S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 15, стр. 1530–1552 (2018) УДК 519.21

DOI 10.33048/semi.2018.15.127 MSC 62G07

ON SUFFICIENT CONDITIONS FOR A GAUSSIAN APPROXIMATION OF KERNEL ESTIMATES FOR DISTRIBUTION DENSITIES

A.S. KARTASHOV, A.I. SAKHANENKO

Abstract. Recently E. Gine, V. Koltchinskii and L. Sakhanenko (Ann. Probab., 2004) investigated necessary and sufficient conditions for weak convergence to the double exponential distribution of a normalized random variable $\sup_{t\in\mathbb{R}} |\psi(t)(f_n(t) - \mathbf{E}f_n(t))|$ with some weight function $\psi(t)$, where f_n is a kernel density estimator. The proof of their results consists of a large number of technically difficult stages and uses more than fifteen bulky assumptions. In this work we prove that sufficiency of convergence can be obtained under simpler and wider assumptions.

Keywords: kernel density estimators, brownian motion, function of bounded variation

1. Introduction

Let $X, X_1, X_2, ...$ be independent identically distributed random variables with unknown density $f(\cdot)$. Consider the random function:

$$f_n(t) = \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{t - X_i}{h_n}\right), \tag{1}$$

where bandwidths $h_n > 0$ are such that

$$h_n \to 0 \quad \text{and} \quad nh_n \to \infty \quad \text{as} \quad n \to \infty, \tag{2}$$

Kartashov, A.S., Sakhanenko, A.I., On sufficient conditions for a Gaussian approximation of kernel estimates for distribution densities.

The work is supported by Russian Science Foundation (project No. 17-11-01173).

Received September, 26, 2018, published December, 3, 2018.

and let $K(\cdot)$ be a function with finite total variation. Assume also that

$$K_1 := \int |dK| < \infty \quad \text{and} \quad 0 < K_2^2 = \int_{-\infty}^{\infty} K^2(x)dx < \infty. \tag{3}$$

If $\int K(x)dx = 1$, then $f_n(\cdot)$ can be considered as an estimator of unknown density $f(\cdot)$. Nevertheless obtained results are still valid without this condition.

In this work we are interested in asymptotic behavior of the distribution of random variable $\beta_n(\mathbb{R})$, where

$$\beta_n(B) := \sqrt{nh_n} \sup_{t\in B} |\psi(t)(f_n(t) - \mathbf{E}f_n(t))|\,, \tag{4}$$

and ψ is some weight function such that

$$0 < \overline{w}(\mathbb{R}) < \infty \quad \text{with} \quad \overline{w}(B) := \sup_{t\in B} \left|\psi(t)\sqrt{f(t)}\right| \tag{5}$$

for $B \subset \mathbb{R}$. But instead of functions $K(\cdot)$ and $\psi(\cdot)$ we may always use $K(\cdot)/K_2$ and $|\psi(\cdot)|/\overline{w}(\mathbb{R})$, respectively. Due to this reason, later on, without loss of generality, we assume that

$$K_2 = 1 = \overline{w}(\mathbb{R}) \qquad \text{and} \qquad \psi(\cdot) \geq 0. \tag{6}$$

It has been shown in [1, Theorem 6], that under serious additional conditions introduced earlier in [2] the following convergence holds

$$\mathbf{P}\left(A_n\left(\beta_n(\mathbb{R}) - A_n\right) \leq x\right) \longrightarrow e^{-e^{-x}} \quad \text{as} \quad n \to \infty, \tag{7}$$

where A_n is defined as the solution of some functional equation. In particular (see [3])

$$A_n \sim \sqrt{2\,|\log h_n|} \to \infty.$$

Moreover, it was shown in [1] that the following assumption (for $\varepsilon = 1$)

$$\lim_{n\longrightarrow\infty} n\mathbf{P}\left(|\psi(X)| > \varepsilon\lambda_n\right) = 0, \quad \text{where} \quad \lambda_n = \sqrt{nh_n \log|h_n|}, \tag{8}$$

is, in some sense, necessary and sufficient for (7) to hold.

Our general aim is to show that the last assertion holds under fairly simpler conditions in comparison with the ones from [1]. The first step in solving this problem was made by the authors earlier, in [4]. In that work we avoided the introduction of most heavy restrictions from [1] and [2] but we had to introduce a more complex accompanying Gaussian process than in [1]. In this paper we want to carry out the next step in solving this problem which consists in presenting general sufficient conditions for (7) to hold. Such conditions will be described in Subsection 2.1; and at the end of Subsection 2.3 it will be shown that they are essentially simpler (and a bit more general) than ones from [1].

One of the main differences of our conditions from analogical conditions in [1] consists in using instead of density $f(\cdot)$ and weight function $\psi(\cdot)$ the following functions

$$p_h(t) := \mathbf{P}(t - h < X < t + h), \quad \overline{\psi}_h(t) := \sup_{|y|\leq h} |\psi(t + y)|. \tag{9}$$

In particular, instead of condition (8) we will use (14).

Our main results — Theorem 1, Corollary 2 and Proposition 3 — will be presented in Subsections 2.2 and 2.3.

2. Main results

2.1. **Main assumptions.** Let us introduce the main conditions which will be used further. We use later notations K_1, K_2, $\overline{\omega}(B), \overline{\psi}_h(\cdot), p_h(\cdot)$ defined in (1) – (9).

(A) $X, X_1, X_2, ...$ are independent and identically distributed random variables with density function $f(\cdot)$ on $\mathbb{R}$ with support $B_f = \{f > 0\}$; function $\psi(\cdot) \geq 0$ is also defined on $\mathbb{R}$ and $\overline{w}(\mathbb{R}) = 1$;

(H) $h_n = n^{-\eta} s(n)$ with $\eta \in (0,1)$, where $s(t)$ is a slowly varying function as $t \to \infty$;

(K) function $K(\cdot)$ has support in $[-1/2, 1/2]$; it satisfies (3) with $K_2 = 1$; and for some $0 < \alpha \leq 2$ and $C_1 > 0$ as $t \to 0$

$$\int_{-\infty}^{\infty} (K(u+t) - K(u))^2 du \sim C_1 K_2^2 |t|^\alpha = C_1 |t|^\alpha ;$$

(D) there exists a set D which consists of a finite number of disjoint closed bounded intervals of positive lengths and such that $\overline{w}(D) = \overline{w}(\mathbb{R}) = 1$; function $w(x) := \psi(x)\sqrt{f(x)}$ is piecewise monotone on D; functions $f(\cdot)$ and $\psi(\cdot)$ are positive on D and, as $h \to 0$,

$$\sup_{\substack{t,t+y \in D \\ |y| \leq h}} \left| \frac{\sqrt{f(t+y)}}{\sqrt{f(t)}} - 1 \right| = o\left(\frac{1}{|\log h|}\right), \tag{10}$$

$$\sup_{\substack{t,t+y \in D \\ |y| \leq h}} \left| \frac{\psi(t+y)}{\psi(t)} - 1 \right| = o\left(\frac{1}{|\log h|}\right); \tag{11}$$

(∂D) for all points t from the boundary ∂D of the set D

$$\hat{w}_n(t) := \frac{\overline{\psi}^2_{h_n/2}(t) p_{h_n}(t)}{h_n} = o(|\log h_n|);$$

($\overline{D}$) for some $\varepsilon_1 > 0$ and $N_1 < \infty$

$$\sup\left\{\hat{w}_n(t) : (t - h_n, t + h_n) \subset \overline{D},\ \overline{\psi}_{2h_n}(t) \leq \varepsilon_1 \lambda_n,\ n \geq N_1\right\} \leq \eta c_1, \tag{12}$$

where $c_1 := 1/(3K_1^2)$ and $\overline{D} = \mathbb{R} \setminus D$ is the compliment of the set D.

Now for $u > 0$ introduce functions:

$$\psi_\alpha(u) := \sqrt{\frac{2}{\pi}} C^{1/\alpha}(\alpha) H(\alpha) u^{2/\alpha - 1} e^{-u^2/2}, \quad \Lambda_\alpha(u) := \int_D \psi_\alpha(u/w(y)) dy,$$

where $H(\alpha)$ is constant of Pickands. Denote by $\{A_n\}$ any sequence of numbers satisfying

$$\Lambda_\alpha(A_n) \sim h_n; \quad A_n \sim \sqrt{2|\log h_n|} \longrightarrow \infty. \tag{13}$$

Existence of such sequence is shown in [1] and [3]. In particular, we can choose A_n as a maximal solution of the equation $\Lambda_\alpha(A_n) = h_n$.

2.2. **Main Results.**

Theorem 1. *Suppose that all assumptions from Subsection 2.1 are valid and that the next condition holds for* $\varepsilon = \varepsilon_1 > 0$*:*

$$n\mathbf{P}\left(\overline{\psi}_{2h_n}(X) > \varepsilon\lambda_n\right) \longrightarrow 0 \quad as \quad n \to \infty, \tag{14}$$

where ε_1 *is the same as in condition* (D)*. Then convergence* (7) *takes place.*

Now, instead of conditions (∂D) and $(\overline{D})$, we want to introduce the following simpler assumption

$(\overline{D}+)$ For some $\varepsilon_2 > 0$ and $N_2 < \infty$

$$\sup\{\overline{\psi}_{2h_n}(t)/\psi(t) : \psi(t) \le \varepsilon_2\lambda_n,\ n \ge N_2\} \le C_2,$$

where the constant C_2 is such that $2C_2^2\overline{w}^2(\overline{D}) \le \eta c_1 = \eta/3K_1^2$.

Corollary 2. *Assume that conditions* $(\overline{D}+)$ *and* (8) *hold for the same* $\varepsilon = \varepsilon_2 > 0$ *and suppose that assumptions* (A)*,* (H)*,* (K) *and* (D) *from Subsection 2.1 are valid. Then convergence* (7) *takes place.*

2.3. **Comparison of conditions.** As we said above, Theorem 1 uses conditions that are weaker than similar conditions in [1, Theorem 6] and now we want to clarify the differences. Here we begin with the list of assumptions used in Theorem 6 from [1].

$(AHK+)$ Assumptions (A), (H) and (K) from subsection (2.1) hold with a non-negative function $K(\cdot) \ge 0$;

$(F1)$ B_f consists of finite union of non-trivial disjoint intervals (half lines not excluded), and f and ψ are both piecewise monotone on B_f. Moreover, f is bounded and Holder continuous of some positive order on B_f; in particular, $\lim_{a\to\infty}\sup_{|t|>a} f(t) = 0$;

$(w1)$ the support W of $\psi f^{1/2}$ consists of a finite number of disjoint closed intervals or half-lines; w is piecewise monotone on its support and such that $w(t) \longrightarrow 0$ as $|t| \longrightarrow \infty$;

$(w2)$ $\int_W \psi_\alpha(u/w(y))dy < \infty$ for some $u_0 < \infty$;

$(w3)$ we have as $h \longrightarrow 0$ that

$$\sup_{\substack{t\in W, t+\tau\in W\\ |\tau|\le h}} \left|\frac{w(t+\tau)}{w(t)} - 1\right| = o\left(\frac{1}{|\log h|}\right).$$

The mentioned above assumptions were introduced in [1]. To minimize their number we summarized several of them in a new assumption (AHK+). But the majority of conditions in Theorem 6 from [1] was introduced earlier in [2]. Here is the first of them.

$(W.c)$ For all $r > 0$

$$\lim_{h\to 0} \sup_{\substack{x,y:\psi(x)\le h^{-r},\\ x+y\in B_f, |y|\le h}} \left|\frac{\psi(x+y)}{\psi(x)} - 1\right| = 0.$$

Proposition 3. *If we choose $D = [-a, a] \cap W$ for sufficiently large a, then all assumptions used in Theorem 1 and Corollary 2 follow from the assumptions listed above in the present subsection.*

Thus, under the listed above assumptions Theorem 6 from [1] takes place; i.e. convergence (7) holds if condition (8) is true for $\varepsilon = 1$.

But in reality, in Theorem 6 in [1] more conditions were used, which were introduced earlier in [2]. Here are the main of them.

(H_1) $h_t, t \geq 1$, is a monotonically decreasing to 0 and th_t is a strictly increasing function diverging to infinity as $t \to \infty$;

$(D.b)$ for all $\delta > 0$ there exist $c \in (0, \infty)$ and $h_0 > 0$ such that, for all $|y| \leq h_0$ and all $x \in B_f, x + y \in B_f$,

$$\frac{1}{c} f^{\delta}(x) \leq \frac{f(x+y)}{f(x)} \leq c f^{-\delta}(x);$$

$(W.b)$ for all $\delta > 0$ there exist $c \in (0, \infty)$ and $h_0 > 0$ such that, for all $|y| \leq h_0$ and all $x \in B_f, x + y \in B_f$,

$$\frac{1}{c} \psi^{\delta}(x) \leq \frac{\psi(x+y)}{\psi(x)} \leq c \psi^{-\delta}(x);$$

$(WD.b)$ for all $r > 0$,

$$\lim_{h \to 0} \sup_{\substack{x,y : \psi(x) \leq h^{-r}, \\ x+y \in B_f, |y| \leq h}} \left| \frac{f(x+y)}{f(x)} - 1 \right| = 0;$$

$(WD.a)_\beta$ $\|f^\beta\|_{\psi,\infty} := \sup_{t \in B_f} \left|\psi(x) f^\beta\right| < \infty$ for some $\beta \in (0, 1/2)$;

(Z) either $B_f = \mathbb{R}$ or $K(0) = \sup_{x \in \mathbb{R}} |K(x)|$.

Remark 4. *Theorem 6 from* [1] *takes place under all listed above assumptions together with conditions (D.a), (D.c) and (W.a). But we omitted conditions (D.a) and (W.a) since they follow from (F1), whereas (D.c) follows from* $(WD.a)_\beta$ *and (W D.b) as it was mentioned in* [2].

Thus, we can say that, in comparison with the conditions of work [1], our conditions are much weaker and are also much simpler.

The rest of the paper is devoted to proofs of Theorem 1, Corollary 2 and Proposition 3.

3. Proofs of main results

3.1. **Main auxiliary propositions.** For any $\varepsilon > 0$ later on we denote by $\partial^\varepsilon D$ the open ε-neighbourhood of the boundary ∂D of the set D. For all $n > 0$ introduce sets:

$$D_n := D \setminus \partial^{h_n/2} D \quad \text{and} \quad \overline{D}_n = \mathbb{R} \setminus D_n. \tag{15}$$

Thus, D_n is the closed $h_n/2$-interior of the set D whereas its complement $\overline{D}_n$ is the open $h_n/2$-neighbourhood of the set $\overline{D} = \mathbb{R} \setminus D$.

Proposition 5. *Suppose that assumptions (A), (K), (D) from Section 2 hold with positive numbers* $h_n \to 0$. *Then for all* x

$$\mathbf{P}\left(A_n\big(\beta_n(D_n)-A_n\big)\le x\right)\longrightarrow e^{-e^{-x}} \quad as \quad n\to\infty.$$

This proposition will be derived in Section 5 from Corollary 4 in [1].

Proposition 6. *Under assumptions of Theorem 1*

$$\mathbf{P}\left(\beta_n\left(\overline{D}_n\right)>0.9A_n\right)\longrightarrow 0 \quad as \quad n\to\infty.$$

This proposition will be proved in the Section 4 by a method different from ones in [1] and [2].

3.2. **Proof of Theorem 1.** The idea of the proof relies on the following inequalities. First: $\beta_n(D_n)\le\beta_n(D)$ since $D_n\subset D$. Hence

$$\mathbf{P}\left(A_n\big(\beta_n(\mathbb{R})-A_n\big)\le x\right)\le\mathbf{P}\left(A_n\big(\beta_n(D_n)-A_n\big)\le x\right),$$

the proof of which is obvious. Second:

$$\begin{aligned}&\mathbf{P}\left(A_n\big(\beta_n(D_n)-A_n\big)\le x\right)-\mathbf{P}\left(A_n\big(\beta_n(\mathbb{R})-A_n\big)\le x\right)\\ &\le\mathbf{P}\left(A_n\big(\beta_n(\overline{D}_n)-A_n\big)>x\right)\le\mathbf{P}\left(\beta_n\left(\overline{D}_n\right)>0.9A_n\right)\to 0\end{aligned}\tag{16}$$

for all $x\ge -A_n^2/10\to-\infty$. The convergence in (16) follows from Proposition 6.

Thus, applying proposition 5 we get the desired convergence (7).

3.3. **Proof of Corollary 2.** It follows from Theorem 1 and the next

Lemma 7. *Under assumptions* $(\overline{D}+)$ *and* (8) *we have conditions* (∂D), $(\overline{D})$ *and* (14) *for* $\varepsilon_1=C_2\varepsilon_2$ *and* $N_1=N_2$.

Proof. First, under assumptions $(\overline{D}+)$

$$n\mathbf{P}\left(\overline{\psi}_{2h_n}(X)>C_2\varepsilon_2\lambda_n\right)\le n\mathbf{P}\left(C_2\psi(X)>C_2\varepsilon_2\lambda_n\right)=n\mathbf{P}\left(\psi(X)>\varepsilon_2\lambda_n\right)\longrightarrow 0$$

and (14) follows from $(\overline{D}+)$ and (8) with $\varepsilon_1=C_2\varepsilon_2$.

Next, if $n\ge N_2$ then from $(\overline{D}+)$ we have for all t with $\overline{\psi}_{h_n}(t)\le\varepsilon_1\lambda_n$ that

$$\begin{aligned}\hat{w}_n(t)&=\frac{1}{h_n}\int_{t-h_n}^{t+h_n}\overline{\psi}^2_{h_n/2}(t)f(x)dx\le\frac{1}{h_n}\int_{t-h_n}^{t+h_n}\overline{\psi}^2_{2h_n}(x)f(x)dx\\ &\le\frac{C_2^2}{h_n}\int_{t-h_n}^{t+h_n}\psi^2(x)f(x)dx\le 2C_2^2\overline{w}^2\big((t-h_n,t+h_n)\big).\end{aligned}\tag{17}$$

Hence, $\hat{w}_n(t)\le 2C_2^2\overline{w}^2\big((t-h_n,t+h_n)\big)\le 2C_2^2\overline{w}^2\big(\mathbb{R}\big)=o(|\log h_n|)$ and, so, we obtained (∂D) from $(\overline{D}+)$.

At last, if $(t-h_n,t+h_n)\subset\overline{D}$, then we have from (17) and $(\overline{D}+)$ that

$$\hat{w}_n(t)\le 2C_2^2\overline{w}^2\big((t-h_n,t+h_n)\big)\le 2C_2^2\overline{w}^2\big(\overline{D}\big)\le\eta c_1.$$

Hence, condition $(\overline{D})$ also follows from $(\overline{D}+)$. □

3.4. **Proof of Proposition 3.** We first prove

Lemma 8. *If conditions* (10) *and* (11) *take place then*

$$\sup_{\substack{t,t+y\in D\\ |y|\le h}}\left|\frac{w(t+y)}{w(t)}-1\right| = o\left(\frac{1}{|\log h|}\right). \tag{18}$$

On the other hand, (11) *follows from* (18) *and* (10).

This fact obviously follows from the representation $w(t) = \psi(t)\sqrt{f(t)}$.

Lemma 9. *Conditions* (D) *and* $(\overline{D}+)$ *follows from assumptions (AHK+), (F1), (w1), (w3) and (W.c).*

Proof. Since $w(t) \to 0$ as $t \to \infty$ by (w1), there exists sufficiently large $a \in (0,\infty)$ such that the set $D = [-a, a] \cap W$ satisfies the following condition:

$$2C_2^2\overline{w}^2(\overline{D}) \le \min\{\eta c_1, 1/2\} < 1 \qquad \text{for} \quad C_2 = 2. \tag{19}$$

Using (19) we obtain from Condition (w1) that such D consists of a finite number of disjoint closed bounded intervals of positive lengths with $\overline{w}(D) = \overline{w}(\mathbb{R}) = 1$; function $w(x) := \psi(x)\sqrt{f(x)}$ is piecewise monotone on D; functions $f(\cdot)$ and $\psi(\cdot)$ are positive on D.

Next, by assumption (F1) density f is positive and Holder continuous on $D \subset B_f$. This fact implies (10). By Lemma 8 from (10) and (w3) we have (18) and (11). Thus, all assumptions from condition (D) are true.

At last, from condition (H) we have $n = h_n^{-1/\eta}s^{1/\eta}(n)$ and hence:

$$\lambda_n = \sqrt{nh_n\log|h_n|} = \sqrt{h_n^{-1/\eta}s^{1/\eta}(n)h_n\log|h_n|} = o(h_n^{-r})$$

for any positive $r < (1/\eta - 1)/2$. Hence, the first assumption in $(\overline{D}+)$ follows from (W.c) with any number $C_2 > 1$. This fact and (19) imply condition $(\overline{D}+)$ with $C_2 = 2$. Lemma is proved. □

Now all assertions of Proposition 3 follow from Lemmas 9 and 7.

The rest of the paper is devoted to proofs of Propositions 5 and 6.

3.5. **Corollaries from Condition** (K)**.** Note first of all that

$$1 = K_2 = \left(\int_{-1/2}^{1/2} K^2(x)dx\right)^{1/2} \le \sup_x |K(x)| \le \int |dK|/2 = K_1/2 < \infty. \tag{20}$$

So, $K_1 \in [2,\infty)$ and $c_1 \le 1/12$. We also need

Lemma 10. *If* $h, H > 0$ *then for any real function* ρ *and any real* $t, \tilde{t}$

$$\sup_{x:|x-t|\le H}\left|\int \rho(x-vh)dK(v)\right| \le K_1 \sup_{x:|x-t|\le H+h/2}\left|\rho(x)-\rho(\tilde{t})\right|. \tag{21}$$

Proof. Since $\int \rho(\tilde{t})dK(v) = \rho(\tilde{t})\int dK(v) = 0$, we have

$$\left|\int \rho(x-vh)dK(v)\right| = \left|\int [\rho(x-vh)-\rho(\tilde{t})]dK(v)\right|$$
$$\le K_1 \sup_{|v|\le 1/2}|\rho(x-vh)-\rho(\tilde{t})| = K_1 \sup_{y:|y-x|\le h/2}|\rho(y)-\rho(\tilde{t})|.$$

Now, taking a supremum over $\{x : |x-t| \le H\}$ we obtain (21). □

3.6. **Corollaries from Condition** (D)**.** By condition (D) the set D may be represented in the following form

$$D = \cup_{i=1}^{k} [a_{2i-1}, a_{2i}] \quad \text{with} \quad d_* = \min_{1\le i\le 2k-1} |a_{i+1} - a_i| > 0 \tag{22}$$

for some integer $k \ge 1$.

Lemma 11. *If* $h_n < d_*/3$ *then there exists a countable set*

$$T(n) = \{t_{j,n} : j = 0, \pm 1, \pm 2, \dots\}$$

which contains all points of the form

$$a_i - h_n \in T(n),\ a_i \in T(n),\ a_i + h_n \in T(n), \qquad i = 1, \dots, 2k; \tag{23}$$

and, in addition,

$$t_{j,n} < t_{j+1,n} \quad \text{and} \quad h_n/2 < t_{j+1,n} - t_{j,n} \le h_n \quad \forall j = 0, \pm 1, \pm 2, \dots. \tag{24}$$

In particular, in this case, each real t *belongs to at least one interval of the form* $[t_{j,n} - h_n/2, t_{j,n} - h_n/2]$ *and it belongs to at most four intervals of the form*

$$\Delta_{j,n} := (t_{j,n} - h_n, t_{j,n} - h_n), \qquad j = 0, \pm 1, \pm 2, \dots. \tag{25}$$

Proof. First, together with the points from (23) we also include into $T(n)$ also the following two countable sets of points:

$$a_1 - lh_n,\ a_{2k} + lh_n, \qquad l = 2, 3, 4, \dots.$$

Second, divide each interval $[a_i + h_n, a_{i+1} - h_n]$ into $k_i + 1$ equal parts of the length d_i with k_i equal to the integer part of the number $(a_{i+1} - a_i - 2h_n)/h_n$. It follows from condition $h_n < d_*/3$ that $(a_{i+1} - a_i - 2h_n)/h_n > 1$ and hence $h_n > d_i > h_n/2$ for all i. Now for each i we add the points

$$a_i + h_n + ld_i, \qquad l = 1, \dots, k_i$$

to the set $T(n)$.

Thus, we constructed a desirable set $T(n)$ with properties (23) and (24). In particular, by (24), each real t belongs to at least one interval of the form $[t_{j,n} - h_n/2, t_{j,n} - h_n/2]$.

Now for a fixed real t introduce the interval number $l := \min\{j : t \in \Delta_{j,n}\}$. It is clear that

$$t \le t_{l,n} + h_n = t_{l,n} + 4(h_n/2) - h_n < t_{l+4,n} - h_n.$$

Hence, $t \notin \Delta_{l+4,n}$. So, t may belong only to at most four intervals $\Delta_{j,n}$ with numbers $j = l, l+1, l+2, l+3$. □

3.7. **Corollaries from Condition** (H)**.** Consider a question if there exists a number $m = m(n, \varepsilon/\varepsilon_0)$ such that

$$\mathbf{P}\left(\overline{\psi}_{2h_n}(X) > \varepsilon\lambda_n\right) \le \mathbf{P}\left(\overline{\psi}_{2h_m}(X) > \varepsilon_0\lambda_m\right) \tag{26}$$

for some $\varepsilon > 0$. The main difficulty here is that we allow for $\varepsilon/\varepsilon_0$ depend on n and allow that $\varepsilon/\varepsilon_0 \to 0$ as $n \to \infty$.

Lemma 12. *There exists a number* $N_3 < \infty$ *such that if*

$$\varepsilon, \varepsilon_0 > 0, \quad N_3 \le m \le n/2 \quad \text{and} \quad m \le n(\varepsilon/\varepsilon_0)^{4/(1-\eta)} \tag{27}$$

then (26) takes place.

Proof. By Karamata Theorem (see, for example, [6]), for slowly varying function $s(\cdot)$ we have that for all sufficiently large $n \geq m$ the following representation holds:

$$\log \frac{s(n)}{s(m)} = g_1(n) - g_1(m) + \int_m^n \frac{g_2(t)}{t} dt,$$

where functions g_1 and g_2 are such that

$$\sup_{n \geq m} |g_1(n) - g_1(m)| + \sup_{t \geq m} |g_2(t)| \longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

Hence, for each real $\delta > 0$ we have for all sufficiently large $n \geq m$ that

$$\left| \log \frac{s(n)}{s(m)} \right| \leq \delta \log 2 + \delta \log \frac{n}{m}$$

and we may also choose $h_m \leq 1$ because $h_n \to 0$. In particular, there exists a finite N_3 such that for all $n \geq m \geq N_3$

$$\frac{1}{2^\delta} \left(\frac{m}{n} \right)^\delta \leq \frac{s(n)}{s(m)} \leq 2^\delta \left(\frac{n}{m} \right)^\delta, \quad \text{with} \quad \delta = \min \left\{ \frac{\eta}{2}, \frac{1-\eta}{4} \right\}. \tag{28}$$

Since $\eta \geq 2\delta$, we have for $N_3 \leq m \leq n/2$ that

$$\frac{h_n}{h_m} = \left(\frac{m}{n} \right)^\eta \frac{s(n)}{s(m)} \leq 2^\delta \left(\frac{m}{n} \right)^{\eta - \delta} \leq 2^\delta \left(\frac{m}{n} \right)^\delta \leq 1.$$

This fact for $h_m \leq 1$ implies that

$$1 \geq h_m \geq h_n \quad \text{and} \quad |\log h_n| = \log \frac{1}{h_n} \geq \log \frac{1}{h_m} = |\log h_m|,$$

when $n/2 \geq m \geq N_3 \geq N_0$. But the latter and (28) yield

$$\left(\frac{\varepsilon \lambda_n}{\varepsilon_0 \lambda_m} \right)^2 \geq \frac{\varepsilon^2 n h_n}{\varepsilon_0^2 m h_m} \geq \frac{1}{2^\delta} \left(\frac{\varepsilon}{\varepsilon_0} \right)^2 \left(\frac{n}{m} \right)^{1-\eta} \left(\frac{m}{n} \right)^\delta = \frac{1}{2^\delta} \left(\frac{\varepsilon}{\varepsilon_0} \right)^2 \left(\frac{n}{m} \right)^{1-\eta-\delta}$$
$$\geq \left(\frac{\varepsilon}{\varepsilon_0} \right)^2 \left(\frac{n}{m} \right)^{1-\eta-2\delta} \geq \left(\frac{\varepsilon}{\varepsilon_0} \right)^2 \left(\frac{n}{m} \right)^{1-\eta-\frac{1-\eta}{2}} = \left(\frac{\varepsilon}{\varepsilon_0} \right)^2 \left(\frac{n}{m} \right)^{\frac{1-\eta}{2}} \geq 1$$

under assumptions (27). Here we also used the fact that $1 - \eta \geq 4\delta$.

Thus, (26) is proved under assumptions (27). □

Remark 13. *Suppose that condition* (14) *holds for some* $\varepsilon = \varepsilon_0 > 0$. *Then, under assumption* (H), *condition* (14) *takes place also for all* $\varepsilon > 0$.

Indeed, this statement follows from Lemma 12 with

$$n(\varepsilon/\varepsilon_0)^{C_3} \geq m = m(n) > n(\varepsilon/\varepsilon_0)^{C_3} - 1 \to \infty, \quad \textit{where} \quad C_3 := 4/(1-\eta).$$

Emphasize also that we may evidently take $m = n$ *in* (26) *when* $\varepsilon \geq \varepsilon_0 > 0$.

3.8. **On technical assumptions.** Later on we use next constants:

$$c_1 := \frac{1}{3K_1^2}, \quad c_2 := \frac{\eta}{3K_1}, \quad C_3 := \frac{4}{1-\eta} > 4, \quad c_3 := \frac{1}{(C_3+1)K_1} < \frac{1}{10}. \tag{29}$$

We need also a number $N_0 < \infty$ with several elementary properties.

Lemma 14. *Under assumptions of Theorem 1*

$$\sup_{t\in\mathbb{R}} \int_{t-h_n}^{t+h_n} f(x)dx \to 0, \quad l_n \sim \eta \log n, \quad \max_{1\le i\le 2k} w(a_i)/l_n \to 0, \tag{30}$$

where $l_n := |\log h_n|$*. In particular, it follows from* (30) *that there exists a finite number* $N_0 \ge \max\{N_1, 2N_3\}$ *such that*

$$\forall n \ge N_0 \quad h_n < d_*/3, \quad l_n \ge 0.9\eta \log n \ge 50\eta, \quad 1.5\sqrt{l_n} \ge A_n \ge 1.4\sqrt{l_n}, \tag{31}$$

$$\forall n \ge N_0 \quad \sup_{t\in\mathbb{R}} p_{h_n}(t) \le \frac{1}{20}, \quad \frac{n}{l_n^{C_3}} \ge \max\{2, N_0\}, \quad \max_{1\le i\le 2k} w(a_i) \le \frac{c_1 l_n}{50}. \tag{32}$$

Proof. From condition (H) we have the second relation in (30), which gives immediately the second inequalities in (31) and (32) because $n/l_n^{C_3} \sim n/(\eta\log n)^{C3} \to \infty$. Now the third property in (31) follows from (13).

From the second relation in (30) we also have that $h_n \to 0$. In particular, we have the first property in (31).

Since $\int f(x)dx = 1$ and $h_n \to 0$ we obtain the first convergence in (30) from the known property of integrable functions. This convergence implies the first property in (32).

At last, the third convergence in (30) follows from (12) and it implies the third property in (32). □

In reality, without loss of generality everywhere in the paper we may assume that all $n \ge N_0$.

4. Proof of Proposition 6

In this section without specifying more we suppose that all assumptions from Proposition 6 hold. The main idea here is to cover the set $\overline{D}_n$ by small intervals of the form $[t - h_n/2, t + h_n/2]$ and to estimate the following probabilities

$$P_n(t) := \mathbf{P}\left(\hat{f}_n(t) > 0.9A_n\right), \tag{33}$$

where

$$\hat{f}_n(t) := \sqrt{nh_n} \sup_{x:|x-t|\le h_n/2} |\psi(x)\left(f_n(x) - \mathbf{E}f_n(x)\right)|. \tag{34}$$

Appropriate estimates for probabilities from (33) will be obtained in a few steps in subsections 4.1 and 4.2.

4.1. **Corollaries from an estimate of Stute.** Introduce notations:

$$F(x) = \mathbf{P}(X \le x), \quad \xi_i = F(X_i), \quad F_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{[0,x]}(\xi_i),$$

$$F_n^{(q)}(x) = F_n(q+x) - F_n(q) - x, \quad 0 \le q \le q+x \le 1.$$

In this case $\{\xi_i\}$ is a sequence of independent random variables uniformly distributed over the interval $[0,1]$ and the sum $nF_n(x) = \sum_{i=1}^{n} \mathbb{1}_{[q,p+q]}(\xi_i)$ has a binomial distribution with parameters p and n.

Proposition 15. *If*

$$0 < a \le \delta/4 < 1/4, \qquad 8a \le n\omega^2\delta^2, \qquad 0 \le q \le q + a \le 1, \tag{35}$$

then we have

$$\mathbf{P}\left(\sup_{0\le p\le a}\left|F_n^{(q)}(p)\right| > \omega\right) \le 2\mathbf{P}\left(\left|F_n^{(0)}(a)\right| > \omega(1-\delta)\right). \tag{36}$$

Proof. Since $F_n^{(0)}(a) = F_n(a) - a$, the result follows from remark after Lemma 2.1 and Lemma 2.2 in [5] with $h(t) \equiv 1$. □

Lemma 16. *For all $n \ge 1$ and $q = F(t - h_n)$ the next inequality takes place for any real t:*

$$\hat{f}_n(t) \le \frac{K_1\overline{\psi}_{h_n/2}(t)}{\sqrt{nh_n}} n \sup_{0\le p\le p_{h_n}(t)}\left|F_n^{(q)}(p)\right|. \tag{37}$$

Proof. Integrating (1) by parts (see details, for example, in [4, Lemma 3]) we obtain the following known representation

$$f_n(x) - \mathbf{E}f_n(x) = \frac{1}{h_n}\int F_n^{(0)}\big(F(x - vh_n)\big)dK(v)$$

Then from definition (33) we have:

$$\hat{f}_n(t) \le \frac{\overline{\psi}_{h_n/2}(t)}{\sqrt{nh_n}} n \sup_{x:|x-t|\le h_n/2}\left|\int F_n^{(0)}\big(F(x - vh_n)\big)dK(v)\right|.$$

Now we are going to apply Lemma 10 with

$$\rho(x) := F_n^{(0)}\big(F(x)\big) \qquad \text{and} \qquad \tilde{t} := t - h_n.$$

In this case

$$\rho(x) - \rho(\tilde{t}) = F_n^{(q)}(p_x) \quad \text{for} \quad q = F(\tilde{t}) \quad \text{and} \quad p_x = F(x) - F(\tilde{t}).$$

So, we have from Lemma 10 that

$$\hat{f}_n(t) \le \frac{K_1\overline{\psi}_{h_n/2}(t)}{\sqrt{nh_n}} n \sup_{x:|x-t|\le h_n}\left|F_n^{(q)}(p_x)\right|. \tag{38}$$

Note that

$$\{x : |x-t| \le h_n\} = \{x : \tilde{t} \le x \le \tilde{t} + 2h_n\} = \{x : 0 \le p_x \le p_{h_n}(t)\}.$$

This fact and (38) imply (37). □

Recall that

$$\hat{w}_n(t) = \frac{\overline{\psi}^2_{h_n/2}(t)p_{h_n}(t)}{h_n} \qquad \text{and} \qquad l_n = |\log h_n|.$$

Lemma 17. *If for some* $t \in \mathbb{R}$

$$\hat{w}_n(t) \le c_1 l_n/50 \qquad \text{and} \qquad n \ge N_0, \tag{39}$$

then the following inequality is true

$$P_n(t) = \mathbf{P}\left(\hat{f}_n(t) > 0.9A_n\right) \le 2\mathbf{P}\left(n\overline{\psi}_{h_n/2}(t)\left|F_n^{(0)}(p_{h_n}(t))\right| > \lambda_n/K_1\right). \tag{40}$$

Proof. We are going to use Proposition 15 with

$$\delta := 0.2, \quad a := p_{h_n}(t), \quad \omega := \frac{\lambda_n}{n\overline{\psi}_{h_n/2}(t)K_1(1-\delta)}, \quad q = F(t-h_n). \tag{41}$$

Since $0.9A_n > \frac{0.9\cdot 1.4\lambda_n}{\sqrt{nh_n}} > \frac{\lambda_n}{0.8\sqrt{nh_n}} = \frac{nK_1\overline{\psi}_{h_n/2}(t)\omega}{\sqrt{nh_n}}$ by (31), we have

$$\mathbf{P}\left(\hat{f}_n(t) > 0.9A_n\right) \le \mathbf{P}\left(\hat{f}_n(t) > nK_1\overline{\psi}_{h_n/2}(t)\omega/\sqrt{nh_n}\right).$$

This fact and Lemma 16 yield:

$$\mathbf{P}\left(\hat{f}_n(t) > 0.9A_n\right) \le \mathbf{P}\left(\frac{K_1\overline{\psi}_{h_n/2}(t)}{\sqrt{nh_n}} n \sup_{0\le p\le a}\left|F_n^{(q)}(p)\right| > \frac{nK_1\overline{\psi}_{h_n/2}(t)\omega}{\sqrt{nh_n}}\right)$$

$$= \mathbf{P}\left(\sup_{0\le p\le a}\left|F_n^{(q)}(p)\right| > \omega\right). \tag{42}$$

In order to use the statement of Proposition 15 we have to verify fairness of (35) for δ, a, ω and q introduced in (41). Under assumptions (39), the first condition in (35) follows from (32); the second condition in (35) coincides with

$$\hat{w}_n(t) \le l_n/(128K_1^2) = (3/128)c_1 l_n,$$

and, hence, it follows from (39); the last condition in (35) is evident with q from (41).

Thus, under assumptions (39) we have the right to apply statement (36) of Proposition 15 for δ, a, ω and q introduced in (41). So, statement (40) of Lemma 17 follows immediately from (42) and (36) since $(1-\delta)\omega := \frac{\lambda_n}{nK_1\overline{\psi}_{h_n/2}(t)}$ in our case. □

4.2. **Corollaries from an estimate of Bennett.** From Bennett inequality we obtain the following result:

Proposition 18. *For any real* $p \in [0,1]$, $x > 0$, $y > 0$ *and* ψ *with* $|\psi| \le y$ *we have*

$$\mathbf{P}\left(n\left|\psi F_n^{(0)}(p)\right| > x\right) \le 2\exp\left\{\frac{x}{y} - \frac{x}{y}(1+b)\log\left(1+\frac{1}{b}\right)\right\} \tag{43}$$

for $b = \frac{n\psi^2 p(1-p)}{xy}$. *In particular,*

$$Q := \mathbf{P}\left(n\left|\psi F_n^{(0)}(p)\right| > x\right) \le Q_1 := 2(eb)^{x/y} \le 2\left(\frac{en\psi^2 p}{xy}\right)^{x/y}, \tag{44}$$

$$Q \le Q_2 := 2\exp\left\{-\frac{x}{2y(b+1/3)}\right\} \le 2\exp\left\{-\frac{x^2}{2n\psi^2 p + 2xy/3}\right\}. \tag{45}$$

Moreover, if, in addition, $z \ge 0$ *and* $x \ge zy$ *then*

$$Q \le (e^{5/2}b)^z Q_2 \le 2\left(\frac{e^{5/2}n\psi^2 p}{xy}\right)^z \exp\left\{-\frac{x^2}{2n\psi^2 p + 2xy/3}\right\}. \tag{46}$$

Proof. Put $\tilde{\xi}_i := \mathbb{1}_{[0,p]}(\xi_i)$, $\tilde{x}_i := \psi(\tilde{\xi}_i - p)$. In this case $\{\tilde{x}_i\}$ are i.i.d. random variables with

$$\mathbf{E}\tilde{x}_i = 0, \quad \mathbf{E}\tilde{x}_i^2 = \psi^2 p(1-p) \le \psi^2 p, \quad |\tilde{x}_i| \le |\psi|, \quad n\psi F_n^{(0)}(x) = \sum_{i=1}^n \tilde{x}_i.$$

Thus, we may apply Theorem 3 from [7] and as a result we obtain inequality (43) (which belongs to Bennett). Inequalities (44) and (45) follow from (43) (see, for example [8]). At last for $x \ge zy \ge 0$

$$Q = Q^{zy/x} Q^{1-zy/x} \le Q_1^{zy/x} Q_2^{1-zy/x} = (eb)^z Q_2 e^{z/(2b+2/3)} \le (e^{5/2}b)^z Q_2$$

because $e^{z/(2b+2/3)} \le e^{z/(2/3)} = (e^{3/2})^z$. □

Consider now the set $T(n) = \{t_{j,n} : j = 0, \pm 1, \pm 2, \dots\}$ introduced in Lemma 11. To simplify further calculations we will below use the following simplified notations connected with the points from $T(n)$:

$$(47) \qquad t_j := t_{j,n}, \quad \psi_j := \overline{\psi}_{h_n}(t_{j,}), \quad p_j := p_{h_n}(t_j), \quad w_j = \hat{w}_n(t_j), \quad \Delta_j = \Delta_{j,n}.$$

In this case

$$(48) \qquad \psi_j^2 p_j = w_j h_n \qquad \text{and} \qquad p_j = \mathbf{P}(X \in \Delta_j).$$

Lemma 19. *Fix arbitrary $t_j \in \mathbb{R}$ and suppose that*

$$(49) \qquad n \ge N_0 \qquad \textit{and} \qquad w_j \le c_1 l_n/50.$$

Then for all $C \ge 1$

$$(50) \qquad P_n(t_j) \le 4\left(\frac{eCK_1^2 n\psi_j^2 p_j}{\lambda_n^2}\right)^C \le 4enp_j\left(\frac{eCK_1^2 w_j}{l_n}\right)^{C-1} \qquad \textit{if} \quad \psi_j \le \frac{\lambda_n}{CK_1}.$$

Furthermore,

$$(51) \qquad P_n(t_j) \le \frac{20p_j}{l_n} \qquad \textit{if} \quad w_j \le \eta c_1 \quad \textit{and} \quad \psi_j \le \frac{c_2\lambda_n}{l_n},$$

where constants c_1 and c_2 may be found in (29).

Proof. In notation (47) conditions (48) and (39) coincide. Hence, the assertion (40) of Lemma 17 takes place and may be rewritten in the following way

$$(52) \qquad P_n(t_j) \le 2\mathbf{P}\left(n\psi_j\left|F_n^{(0)}(p_j)\right| > \lambda_n/K_1\right).$$

Now we are going to apply the assertions of Proposition 18 when

$$(53) \qquad z = 1, \quad x = \lambda_n/K_1, \quad y = x/C \le x, \quad p = p_j, \quad \psi = \psi_j.$$

In this case, (50) follows from (52) and (44), whereas from (52) and (46) we obtain

$$P_n(t_j) \le 4\frac{e^{5/2}n\psi_j^2 p_j}{xy}\exp\left\{-\frac{\lambda_n^2}{2nK_1^2\psi_j^2 p_j + 2\lambda_n^2/(3C)}\right\}$$

$$(54) \qquad \le \frac{4e^{5/2}np_j}{C}\exp\left\{-\frac{l_n}{2K_1^2 w_j + 2l_n/(3C)}\right\}.$$

Here we also used (48) and estimate $\psi_j \le y \le x/C$.

Note that the last assumption in (51) corresponds to $C = 3l_n/\eta$ in (53). But in this case we have from (31) that

$$C > 1 \quad \text{and} \quad l_n > 0.9\eta \log n \quad \text{for all} \quad n \ge N_0.$$

Using again assumption $w_j \le \eta c_1 = \eta/3K_1^2$ from (51) we obtain

$$\frac{l_n}{2K_1^2 w_j + 2l_n/(3C)} \ge \frac{0.9\eta \log n}{2\eta/3 + 2\eta/9} = \frac{0.81 \log n}{8} > \log n.$$

This fact and (54) yield

$$P_n(t_j) \le \frac{4e^{5/2} n p_j}{C} \exp\{-\log n\} = \frac{4e^{5/2} p_j}{C} = \frac{4e^{5/2} \eta p_j}{3l_n}.$$

Thus (51) is proved since $\eta < 1$ and $e^{5/2} < 15$. □

4.3. **Proof of Proposition 6.** Introduce several subsets from the set $T(n) = \{t_{j,n} : j = 0, \pm 1, \pm 2, \dots\}$ defined in Lemma 11:

$$T_1 := \left\{t_j \in T(n) \cap \partial^{h_n/2} D : \psi_j \le c_3 \lambda_n < \lambda_n / K_1\right\}, \tag{55}$$

$$T_* := T(n) \cap (\overline{D} \setminus \partial^{h_n/2} D), \quad T_2 := \{t_j \in T_* : 0 < \psi_j \le c_2 \lambda_n / l_n\}, \tag{56}$$

$$T_3 := \{t_j \in T_* : c_2 \lambda_n / l_n < \psi_j \le c_3 \lambda_n\}, \quad T_4 = \{t_j \in T(n) : \psi_j > c_3 \lambda_n\}. \tag{57}$$

This definition implies that

$$T(n) \cap \overline{D}_n \subset T^* := T_1 \cup T_2 \cup T_3 \cup T_4, \quad \overline{D}_n \subset \cup_{j : t_j \in T^*} \Delta_j. \tag{58}$$

Later on we will also use the following notation

$$\Sigma_n(T) := \sum_{j : t_j \in T} P_n(t_j) \qquad \forall\, T \subset T(n). \tag{59}$$

Lemma 20. *If $n \ge N_0$ then*

$$\Sigma_n(T_1) \le 8keK_1^2 \max_{1 \le i \le 2k} w(a_i)/l_n. \tag{60}$$

In particular, $\Sigma_n(T_1) \to 0$.

Proof. From definitions (23), (24) and (55) we have that $T_1 \subset \partial D$. Thus,

$$\max_{t \in T_1} \hat{w}_n(t) \le \max_{t \in \partial D} \hat{w}_n(t) = \max_{1 \le i \le 2k} w(a_i) \le c_1 l_n / 50$$

for $n \ge N_0$ as it follows from (32). Hence condition (49) of Lemma 19 is satisfied and we have the right to apply inequality (50) for $C = 1$. We obtain for $a_i = t_j$ that

$$P_n(a_i) \le 4 \frac{eK_1^2 n \psi_j^2 p_j}{\lambda_n^2} = \frac{4eK_1^2 n h_n w_j}{\lambda_n^2} = \frac{4eK_1^2 w_j}{l_n}.$$

This fact implies (60) since set T_1 consists from at most $2k$ points as it was supposed in (22). Convergence $\Sigma_n(T_1) \to 0$ follows from (12) or (30). □

Lemma 21. *For each $n \ge N_0$ and all c*

$$\Sigma_{(c)} := \sum_{j : t_j \in T_{(c)}} p_j \le 4\mathbf{P}(\overline{\psi}_{2h_n}(X) > c), \quad \textit{where} \quad T_{(c)} = \{t_j : \psi_j > c\}. \tag{61}$$

Proof. It follows from Lemma 11 that

$$\Sigma_{(c)} := \sum_{j : t_j \in T_{(c)}} \mathbf{P}(X \in \Delta_j) \le 4\mathbf{P}(X \in U(c)) \quad \text{with} \quad U(c) := \cup_{j : t_j \in T_{(c)}} \Delta_j. \tag{62}$$

But if $t \in U(c)$ then there exists a number $j = j(t)$ such that

$$t \in (t_j - h_n, t_j + h_n) \subset (t - 2h_n, t + 2h_n).$$

It means that $\overline{\psi}_{2h_n}(t) \geq \overline{\psi}_{h_n/2}(t_j) = \psi_j > c$. Hence, $U(c) \subset \{t : \overline{\psi}_{2h_n}(t) > c\}$. But this fact and (62) yield (61). □

Lemma 22. *For each* $n \geq N_0$

$$\Sigma_n(T_4) \leq 4n\mathbf{P}(\overline{\psi}_{2h_n}(X) > c_3\lambda_n). \tag{63}$$

Proof. Fix t_j. If $|X_i - t_j| > h_n$ for all $i = 1, \ldots .n$ than $|X_i - t| > h_n/2$ for all $t \in [t - h_n/2, t + h_n/2]$ for the same i, and $\hat{f}(t_j) = 0$ by (34). Hence, by (33)

$$P_n(t_j) \leq \mathbf{P}(\hat{f}(t_j) > 0) \leq \mathbf{P}(\exists i \leq n : |X_i - t_j| \leq h_n)$$
$$\leq n\mathbf{P}(|X - t_j| \leq h_n) = n\mathbf{P}(|X - t_j| < h_n) = np_j.$$

Now (63) follows from (61) with $c = c_3\lambda_n$. □

In Lemmas 23 and 24 we will use the fact that

$$\forall\, n \geq N_0 \quad \forall\, t_j \in T_2 \cup T_3 \quad w_j \leq \eta c_1 \leq c_1 l_n/50. \tag{64}$$

The first inequality here follows from condition (D) whereas the second one was pointed out in (31).

Thus, for all $t_j \in T_2 \cup T_3$ condition (49) of Lemma 19 is satisfied for each $n \geq N_0$.

Lemma 23. *For all* $n \geq N_0$ *we have* $\Sigma_n(T_2) \leq 80/l_n$.

Indeed, from assertion (51) of Lemma 19 and from Lemma 21 with $c = 0$ we have:

$$\Sigma_n(T_2) \leq 20\Sigma_{(0)}/l_n \leq 80\mathbf{P}(\overline{\psi}_{2h_n}(X) > 0)/l_n \leq 80/l_n.$$

Lemma 24. *For all* $n \geq N_0$

$$\Sigma_n(T_3) \leq 32e(C_3 + 1)^{C_3} m(n)\mathbf{P}(\overline{\psi}_{2h_{m(n)}}(X) > c_2\lambda_{m(n)}). \tag{65}$$

where the integer $m(n)$ *is such that*

$$n/2 \geq \mu_n := n/l_n^{C_3} \geq m(n) > \mu_n - 1 \geq \mu_n/2 \to \infty. \tag{66}$$

Proof. First, it follows from (64) that for all $n \geq N_0$ and $t_j \in T_3$ we have the right to apply assertion (50) of Lemma 19 with $C = C_3 + 1 > 5$. As a result we have:

$$P_n(t_j) \leq 4enp_j \left(\frac{e(C_3 + 1)K_1^2 w_j}{l_n}\right)^{C_3} \leq 4enp_j \frac{(C_3 + 1)^{C_3}}{l_n^{C_3}}, \tag{67}$$

since $w_j < 1/(eK_1^2)$ by (64) and (29).

Second, since $T_3 \subset T_{(c)}$ for $c = c_2\lambda_n/l_n$, we have from (61) and (67) that

$$\Sigma_n(T_3) \leq 4en\Sigma_{(c)} \frac{(C_3 + 1)^{C_3}}{l_n^{C_3}} \leq 16en\mathbf{P}\left(\overline{\psi}_{2h_n}(X) > \frac{c_2\lambda_n}{l_n}\right) \frac{(C_3 + 1)^{C_3}}{l_n^{C_3}}. \tag{68}$$

Next, for $n \geq N_0$ we have from (32) and (30) that $\mu_n \to \infty$ and $\mu_n - 1 \geq \mu_n/2$. Hence for $n \geq N_0$ number $m(n)$ satisfying (66) is uniquely defined.

At last, we are going to apply Lemma 12 with $\varepsilon_0 = c_2$ and $\varepsilon = \varepsilon_0/l_n$. Emphasize, that in this case condition (27) of Lemma 12 follows for $m = m(n)$ from condition (66) of the present lemma. Since $n/l_n^{C_3} \leq 2m(n)$ by (66), we obtain (65) from (68) applying Lemma 12. □

Proof of Proposition 6. It follows from (58) that set $\overline{D}_n$ is covered by intervals Δ_j with centers in $T^* = T_1 \cup T_2 \cup T_3 \cup T_4$. Using notation introduced in (59) we obtain from definitions (4), (34) and (33) that

$$\mathbf{P}\left(\beta_n\left(\overline{D}_n\right) > 0.9A_n\right) \leq \Sigma_n(T^*) \leq \Sigma_n(T_1) + \Sigma_n(T_2) + \Sigma_n(T_3) + \Sigma_n(T_4).$$

Substituting estimate obtained in Lemmas 20 –24 we obtain:

$$\mathbf{P}\left(\beta_n\left(\overline{D}_n\right) > 0.9A_n\right) \le \Sigma_n(T_1) + 80/l_n \tag{69}$$
$$+4n\mathbf{P}(\overline{\psi}_{2h_n}(X) > c_3\lambda_n) + 32e(C_3+1)^{C_3}m(n)\mathbf{P}(\overline{\psi}_{2h_{m(n)}}(X) > c_2\lambda_{m(n)}).$$

But all the the terms in the right hand side of (69) tend to zero as $n \to \infty$. Indeed, for the first term it follows from Lemma 20. The convergence of both last terms follows from Remark 13 since $m(n) \to \infty$.

Thus, the assertion of Proposition 6 is proved.

5. Proof of Proposition 5

We will follow the scheme of proof of Theorem 6 in [1] with changes due to the facts that we are using less restrictive assumptions. In this section without specifying more we suppose that all assumptions from Proposition 5 hold.

5.1. **Main considerations.** The following assertion will be proved below in subsection 5.2 as a simple partial case of Theorem 1 from [4] .

Proposition 25. *For each $n \ge 1$ on the same probability space with the sample $X_1, X_2, ...$ we can construct a Wiener process $B_n(\cdot)$ and the process*

$$\gamma_n(t) = \frac{1}{\sqrt{nh_n}} \int_{-\infty}^{\infty} B_n(F(t - vh_n))dK(v) \tag{70}$$

so that the difference

$$\delta_n(t) = f_n(t) - \mathbf{E}f_n(t) - \gamma_n(t) \tag{71}$$

satisfies the following relation

$$\Delta_{n,1} := A_n\sqrt{nh_n} \sup_{t\in D_n} |\psi(t)\delta_n(t)| \xrightarrow{\mathbf{P}} 0. \tag{72}$$

Now we introduce two new Wiener processes

$$\tilde{B}(t) := \int_{-\infty}^{t} \frac{1}{\sqrt{f(x)}} dB_n(F(x)), \quad \tilde{B}_n(t) := \tilde{B}(th_n)/\sqrt{h_n} \tag{73}$$

and let

$$\xi(t) := \int_{-\infty}^{\infty} \tilde{B}_n\,(t - v)\ dK(v), \quad \tilde{\delta}_n(t) := \sqrt{nh_n}\gamma_n(t) - \sqrt{f(t)}\xi(t/h_n). \tag{74}$$

The main idea of our proof of Proposition 5 is based on the following representation:

$$\sqrt{nh_n}\psi(t)(f_n(t) - \mathbf{E}f_n(t)) - w(t)\xi(t/h_n) = \sqrt{nh_n}\psi(t)\delta_n(t) + \psi(t)\tilde{\delta}_n(t). \tag{75}$$

For each $B \subset \mathbb{R}$ define

$$\tilde{\beta}_n(B) := \sup_{t\in B} w(t)\,|\xi\,(t/h_n)| \quad \text{and} \quad \tilde{\alpha}_n(B) := A_n\left(\tilde{\beta}_n(B) - A_n\right). \tag{76}$$

In subsection 5.3 we easily prove

Proposition 26. *Under assumptions of Proposition 5 all conditions of Corollary 4 in* [1] *are satisfied. In particular*

$$\tilde{\alpha}_n(D) \Rightarrow \zeta \quad \textit{with} \quad \mathbf{P}(\zeta \le x) = e^{-e^{-x}} \quad \forall\ x \in \mathbb{R}.$$

The following two assertions will be proved in subsections 5.5 and 5.6 using lemmas from subsection 5.4.

Proposition 27. *We have* $\tilde{\alpha}_n(D_n) \Rightarrow \zeta$.

Proposition 28. *We have also* $\tilde{\Delta}_n := A_n\left(\beta_n(D_n) - A_n\right) - \tilde{\alpha}_n(D_n) \xrightarrow{\mathbf{P}} 0$.

Proposition 5 follows immediately from Propositions 25 and 26 because in this case evidently

$$A_n\left(\beta_n(D_n) - A_n)\right) = \tilde{\alpha}_n(D_n) + \tilde{\Delta}_n \Rightarrow \zeta.$$

The rest of the section is devoted to proofs of Propositions 25 – 28.

5.2. **Proof of proposition 25.** Introduce processes

$$W_n(t) = B_n(t) - tB_n(1) \quad \text{and} \quad \gamma_{0n}(t) = \frac{1}{\sqrt{n}h_n}\int W_n(t - vh_n)dK(v). \tag{77}$$

Since $W_n(\cdot)$ is a Brownian Bridge, we may use Theorem 1 from [4]. As a result we obtain that for any number $C_{(n)}$ such that

$$C_{(n)} = o\left(\frac{\sqrt{nh_n}}{A_n \log n}\right) \tag{78}$$

we can construct on the same probability space with observations $\{X_i\}$ a Brownian Bridge $W_n(\cdot)$ so that

$$\Delta_{n,2}(C_{(n)}) := A_n\sqrt{nh_n}\sup_{t\in\mathbb{R}} |\min\{\psi(t), C_{(n)}\delta_{0n}(t)\}| \xrightarrow{\mathbf{P}} 0, \tag{79}$$

where

$$\delta_{0n}(t) = f_n(t) - \mathbf{E}f_n(t) - \gamma_{0n}(t). \tag{80}$$

From conditions $(D), (H)$ and (A_n) we have

$$C_4 := \sup_{t\in D}\psi(t) + \sup_{t\in D} f(t) < \infty \quad \text{and} \quad \frac{\sqrt{nh_n}}{A_n \log n} \sim \frac{n^{(1-\eta)/2}\sqrt{s(n)}}{\eta \log^2 n} \to \infty. \tag{81}$$

So, $C_{(n)} = C_4$ satisfies (78). Hence

$$\Delta_{n,3} := A_n\sqrt{nh_n}\sup_{t\in D_n} |\psi(t)\delta_{0n}(t)| \le \Delta_{n,2}(C_4) \xrightarrow{\mathbf{P}} 0. \tag{82}$$

Next, from (15), (23) and (24) we have that

$$\forall t \in D_n \quad [t - h_n/2, t + h_n/2] \subset D. \tag{83}$$

It follows from (70), (77) and (80) that

$$\delta_{0n}(t) = \delta_n(t) + a_n(t) \quad \text{with} \quad a_n(t) = \frac{1}{\sqrt{n}h_n}\int F(t - vh_n)dK(v). \tag{84}$$

Since $a_n(t) = \mathbf{E}f_n(t)/\sqrt{n}$ (see, for example, proof of Lemma 6 in [4]), we obtain from (1) and (83) that

$$|a_n(t)| \le \frac{1}{\sqrt{n}h_n}\int\limits_{t-h_n/2}^{t+h_n/2} K_1 f(x)dx \le \frac{K_1C_1}{\sqrt{n}} \quad \forall t \in D_n.$$

Thus,

$$\Delta_{n,4} := A_n\sqrt{nh_n}\sup_{t\in D_n} |\psi(t)a_n(t)| \le A_n\sqrt{h_n}C_1^2K_1 \longrightarrow 0$$

due to (13). This fact together with (71), (82) and (84) yield

$$\Delta_{n,1} \le \Delta_{n,3} + \Delta_{n,4} \longrightarrow 0.$$

So, (72) is proved.

5.3. **Proof of proposition 26.**

Lemma 29. *$\xi(t)$ is a centered, stationary Gaussian process satisfying all conditions from Corollary 4 in* [1].

Proof. As $\xi(t)$ is an integral of Ito, it is centered and stationary Gaussian process. Next, using Ito's isometry we get for all $t, s \in \mathbb{R}$

$$\mathbf{E}\xi^2(t) = \int_{-\infty}^{\infty} K^2(t-y)dy = \int_{-\infty}^{\infty} K^2(x)dx = K_2^2 = 1, \tag{85}$$

$$\mathbf{E}(\xi(t+s)-\xi(t))^2 = \int_{-\infty}^{\infty} (K(t+s-y)-K(t-y))^2 dy = \int_{-\infty}^{\infty} (K(x+t)-K(x))^2 dx.$$

Finally we derive from Condition (K) and (6) that as $t \to 0$

$$\mathbf{E}(\xi(t+s)-\xi(t))^2 \sim C_1 K_2^2 \left|t\right|^{\alpha} = C_1 \left|t\right|^{\alpha}. \tag{86}$$

Now all assertions of Lemma 29 follow from (85) and (86). □

It follows from Lemma 8 that function $w(\cdot)$, as a function on a set D, satisfies all conditions from Corollary 4 in [1]. This fact and Lemma 29 show that proposition 26 is a particular case of Corollary 4 from [1].

5.4. **Main lemmas.** Further we will essentially use a famous inequality for the supremum of Brownian motion $B(\cdot)$:

$$\mathbf{P}\left(\sup_{u\in[0,\sigma^2]} |B(u)| > a\right) \le 2\mathbf{P}\left(|B(\sigma^2)| > a\right) = 4\mathbf{P}\left(\eta > \frac{a}{\sigma}\right) \le 2e^{-a^2/(2\sigma^2)} \tag{87}$$

for all $a, \sigma > 0$, where η is a standard normal random variable.

Lemma 30. *For all $z > 0$ and $t \in \mathbb{R}$ we have*

$$\mathbf{P}(\sup_{x:|x-t|\le h_n/2} |\xi(x/h_n)| > K_1 z) \le 4\mathbf{P}\left(\eta > z/\sqrt{2}\right) \le 2e^{-z^2/4}. \tag{88}$$

Proof. By (74) with $h = h_n$ we may apply Lemma 10 for $\rho(x) = \tilde{B}_n(x/h_n)$ with $\tilde{t} = t - h_n, H = h_n/2$. We obtain:

$$\sup_{x:|x-t|\le h_n/2} |\xi(x/h_n)| = \sup_{x:|x-t|\le h_n/2} \Big| \int \tilde{B}_n\Big(\frac{x-vh_n}{h_n}\Big) dK(v)\Big|$$
$$\le K_1 \sup_{x:|x-t|\le h_n} \Big|\tilde{B}_n\Big(\frac{x}{h_n}\Big) - \tilde{B}_n\Big(\frac{t-h_n}{h_n}\Big)\Big| = K_1 \sup_{v\in[0,2h_n]} \Big|B\Big(\frac{v}{h_n}\Big)\Big|,$$

where $B(v) = \tilde{B}_n\Big(t - h_n + v\Big) - \tilde{B}_n\Big(t-h_n\Big)$ is a new Wiener process. Hence,

$$\mathbf{P}\Big(\sup_{x:|x-t|\le h_n/2} |\xi(x/h_n)| > K_1 z\Big) \le \mathbf{P}\Big(\sup_{v\in[0,2h_n]} |B(v/h_n)| = \sup_{u\in[0,2]} |B(u)| > z\Big)$$

and (88) follows from (87) with $\sigma^2 = 2$. □

Lemma 31. *For all $h, z > 0$ and $t \in \mathbb{R}$*

$$\tilde{Q}_n(t,z) := \mathbf{P}\left(\sup_{x\in[t-h_n/2,t+h_n/2]} \left|\tilde{\delta}_n(x)\right| > 4zK_1\right) \le 4e^{-z^2/M^2(t,h)}, \tag{89}$$

where

$$M(t,h) = \sup_{x\in[t-h,t+h]} |\sqrt{f(x)} - \sqrt{f(t-h)}|. \tag{90}$$

Proof. We fix numbers t, n and $h = h_n > 0$ and introduce the following notations:

$$t_- = t - h, \quad t_+ = t + h, \quad g(y) = \sqrt{f(y)} - \sqrt{f(t_-)}, \quad M = M(t,h),$$

$$V_1(x) := \int_{t_-}^{x} g(y)\, d\tilde{B}(y), \qquad V_2(x) := \int_{t_-}^{x} d\tilde{B}(y).$$

We have from (73) and (74) that

$$\xi\left(\frac{x}{h}\right) := \int_{-\infty}^{\infty} \tilde{B}_n\left(\frac{x - vh}{h}\right)\, dK(v) = \frac{1}{\sqrt{h}} \int_{-\infty}^{\infty} \tilde{B}\left(t - vh\right)\, dK(v).$$

It follows from (73) and (70) that

$$B(F(x)) = \int_{-\infty}^{x} \sqrt{f(y)} d\tilde{B}(y), \quad \sqrt{n} h_n \gamma_n(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{x-vh} \sqrt{f(y)}\, d\tilde{B}(y)\, dK(v). \tag{91}$$

Hence for $x \ge t_-$ we have from (74) that

$$\sqrt{h_n}\tilde{\delta}_n(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{x-vh} \left(\sqrt{f(y)} - \sqrt{f(x)}\right)\, d\tilde{B}(y)\, dK(v)$$

$$= \int_{-\infty}^{\infty} \int_{t_-}^{x-vh} \left(\sqrt{f(y)} - \sqrt{f(t_-)} + \sqrt{f(t_-)} - \sqrt{f(x)}\right)\, d\tilde{B}(y)\, dK(v).$$

So, $\sqrt{h_n}\tilde{\delta}_n(x) = \int_{-\infty}^{\infty} (V_1(x - vh) - g(x)V_2(x - vh))\, dK(v)$. Hence,

$$\left|\sqrt{h_n}\tilde{\delta}_n(x)\right| \le varK \sup_{v\in[-\frac{1}{2},\frac{1}{2}]} \left|V_1(x - vh) - g(x)V_2(x - vh)\right|.$$

Thus,

$$\sup_{x\in[t_-,t_+]} \left|\sqrt{h_n}\tilde{\delta}_n(x)\right| \le K_1(\overline{V_1} + M\overline{V_2}), \tag{92}$$

where

$$\overline{V_1} := \sup_{x\in[t_-,t_+]} \sup_{v\in[-\frac{1}{2},\frac{1}{2}]} |V_1(x - vh)| = \sup_{u\in[0,2h]} \left|V_1(t_- + u)\right|,$$

and similarly

$$\overline{V_2} := \sup_{x\in[t_-,t_+]} \sup_{v\in[-\frac{1}{2},\frac{1}{2}]} |V_2(x - vh)| = \sup_{u\in[0,2h]} \left|V_2(t_- + u)\right|.$$

One can see that $V_1(t_- + u) := \int_{t_-}^{t_-+u} g(y)\, d\tilde{B}(y)$ is a Gaussian process with independent increments and it is identically distributed with $B(d(u))$, where

$$d(u) = \int_{t_-}^{t_-+u} g^2(y)\, dy \le d(2h) \le 2hM^2(t,h) = 2hM^2, \quad \forall\, u \in [t_-, t_+]. \tag{93}$$

Using (87) we obtain:

$$\mathbf{P}\left(\overline{V_1} > a\right) = \mathbf{P}\left(\sup_{u\in[0,2h]} |V_1(t_- + u)| > a\right) = \mathbf{P}\left(\sup_{u\in[0,2h]} |B(d(u))| > a\right)$$

$$= \mathbf{P}\left(\sup_{x\in[0,d(2h)]} |B(x)| > a\right) \le 4\mathbf{P}\left(B(d(2h)) > a\right) = 4\mathbf{P}\left(\eta > \frac{a}{\sqrt{d(2h)}}\right),$$

where η is a standard normal random variable. Then we get from (93) that

$$\mathbf{P}\left(\frac{\overline{V_1}}{\sqrt{h}} > 2z\right) \le 4\mathbf{P}\left(\eta > \frac{2z\sqrt{h}}{\sqrt{d(2h)}}\right) \le 4\mathbf{P}\left(\eta > \frac{2z\sqrt{h}}{\sqrt{2h}M}\right). \tag{94}$$

Similarly, since $B(u) = \int_{t_-}^{t_-+u} d\tilde{B}(x)$ is a Wiener process,

$$\mathbf{P}\left(\overline{V_2} > a\right) = \mathbf{P}\left(\sup_{u\in[0,2h]} \left|\int_{t_-}^{t_-+u} d\tilde{B}(x)\right| > z\sqrt{2h}\right) \le 4\mathbf{P}\left(\eta > \frac{a}{\sqrt{2h}}\right). \tag{95}$$

So finally we have from (92) that

$$\tilde{Q}_n(t,z) \le \mathbf{P}\left(K_1(\overline{V_1} + M\overline{V_2})/\sqrt{h} > 4K_1 z\right)$$

$$\le \mathbf{P}\left(\overline{V_1} > 2z\sqrt{h}\right) + \mathbf{P}\left(\overline{V_2} > 2z\sqrt{h}/M\right) \le 8\mathbf{P}\left(\eta > \frac{\sqrt{2}z}{M}\right) \le 4e^{-x^2/M^2}.$$

Here we used also (94) and (95). Thus, (89) is obtained. □

Lemma 32. *If $n \ge N_0$ then*

$$\forall\, t \in \mathbb{R} \quad Q_n(t) := \mathbf{P}\Big(\sup_{x\in D: |x-t| \le h_n/2} |w(x)\xi(x/h_n)| > 0.9A_n\Big) \le 2h_n^{c_4} \to 0 \tag{96}$$

with $c_4 := 1/(4K_1^2)$. In addition, for all $\varepsilon > 0$ there exists a finite integer $N(\varepsilon) \ge N_0$ such that

$$\forall\, t \in D_n \quad \tilde{Q}_n(t) := \mathbf{P}\Big(A_n \sup_{x: |x-t| \le h_n/2} |\psi(x)\tilde{\delta}_n(x)| > \varepsilon\Big) \le 4h_n^2. \tag{97}$$

Proof. Inequality (96) follows from (88) with $z = \sqrt{l_n}/K_1 = \sqrt{\log 1/h_n}/K_1$ because $w(t) \le 1$ for all $t \in D$ and $0.9A_n > \sqrt{l_n}$ for all $n \ge N_0$ as it follows from (31).

Since $\sup_{t\in D} \psi(t) \le C_4 < \infty$ by (81), we have from (89) and (97) with $4A_nC_4K_1z = \varepsilon$ that

$$Q_n(t) \le \mathbf{P}\left(A_nC_4 \sup_{x\in[t-h_n/2, t+h_n/2]} \left|\tilde{\delta}_n(x)\right| > \varepsilon\right) \le 4\exp\left\{-\frac{\varepsilon^2}{16A_n^2C_4^2K_1^2M^2(t,h)}\right\}. \tag{98}$$

But $\sup_{t\in D} p(t) \le C_4 < \infty$ by (81). Hence we have from (10) that

$$\sup_{t\in D_n} D(t,h_n) \le \sqrt{C_4} \sup_{\substack{t,t+y\in D\\ |y|\le h_n/2}} \left| \frac{\sqrt{f(t+y)}}{\sqrt{f(t)}} - 1 \right| = o\left(\frac{1}{|\log h_n|}\right). \tag{99}$$

Here we used also (83). Since $A_n^2 \sim 2|\log h_n|$ by (13) we have from (99) that

$$A_n^2 \sup_{t\in D_n} D^2(t,h_n) = o\left(\frac{1}{|\log h_n|}\right).$$

The latter means that there exists a finite integer $N(\varepsilon) \ge N_0$ such that

$$2\cdot 16 A_n^2 C_4^2 K_1^2 A_n^2 \sup_{t\in D_n} D^2(t,h_n) \le \varepsilon^2/|\log h_n|.$$

Thus, we have for all $n \ge N(\varepsilon)$ from (98) that

$$Q_n(t) \le 4\exp\{-2|\log h_n|\} = 4\exp\{-2\log 1/h_n\} = 4h_n^2.$$

So, (97) is proved. □

5.5. **Proof of proposition 27.** Repeating the same arguments from the similar proof of Theorem 1 we have:

$$\mathbf{P}\left(A_n(\tilde\beta_n(D) - A_n) \le x\right) \le \mathbf{P}\left(A_n(\tilde\beta_n(D_n) - A_n) \le x\right),$$

$$\mathbf{P}\left(A_n(\tilde\beta_n(D_n) - A_n) \le x\right) - \mathbf{P}\left(A_n(\tilde\beta_n(D) - A_n) \le x\right)$$

$$\le \mathbf{P}\left(A_n(\tilde\beta_n(D\setminus D_n) - A_n) > x\right) \le \mathbf{P}\left(\tilde\beta_n(D\setminus D_n) > 0.9A_n\right)$$

for all $x \ge -A_n^2/10 \to -\infty$. Thus, using notation $\tilde\alpha_n(\cdot)$ introduced in (76) we obtain:

(100)

$$\forall\ x \ge -A_n^2/10 \quad |\mathbf{P}(\tilde\alpha_n(D_n) \le x) - \mathbf{P}(\tilde\alpha_n(D) \le x)| \le \mathbf{P}\left(\tilde\beta_n(D\setminus D_n) > 0.9A_n\right).$$

Now note that the union of intervals $[a_i - h_n/2, a_i + h_n/2]$, where points a_i were defined in (22), covers set $D\setminus D_n$. This fact and (96) yield

$$\mathbf{P}\left(\tilde\beta_n(D\setminus D_n) > 0.9A_n\right) = \mathbf{P}(\sup_{x\in D\setminus D_n} |w(x)\xi(x/h_n)| > 0.9A_n)$$

$$\le \sum_{i=1}^{2k} Q_n(a_i) \le 2k\cdot 2h_n^{c_4} \to 0.$$

So, this convergence together with inequality (100) imply Proposition 27.

5.6. **Proof of proposition 28.**

Lemma 33. *Under assumptions of Proposition 28*

$$\Delta_{n,5} := A_n \sup_{t\in D_n} |\psi(t)\tilde\delta_n(t)| \xrightarrow{\mathbf{P}} 0. \tag{101}$$

Proof. Consider the set $T_0 := T(n) \cap D_n$. It follows from (23) and (24) that the union of intervals $[t_j - h_n/2, t_j + h_n/2]$ over all t_j from T_0 covers set D_n. This observation and (97) yield for $n \geq N(\varepsilon)$

(102)
$$\mathbf{P}\left(A_n \sup_{t \in D_n} |\psi(t)\tilde{\delta}_n(t)| > \varepsilon\right) \leq \sum_{t_j \in D_n} \tilde{Q}_n(t_j) \leq \frac{a_{2k} - a_1}{h_n/2} \cdot 4h_n^2 = O(h_n) \to 0.$$

Here we also used the fact that set T_0 contains at most $\frac{a_{2k}-a_1}{h_n/2}$ points by Lemma 11.

It is clear that convergence, for all $\varepsilon > 0$, of probabilities in (102) implies convergence (101) in probability. □

Now introduce simplified notations:

$$\eta_n(t) := \sqrt{nh_n}\psi(t)(f_n(t) - \mathbf{E}f_n(t)), \quad \xi_n(t) := w(t)\xi(t/h_n). \tag{103}$$

and rewrite representation (75) in the following way:

$$\eta_n(t) - \xi_n(t) = \sqrt{nh_n}\psi(t)\delta_n(t) + \psi(t)\tilde{\delta}_n(t). \tag{104}$$

From definitions (4) and (76) we now have:

$$\left|\tilde{\Delta}_n\right| = |A_n(\beta_n(D_n) - A_n) - \tilde{\alpha}_n(D_n)| = \left|A_n\beta_n(D_n) - A_n\tilde{\beta}_n(D_n)\right|$$
$$= \left|\sup_{t \in D_n} \eta_n(t) - \sup_{t \in D_n} \xi_n(t)\right| \leq \sup_{t \in D_n} |\eta_n(t) - \xi_n(t)| \leq \Delta_{n,1} + \Delta_{n,5} \xrightarrow{\mathbf{P}} 0,$$

where $\Delta_{n,1}$ were defined in (72). Thus, we get the desired convergence from Proposition 28.

References

[1] E. Gine, V. Koltchinskii, L. Sakhanenko, *Kernel density estimators: convergence in distribution for weighted sup-norms*, Probability Theory and Related Fields, **130**:2 (2004), 167–198. MR2093761

[2] E. Gine, V. Koltchinskii, J. Zinn, *Weighted uniform consistency of kernel density estimators*, Annals of Probability, **32**:3B (2004), 2570–2605. MR2078551

[3] L. Sakhanenko, *Asymptotics of suprema of weighed Gaussian fields with applications to kernel density estimators*, Theory of Probability and its Applications, **59**:3 (2015), 499–541. MR3415976

[4] A. S. Kartashov, A. I. Sakhanenko, *About conditions of gaussian approximation of kernel estimates for distribution density*, Siberian Electronic Mathematical Reports, **12** (2015), 766–776. MR3493803

[5] W. Stute, *The oscillation behavior of empirical processes*, The Annals of Probability, **10**:1 (1982), 86–107. MR0637378

[6] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics, **508**, Berlin-New York: Springer, 1976. MR0453936

[7] W. Hoeffding, *Probability Inequalities for Sums of Bounded Random Variables*, Journal of the American Statistical Association, **58** (301) (1963), 13–30. MR0144363

[8] I. F. Pinelis, A. I. Sakhanenko, *Remarks on Inequalities for Large Deviation Probabilities*, Theory of Probability and its Applications, **30**:1 (1986), 127–131. MR0779438

Andrei Sergeevich Kartashov
Novosibirsk State University,
2, Lyapunov st.,
Novosibirsk, 630090, Russia
E-mail address: andrewkartashov@mail.ru

Alexander Ivanovich Sakhanenko
Sobolev Institute of Mathematics,
4, pr. Koptyuga,
Novosibirsk, 630090, Russia
E-mail address: aisakh@mail.ru