}(\Omega_0; u^*),$$

and an existence proof of solutions to the problem (36) is complete in the case 1.

Case 2. $\delta^n \in \mathbb{R}$, $\delta^n \to \infty$, $n \to \infty$.

Like in the case 1, for any n, there exists a solution of the problem

$$u^n \in K_0, \tag{41}$$

$$\int\limits_{\Omega_0} \sigma(u^n)\varepsilon(v-u^n) - \int\limits_{\Omega_0} f(v-u^n) + \frac{1}{\delta^n}\int\limits_{\gamma_0} [u^n][v-u^n] \ge 0 \quad \forall v \in K_0, \tag{42}$$

and moreover, we have the uniform in n estimate, see Section 3,

$$\|u^n\|^2_{H^1_\Gamma(\Omega_0)^2} \le c.$$

By this estimate, we assume that as $n \to \infty$

$$u^n \to u^\infty \text{ weakly in } H^1_\Gamma(\Omega_0)^2. \tag{43}$$

This convergence allows us to pass to the limit in (41)-(42) as $n \to \infty$. Moreover, the function u^∞ is the solution of the variational inequality (14).

We need a proof of the strong convergence of the sequence u^n. From (41)-(43) and (14), it follows

$$\int_{\Omega_0} \sigma(u^n)\varepsilon(u^n) + \frac{1}{\delta^n}\int_{\gamma_0} [u^n]^2 = \int_{\Omega_0} fu^n \rightarrow \int_{\Omega_0} fu^\infty = \frac{1}{2}\int_{\Omega_0} \sigma(u^\infty)\varepsilon(u^\infty). \tag{44}$$

Taking into account that, as $n \to \infty$,

$$\frac{1}{\delta^n}\int_{\gamma_0} [u^n]^2 \to 0$$

we conclude from (44) that

$$\|u^n\|^2_{H^1_\Gamma(\Omega_0)^2} \to \|u^\infty\|^2_{H^1_\Gamma(\Omega_0)^2}. \tag{45}$$

By (43), (45), it provides

$$u^n \to u^\infty \text{ strongly in } H^1_\Gamma(\Omega_0)^2. \tag{46}$$

Taking into account the formulae (29), (30) for the derivatives, by (46), it is possible to complete the proof of Theorem 3 in the case 2 since

$$J_{\delta^n}(\Omega_0; u^n) \to J_\infty(\Omega_0; u^\infty).$$

Hence, $\delta = \infty$ is the solution of the optimal control problem (36) in this case. Theorem 3 is proved.

8. Conclusion

We provide a rigorous mathematical analysis of the model describing an equilibrium state of elastic bodies with defects. The defects are characterized by the damage parameter. Solution existence of the problems considered is proved, and different equivalent problem formulations are proposed. Dependence of solutions on the damage parameter and the defect length is investigated. We derive a formula for the derivative of the energy functional with respect to the defect length and prove that the formula can be written in the form of invariant integral. An optimal control problem is investigated what guarantees an existence of optimal values of the damage parameter provided that the Griffith rapture criterion is used.

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