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FUNK–MINKOWSKI TRANSFORM AND SPHERICAL CONVOLUTION OF HILBERT TYPE IN RECONSTRUCTING FUNCTIONS ON THE SPHERE

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Abstract. The Funk–Minkowski transform $\mathcal{F}$ associates a function f on the sphere $\mathbb{S}^2$ with its mean values (integrals) along all great circles of the sphere. The presented analytical inversion formula reconstruct the unknown function f completely if two Funk–Minkowski transforms, $\mathcal{F}f$ and $\mathcal{F}\nabla f$, are known. Another result of this article is related to the problem of Helmholtz–Hodge decomposition for tangent vector field on the sphere $\mathbb{S}^2$. We proposed solution for this problem which is used the Funk–Minkowski transform $\mathcal{F}$ and Hilbert type spherical convolution $\mathcal{S}$.

Keywords: Funk–Minkowski transform, Funk—Radon transform, spherical convolution of Hilbert type, Fourier multiplier operator, inverse operator, surface gradient, scalar and vector spherical harmonics, tangential spherical vector field, Helmholtz–Hodge decomposition.

1. Introduction

The paper is devoted to the analytical inverse of the Minkowski—Funk transform (F–M transform). This transform was introduced by P. Funk [9, 10, 11], based on the work [25] of H. Minkowski. In literature Funk–Minkowski transform is known also as the Funk transform, Funk—Radon transform or spherical Radon transform. F–M transform associates a function on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ with its mean values (integrals) along all great circles of the sphere. Funk–Minkowski transform is a geodesic transform because the great circles on the sphere are geodesics. In recent time many authors investigate the generalized Funk–Minkowski

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transforms (or nongeodesic Funk–Minkowski transforms) on the sphere $\mathbb{S}^2$, which include nongeodesic paths of integration, such as circles with fixed diameter [28, 37, 46], circles perpendicular to the equator [15, 19, 28, 55] and circles, which obtained by intersections of the sphere with planes passing through a fixed common point $\mathbf{a} \in \mathbb{R}^3$, for example, through the northpole $\mathbf{k} \in \mathbb{S}^2$ [2, 6, 18, 30, 33, 43].

Funk–Minkowski transform plays an important role in the study of other integral transforms on the sphere and has various applications, for example, it is used in the convex geometry, harmonic analysis, image processing and in photoacoustic tomography, see [7, 20, 22, 23, 25, 32, 50, 54, 55].

Let $\mathbb{B}^3$ and $\mathbb{S}^2$ be the unit ball and the unit sphere in $\mathbb{R}^3$, respectively, i.e. $\mathbb{B}^3 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < 1\}$ and $\mathbb{S}^2 = \partial\mathbb{B}^3 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : |\boldsymbol{\xi}| = 1\}$, where $|\cdot|$ denotes the Euclidean norm. Throughout the paper we adopt the convention to denote in bold type the vectors in $\mathbb{R}^3$, and in simple type the scalars in $\mathbb{R}$. By the greek letters $\boldsymbol{\theta}$, $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ and so on we denote the units vectors $\mathbb{S}^2$. We will use for unit vector $\boldsymbol{\xi}$ on the sphere $\mathbb{S}^2$ usual angular coordinates (θ, φ)

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\theta,\varphi) = \mathbf{i}\sin\theta\cos\varphi + \mathbf{j}\sin\theta\sin\varphi + \mathbf{k}\cos\theta = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),$$

where $0 < \theta < \pi$ (the colatitude), $0 < \varphi < 2\pi$ (the longitude) and $t = \cos\theta$ — polar distance.

The plane $\boldsymbol{\xi}^\perp = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \centerdot \boldsymbol{\xi} = 0\}$ is spanned by the two orthonormal vectors $\mathbf{e}_1$, $\mathbf{e}_2$ with representations in polar coordinates

$$\mathbf{e}_1(\boldsymbol{\xi}) = \frac{\partial\boldsymbol{\xi}}{\partial\theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta),\ \mathbf{e}_2(\boldsymbol{\xi}) = \frac{1}{\sin\theta}\frac{\partial\boldsymbol{\xi}}{\partial\varphi} = (-\sin\varphi, \cos\varphi, 0).$$

The vectors $\mathbf{e}_1(\boldsymbol{\xi})$, $\mathbf{e}_2(\boldsymbol{\xi})$ and $\boldsymbol{\xi}$ form the so called local moving triad $\boldsymbol{\xi} \centerdot \mathbf{e}_1 = 0$, $\boldsymbol{\xi} \centerdot \mathbf{e}_2 = 0$, $\mathbf{e}_1 \centerdot \mathbf{e}_2 = 0$, where $\centerdot$ denotes the inner product of two vectors in $\mathbb{R}^3$.

Let denote by f_{even} and f_{odd} the even and odd parts of function f on $\mathbb{S}^2$, respectively, that is, we have

$$f(\boldsymbol{\xi}) = f_{even}(\boldsymbol{\xi}) + f_{odd}(\boldsymbol{\xi}),\ f_{even}(\boldsymbol{\xi}) = \frac{f(\boldsymbol{\xi}) + f(-\boldsymbol{\xi})}{2},\quad f_{odd}(\boldsymbol{\xi}) = \frac{f(\boldsymbol{\xi}) - f(-\boldsymbol{\xi})}{2}.$$

The space of continuous functions on the sphere $\mathbb{S}^2$ is denoted by $C(\mathbb{S}^2)$ and is endowed with the supremum norms

$$||f||_{C(\mathbb{S}^2)} = \sup_{\boldsymbol{\xi}\in\mathbb{S}^2}|f(\boldsymbol{\xi})|.$$

$C(\mathbb{S}^2)$, $C_{even}(\mathbb{S}^2)$ and $C_{odd}(\mathbb{S}^2)$ denote the space of continuous functions on $\mathbb{S}^2$, the space of even continuous functions on $\mathbb{S}^2$ and the space of odd continuous functions on $\mathbb{S}^2$, respectively. The subset of $C_{even}(\mathbb{S}^2)$ ($C_{odd}(\mathbb{S}^2)$) that contains the infinitely differentiable functions will be denoted by $C^\infty_{even}(\mathbb{S}^2)$ ($C^\infty_{odd}(\mathbb{S}^2)$).

Definition 1. *Let f be a continuous function on the sphere $\mathbb{S}^2$, $f \in C(\mathbb{S}^2)$. Then, for a unit vector $\boldsymbol{\xi} \in \mathbb{S}^2$ the Funk–Minkowski transform of a function f is a function $\mathcal{F}f$ on $\mathbb{S}^2$, given by*

$$\{\mathcal{F}f\}(\boldsymbol{\xi}) \equiv \mathcal{F}_{\boldsymbol{\xi}}f = \frac{1}{2\pi}\int_0^{2\pi} f\Big(\mathbf{e}_1(\boldsymbol{\xi})\cos\omega + \mathbf{e}_2(\boldsymbol{\xi})\sin\omega\Big)\,\mathrm{d}\omega. \tag{1}$$

It is clear that the Funk–Minkowski transform is even, $\{\mathcal{F}f\}(-\boldsymbol{\xi}) = \{\mathcal{F}f\}(\boldsymbol{\xi})$, and $\mathcal{F}$ annihilates all odd functions.

The inversion of the Funk–Minkowski transform has been treated by many authors and there are exist several inversion formulas in the literature, see [9, 17, 38, 39, 47]. In [9, 11] P. Funk proved that an even function can be recovered from the knowledge of integrals over great circles and presented two different inversion methods: the first method is based on the spherical harmonic decomposition of the functions f, $\mathcal{F}f$ and the second one utilizes Abel's integral equation, [28].

The inversion formula after P. Funk was obtained by V. Semyanisty in [47, formulas (9) and (11)],

$$f_{even}(\boldsymbol{\theta}) = -\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{1}{(\boldsymbol{\theta}\cdot\boldsymbol{\eta})^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}, \tag{2}$$

where the $\mathrm{d}\boldsymbol{\eta}$ is the surface measure on $\mathbb{S}^2$ with normalization $\int_{\mathbb{S}^2}\mathrm{d}\boldsymbol{\eta} = 4\pi$ and integral is understood in the regularized sense.

In [17, p. 99] S. Helgason gives for (1) the inversion formula of filtered back-projection type

$$f_{even}(\boldsymbol{\theta}) = \frac{1}{2\pi}\frac{d}{du}\int_0^u\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\delta\Big(\boldsymbol{\eta}\cdot\boldsymbol{\theta} - \sqrt{1-v^2}\Big)\mathrm{d}\boldsymbol{\eta}\frac{vdv}{\sqrt{u^2-v^2}}\Big|_{u=1}, \tag{3}$$

where δ denotes the the Dirac delta function.

Another example of inversion formula is due to B. Rubin [38, 39]

$$f_{even}(\boldsymbol{\theta}) = \frac{1}{4\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} + \frac{\Delta_{\boldsymbol{\theta}}}{4\pi}\int_{\mathbb{S}^2}\ln|\boldsymbol{\eta}\cdot\boldsymbol{\theta}|\,\{\mathcal{F}f\}(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\,, \tag{4}$$

here $\Delta_{\boldsymbol{\theta}}$ it the Laplace–Beltrami operator (31).

In our studies, an important role is played by spherical convolution operator $\mathcal{S}$, which is the spherical analogue of Hilbert transform, see [21, 41, 44, 45].

Definition 2. *Let $f \in C(\mathbb{S}^2)$. The spherical convolution operator $\mathcal{S}$ is defined by,*

$$\{\mathcal{S}v\}(\boldsymbol{\theta}) \equiv \mathcal{S}_{\boldsymbol{\theta}}v = p.v.\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{v(\boldsymbol{\eta})}{\boldsymbol{\theta}\cdot\boldsymbol{\eta}}\mathrm{d}\boldsymbol{\eta},\ \ \boldsymbol{\theta}\in\mathbb{S}^2. \tag{5}$$

This transform is odd, $\{\mathcal{S}f\}(-\boldsymbol{\theta}) = -\{\mathcal{S}f\}(\boldsymbol{\theta})$, and $\mathcal{S}$ annihilates all even functions.

The results of this paper are formulated below in Theorems 1 and 2.

Theorem 1. *For any function $f(\boldsymbol{\theta}) \in H^1(\mathbb{S}^2)$ the following identity take place*

$$\begin{aligned} f(\boldsymbol{\theta}) &= \underbrace{\frac{1}{4\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}}_{=f_{00}} + p.v.\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{(\boldsymbol{\eta}+\boldsymbol{\theta})\cdot\Big\{\Big[\mathcal{F},\nabla\Big]f\Big\}(\boldsymbol{\eta})}{\boldsymbol{\eta}\cdot\boldsymbol{\theta}}\mathrm{d}\boldsymbol{\eta} \\ &= f_{00} + \mathcal{S}_{\boldsymbol{\theta}}(\boldsymbol{\eta}+\boldsymbol{\theta})\cdot\Big[\mathcal{F},\nabla\Big]_{\boldsymbol{\eta}}f. \end{aligned} \tag{6}$$

Here operators $\mathcal{F}$ and ∇ are the Funk–Minkowski transform (1) and the surface gradient (21), respectively. Through the square brackets $[.,.]$ we, as usual, denoted the commutator $\Big[\mathcal{F},\nabla\Big]f = \mathcal{F}\nabla f - \nabla\mathcal{F}f$, where the F–M transform $\mathcal{F}$ is applied to vector function ∇f by componentwise.

If we decompose identity (6) on even and odd parts then we can write,

$$f_{even}(\boldsymbol{\theta}) = \frac{1}{4\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} - p.v.\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{\boldsymbol{\theta}\,\boldsymbol{.}\,\{\nabla\mathcal{F}f\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{.}\,\boldsymbol{\eta}}\mathrm{d}\boldsymbol{\eta}$$

$$= \frac{1}{4\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} - \boldsymbol{\theta}\,\boldsymbol{.}\mathcal{S}_{\boldsymbol{\theta}}\nabla\mathcal{F}f\,, \tag{7}$$

$$f_{odd}(\boldsymbol{\theta}) = p.v.\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{\boldsymbol{\eta}\,\boldsymbol{.}\,\{\mathcal{F}\nabla f\}(\boldsymbol{\eta})}{\boldsymbol{\eta}\,\boldsymbol{.}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} = \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{F}_{\boldsymbol{\eta}}\nabla f. \tag{8}$$

The inversion formulas for f_{even} and f_{odd} follow from these equations and if two F–M transformations $g(\boldsymbol{\eta}) = \{\mathcal{F}f\}(\boldsymbol{\eta})$ and $\mathbf{h}(\boldsymbol{\eta}) = \{\mathcal{F}\nabla f\}(\boldsymbol{\eta})$ are known, then the unknown function f can be reconstruct completely,

$$f(\boldsymbol{\theta}) = \frac{1}{4\pi}\int_{\mathbb{S}^2}g(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta} - p.v.\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{\boldsymbol{\theta}\,\boldsymbol{.}\,\nabla g(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{.}\,\boldsymbol{\eta}}\,\mathrm{d}\boldsymbol{\eta} + p.v.\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{\boldsymbol{\eta}\,\boldsymbol{.}\,\mathbf{h}(\boldsymbol{\eta})}{\boldsymbol{\eta}\,\boldsymbol{.}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta}\,. \tag{9}$$

The next problem that we will consider is the problem of Helmholtz–Hodge decomposition for a tangential vector field on the sphere $\mathbb{S}^2$, see [12]. The Helmholtz–Hodge decomposition says that we can write any vector field tangent to the surface of the sphere as the sum of a curl-free component and a divergence-free component

$$\mathbf{f}(\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}}u(\boldsymbol{\theta}) + \boldsymbol{\theta}\times\nabla_{\boldsymbol{\theta}}v(\boldsymbol{\theta}), \tag{10}$$

where $\nabla_{\boldsymbol{\theta}}$ is the surface gradient on the sphere, and rotated gradient $\boldsymbol{\theta}\times\nabla_{\boldsymbol{\theta}}$ means the cross-product of the surface gradient of v with the unit normal vector $\boldsymbol{\theta}$ to the sphere. Here $\nabla_{\boldsymbol{\theta}}u$ is called also as inrrotational, poloidal, electric or potential field and $\nabla^{\perp}_{\boldsymbol{\theta}}v$ is called as incompressible, toroidal, magnetic or stream vector field. Scalar functions u and v are called velocity potential and stream functions, respectively.

In the next theorem we show that decomposition (10) is obtained by use of Funk–Minkowski- transform $\mathcal{F}$ and spherical convolution transform $\mathcal{S}$.

Theorem 2. *Any vector field $\mathbf{f}\in\mathbf{L}_{2,tan}(\mathbb{S}^2)$ that is tangent to the sphere can be uniquely decomposed into a sum (10) of a surface curl-free component and a surface divergence-free component with scalar valued functions $u, v\in H^1(\mathbb{S}^2)/\mathbb{R}$. Functions u and v are velocity potential and stream functions that are calculated unique up to a constant by the formulas*

$$u(\boldsymbol{\theta}) = \Big[\mathcal{S},\boldsymbol{\eta}\,\boldsymbol{.}\,,\mathcal{F}\Big]_{\boldsymbol{\theta}}\mathbf{f} = \Big\{\mathcal{S}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{F}\mathbf{f}\Big\}(\boldsymbol{\theta}) - \Big\{\mathcal{F}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{S}\mathbf{f}\Big\}(\boldsymbol{\theta})$$

$$= \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{F}_{\boldsymbol{\eta}}\mathbf{f} - \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{S}_{\boldsymbol{\eta}}\mathbf{f}, \tag{11}$$

$$v(\boldsymbol{\theta}) = \boldsymbol{\theta}\,\boldsymbol{.}\,\Big[\mathcal{S},\boldsymbol{\eta}\times,\mathcal{F}\Big]_{\boldsymbol{\theta}}\mathbf{f} = \boldsymbol{\theta}\,\boldsymbol{.}\,\Big\{\mathcal{S}\boldsymbol{\eta}\times\mathcal{F}\mathbf{f}\Big\}(\boldsymbol{\theta}) - \boldsymbol{\theta}\,\boldsymbol{.}\,\Big\{\mathcal{F}\boldsymbol{\eta}\times\mathcal{S}\mathbf{f}\Big\}(\boldsymbol{\theta})$$

$$= \boldsymbol{\theta}\,\boldsymbol{.}\,\mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\times\mathcal{F}_{\boldsymbol{\eta}}\mathbf{f} - \boldsymbol{\theta}\,\boldsymbol{.}\,\mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta}\times\mathcal{S}_{\boldsymbol{\eta}}\mathbf{f}, \tag{12}$$

where through $[\mathcal{A},\mathcal{B},\mathcal{C}]$ we denote the generalized commutator,

$$[\mathcal{A},\mathcal{B},\mathcal{C}] = \mathcal{A}\mathcal{B}\mathcal{C} - \mathcal{C}\mathcal{B}\mathcal{A}.$$

As a consequence of this theorem, we can obtain formulas for solving two important problems on the sphere $\mathbb{S}^2$: $\nabla u = \mathbf{f}$ and $\nabla^{\perp}v = \mathbf{g}$. Answers to solve these problems are

$$u(\boldsymbol{\theta}) = (\mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{F}_{\boldsymbol{\eta}} - \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{.}\,\mathcal{S}_{\boldsymbol{\eta}})\mathbf{f} \quad \text{for} \quad \nabla u = \mathbf{f}\in\mathbf{L}_{2,tan}(\mathbb{S}^2)$$

and

$$v(\boldsymbol{\theta}) = \boldsymbol{\theta}\,\boldsymbol{.}\,(\mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\times\mathcal{F}_{\boldsymbol{\eta}} - \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta}\times\mathcal{S}_{\boldsymbol{\eta}})\mathbf{g} \quad \text{for} \quad \nabla^{\perp}v = \mathbf{g}\in\mathbf{L}_{2,tan}(\mathbb{S}^2).$$

2. Basic methods and tools

2.1. **Spherical harmonics (SHs).** In this section we state some properties of complex spherical harmonics. A spherical harmonic $Y_{N\ell}$ of degree N on $\mathbb{S}^2$ is the restriction to $\mathbb{S}^2$ of a homogeneous harmonic polynomial of degree N in $\mathbb{R}^3$.

The Legendre polynomials of the first kind P_N of degree $N \in \mathbb{N}_0$ or simply Legendre polynomials are given by the Rodrigues formula

$$P_N(t) = \frac{1}{N!2^N}\frac{\mathrm{d}^N}{\mathrm{d}t^N}(t^2-1)^N.$$

We recall that Legendre polynomials of the first kind $P_N(t)$ are the orthogonal polynomials on $(-1,1)$ with weight function $w(t)=1$. We define with $C_N^{(3/2)}$ the Gegenbauer polynomial of degree N with parameter $\lambda = 3/2$,

$$C_N^{(3/2)}(t) = \frac{d}{dt}P_{N+1}(t).$$

The following formulas will be used in our calculations ([1])

$$P_{2j}(0) = (-1)^j\frac{\Gamma(j+1/2)}{\sqrt{\pi}j!} = \frac{(-1)^j(2j-1)!!}{(2j)!!}, \tag{13}$$

$$(N+1)P_{N+1}(0) = -NP_{N-1}(0), \tag{14}$$

$$C_{2j}^{(3/2)}(0) = \frac{(-1)^j(2j+1)!!}{(2j)!!} \text{ or } C_{N-1}^{(3/2)}(0) = NP_{N-1}(0),\ N = 2j+1. \tag{15}$$

The following usefull asymptotics holds as j goes to infinity

$$P_{2j}(0) \sim \frac{1}{\sqrt{2j+1}} \text{ and } \frac{1}{C_{2j}^{(3/2)}(0)} = \frac{1}{(2j+1)P_{2j}(0)} \sim \frac{1}{\sqrt{2j+1}} \text{ if } \ j\to\infty. \tag{16}$$

The associated Legendre functions of the first kind P_N^ℓ for non negative $\ell \geq 0$ are defined as

$$P_N^\ell(t) = (1-t^2)^{\frac{\ell}{2}}\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}P_N(t),$$

where $N,\ell \in \mathbb{N}_0$ with $\ell \leq N$ and for the negative order $-\ell$, $P_N^{-\ell}$ are given by

$$P_N^{-\ell}(t) = (-1)^\ell\frac{(N-\ell)!}{(N+\ell)!}P_N^\ell(t),\ \ell \geq 0\,.$$

When the order $\ell = 0$, the associated Legendre function becomes a polynomial in t and instead being written $P_N^0(t)$ it is designated $P_N(t)$, the Legendre polynomial. The complex SHs $Y_{N\ell}$ are related to the associated Legendre functions as follows

$$Y_{N\ell}(\boldsymbol{\xi}) = (-1)^\ell N_{N\ell}e^{\mathrm{i}\ell\varphi}P_N^\ell(\cos\theta),\ |\ell| \leq N,$$

where $N_{N\ell}$ is a normalization constant

$$N_{N\ell} = \sqrt{\frac{2N+1}{4\pi}\frac{(N-\ell)!}{(N+\ell)!}}$$

and the extra factor $(-1)^\ell$ is called the Condon–Shortley phase.

The $Y_{N\ell}$ are complex-valued polynomials of the sines and cosines of θ and φ and for complex conjugate functions the following formula fulfil

$$\overline{Y_{N\ell}(\boldsymbol{\xi})} = (-1)^\ell Y_{N,-\ell}(\boldsymbol{\xi}).$$

The parity rule for spherical harmonic is

$$Y_{N\ell}(-\boldsymbol{\xi}) = (-1)^N Y_{N\ell}(\boldsymbol{\xi}).$$

It is known that the subspace of all spherical harmonics of degree N, $\operatorname{span}\{Y_{N\ell}\}_\ell^N$, is the eigenspace of the Laplace–Beltrami operator (31) corresponding to the eigenvalue $-\lambda_N^2 = -N(N+1)$,

$$\Delta_{\boldsymbol{\xi}} Y_{N\ell}(\boldsymbol{\xi}) = -N(N+1)Y_{N\ell}(\boldsymbol{\xi}).$$

The dimension of this subspace being $2N+1$, so one may choose for it an orthonormal basis in different ways.

The collection of all spherical harmonics $\Big\{Y_{N\ell}, |\ell| \le N\Big\}_{N=0}^{\infty}$ forms an orthonormal basis for $L_2(\mathbb{S}^2;\mathbb{C})$

$$(Y_{N_1\ell_1}, Y_{N_2\ell_2})_{L_2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} Y_{N_1\ell_1}(\boldsymbol{\xi})\overline{Y_{N_2\ell_2}(\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\xi} = \delta_{N_2}^{N_1}\delta_{\ell_2}^{\ell_1}, \tag{17}$$

where δ_j^i is the Kronecker symbol and the space $L_2(\mathbb{S}^2) \equiv L_2(\mathbb{S}^2;\mathbb{C})$ is a Hilbert space of square-integrable functions on $\mathbb{S}^2$ with the hermitian inner product and the finite norm,

$$(u,v)_{L_2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} u(\boldsymbol{\xi})\overline{v(\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\xi},\ \ ||u||^2_{L_2(\mathbb{S}^2)} = (u,u)_{L_2(\mathbb{S}^2)}.$$

The Fourier coefficients for $u \in L_2(\mathbb{S}^2)$ are $u_{N\ell} = (u, Y_{N\ell})_{L_2}$. Then, every function $u \in L_2(\mathbb{S}^2)$ admits a spherical harmonics series expansion in L_2–sense

$$u(\boldsymbol{\xi}) = \sum_{N=0}^{\infty}\sum_{\ell} u_{N\ell} Y_{N\ell}(\boldsymbol{\xi}), \tag{18}$$

$$||u||^2_{L_2(\mathbb{S}^2)} = \sum_{N=0}^{\infty}\sum_{\ell} |u_{N\ell}|^2 . \tag{19}$$

We close this section with Funk–Hecke formula. It was first published by Funk (1916) and a little later by Hecke (1918).

Theorem 3. *[The Funk–Hecke Theorem] Suppose* $f(t) \in L_1(-1,1)$ *is an integrable function. Then for every spherical harmonics of degree* N *we have*

$$\int_{\mathbb{S}^2} f(\boldsymbol{\xi}\cdot\boldsymbol{\eta})Y_{N\ell}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = 2\pi Y_{N\ell}(\boldsymbol{\eta})\int_{-1}^{1} f(t)P_N(t)\,\mathrm{d}t, \tag{20}$$

where $\boldsymbol{\xi}\cdot\boldsymbol{\eta}$ *denotes the inner product of unit vectors* $\boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$*,* P_N *denotes the* N*th order Legendre polynomial.*

The Funk–Hecke formula is useful in simplifying calculations of certain integrals over $\mathbb{S}^2$ and plays an important role in the theory of spherical harmonics. For more details on the Funk–Hecke formula see [3, 45], for example. A general overview on spherical harmonics and the relevant problems can be found in the monographs [1, 5, 12, 13, 27, 52].

2.2. **Surface differential operators on the sphere $\mathbb{S}^2$.** Here we briefly recall the definitions and some properties of surface differential operators.

The space $\mathbf{L}_2(\mathbb{S}^2) \equiv \mathbf{L}_2(\mathbb{S}^2;\mathbb{C})$ is a Hilbert space of square-integrable vector functions on $\mathbb{S}^2$ with the inner product and the finite norm,

$$(\mathbf{u},\mathbf{v})_{\mathbf{L}_2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} \mathbf{u}(\boldsymbol{\xi}) \cdot \overline{\mathbf{v}(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi},\ \ ||\mathbf{u}||^2_{\mathbf{L}_2(\mathbb{S}^2)} = (\mathbf{u},\mathbf{u})_{\mathbf{L}_2(\mathbb{S}^2)}.$$

Definition 3. *The tangential gradient or the surface gradient, denoted by $\nabla \equiv \nabla_{\boldsymbol{\xi}}$ and the tangential rotated gradient (the surface curl-gradient), denoted by $\nabla^{\perp} \equiv \nabla^{\perp}_{\boldsymbol{\xi}}$, are defined accordingly as*

$$\nabla_{\boldsymbol{\xi}} u = \frac{\partial u}{\partial \theta}\mathbf{e}_1(\boldsymbol{\xi}) + \frac{1}{\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_2(\boldsymbol{\xi}), \tag{21}$$

$$\nabla^{\perp}_{\boldsymbol{\xi}} u = \boldsymbol{\xi} \times \nabla_{\boldsymbol{\xi}} u = -\frac{1}{\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_1(\boldsymbol{\xi}) + \frac{\partial u}{\partial \theta}\mathbf{e}_2(\boldsymbol{\xi}), \tag{22}$$

where $\boldsymbol{\xi} = \mathbf{i}\sin\theta\cos\varphi + \mathbf{j}\sin\theta\sin\varphi + \mathbf{k}\cos\theta$.

Obviously, we have $\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = 0$, $\boldsymbol{\xi} \cdot \nabla^{\perp}_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = 0$ and $\nabla u \cdot \nabla^{\perp} u = 0$, thus ∇u and $\nabla^{\perp} u$ are will be tangential vector fields on the sphere $\mathbb{S}^2$ with $\nabla^{\perp}$ is rotation by $\pi/2$ in the tangent plane.

We must note here that integration by parts formulas on the sphere for operators (21) and (22) are differ. Namely, for $u, v \in C^1(\mathbb{S}^2)$, we have

$$\int_{\mathbb{S}^2} u(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}} v(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} = -\int_{\mathbb{S}^2} v(\boldsymbol{\xi})\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} + 2\int_{\mathbb{S}^2} \boldsymbol{\xi} u(\boldsymbol{\xi}) v(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \tag{23}$$

$$\int_{\mathbb{S}^2} u(\boldsymbol{\xi})\nabla^{\perp}_{\boldsymbol{\xi}} v(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} = -\int_{\mathbb{S}^2} v(\boldsymbol{\xi})\nabla^{\perp}_{\boldsymbol{\xi}} u(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}. \tag{24}$$

Definition 4. *In canonical coordinates, the surface divergence* $\mathrm{div}_{\boldsymbol{\xi}}$ *of vector-valued function $\mathbf{v}(\boldsymbol{\xi}) = v^1\mathbf{e}_1(\boldsymbol{\xi}) + v^2\mathbf{e}_2(\boldsymbol{\xi}) + v^3\boldsymbol{\xi}$ on the sphere $\mathbb{S}^2$ is written as,*

$$\mathrm{div}_{\boldsymbol{\xi}}\mathbf{v} = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(v^1\sin\theta) + \frac{\partial}{\partial\varphi}v^2\right) + 2v^3\,. \tag{25}$$

For tangent vector field $\mathbf{v}$ we define the scalar surface rotation (or scalar curl operator) $\mathrm{curl}_{\boldsymbol{\xi}}$ *by*

$$\mathrm{curl}_{\boldsymbol{\xi}}\mathbf{v} = -\mathrm{div}_{\boldsymbol{\xi}}(\boldsymbol{\xi}\times\mathbf{v}) = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(v^2\sin\theta) - \frac{\partial}{\partial\varphi}v^1\right). \tag{26}$$

If $u \in C^1(\mathbb{S}^2)$ and tangential vector field $\mathbf{v} \in \mathbf{C}^1(\mathbb{S}^2)$, then we have integral formulas, which are also understood as inner products

$$\int_{\mathbb{S}^2} \mathbf{v}(\boldsymbol{\xi}) \cdot \nabla_{\boldsymbol{\xi}}\overline{u(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} = -\int_{\mathbb{S}^2} \overline{u(\boldsymbol{\xi})}\mathrm{div}_{\boldsymbol{\xi}}\mathbf{v}(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} \tag{27}$$

$$\text{or}\quad (\mathbf{v},\nabla u)_{\mathbf{L}_2(\mathbb{S}^2)} = -(\mathrm{div}\,\mathbf{v}, u)_{L_2(\mathbb{S}^2)}, \tag{28}$$

$$\int_{\mathbb{S}^2} \mathbf{v}(\boldsymbol{\xi}) \cdot \nabla^{\perp}_{\boldsymbol{\xi}}\overline{u(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} = -\int_{\mathbb{S}^2} \overline{u(\boldsymbol{\xi})}\mathrm{curl}_{\boldsymbol{\xi}}\mathbf{v}(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} \tag{29}$$

$$\text{or}\quad (\mathbf{v},\nabla^{\perp} u)_{\mathbf{L}_2(\mathbb{S}^2)} = -(\mathrm{curl}\,\mathbf{v}, u)_{L_2(\mathbb{S}^2)}. \tag{30}$$

Definition 5. *Finally, we define the Beltrami operator, which is also called the Laplace–Beltrami operator $\Delta \equiv \Delta_{\boldsymbol{\xi}}$ as*

$$\Delta_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = \mathrm{div}_{\boldsymbol{\xi}}\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi})\,, \tag{31}$$

i.e. the divergence of a gradient is the Laplacian.

One easily checks that

$$\Delta_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = \operatorname{curl}_{\boldsymbol{\xi}} \nabla_{\boldsymbol{\xi}}^{\perp} u(\boldsymbol{\xi}) \tag{32}$$

and also

$$\operatorname{curl}_{\boldsymbol{\xi}} \nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = 0, \quad \operatorname{div}_{\boldsymbol{\xi}} \nabla_{\boldsymbol{\xi}}^{\perp} u(\boldsymbol{\xi}) = 0,$$

thus we say that $\nabla_{\boldsymbol{\xi}} u$ is the curl-free, but $\nabla_{\boldsymbol{\xi}}^{\perp} u$ is the divergence-free vector fields.

The next formula is Green–Beltrami identity or Green's first surface identity, see [3, Proposition 3.3], [24, Theorem 4.12]: for any $u \in C^1(\mathbb{S}^2)$ and any $v \in C^2(\mathbb{S}^2)$ we have,

$$\int_{\mathbb{S}^2} \nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) \cdot \nabla_{\boldsymbol{\xi}} \overline{v(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} = -\int_{\mathbb{S}^2} u(\boldsymbol{\xi}) \Delta_{\boldsymbol{\xi}} \overline{v(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} \tag{33}$$

$$\text{or} \quad (\nabla u, \nabla v)_{\mathbf{L}_2(\mathbb{S}^2)} = -(u, \Delta v)_{L_2(\mathbb{S}^2)}\,. \tag{34}$$

For example, if we take $u = Y_{N_1 \ell_1}$ and $v = Y_{N_2 \ell_2}$, then

$$\begin{aligned}
&(\nabla Y_{N_1\ell_1}, \nabla Y_{N_2\ell_2})_{\mathbf{L}_2(\mathbb{S}^2)} \\
&= \int_{\mathbb{S}^2} \nabla_{\boldsymbol{\xi}} Y_{N_1\ell_1}(\boldsymbol{\xi}) \cdot \nabla_{\boldsymbol{\xi}} \overline{Y_{N_2\ell_2}(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} = -\int_{\mathbb{S}^2} Y_{N_1\ell_1}(\boldsymbol{\xi}) \Delta_{\boldsymbol{\xi}} \overline{Y_{N_2\ell_2}(\boldsymbol{\xi})} \mathrm{d}\boldsymbol{\xi} \\
&= N_2(N_2+1) \int_{\mathbb{S}^2} Y_{N_1\ell_1}(\boldsymbol{\xi}) \overline{Y_{N_2\ell_2}(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} = N_2(N_2+1) \delta_{N_1}^{N_2} \delta_{\ell_1}^{\ell_2}\,.
\end{aligned} \tag{35}$$

For more definitions and properties of these differential operators see e.g. [3, 12, 13, 29, 52].

2.3. **Two systems of vector spherical harmonics (VSHs).** There are vectorial analogues of scalar spherical harmonics called vector spherical harmonics. VSHs can be defined in several ways. In this section we give definitions and properties of the vector spherical harmonics, which are needed in our work. We refer to [8, 12, 26, 29, 52] for more details in this theme.

2.3.1. *Pure–spin vector spherical harmonics.* Let us now define a complete orthogonal set of vectors in $\mathbf{L}_2(\mathbb{S}^2)$.

Definition 6. *The vector spherical harmonics (or pure–spin VSHs) are arranged in three families:* $\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi})$, $\mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi})$ *and* $\mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\xi})$. *For* $\boldsymbol{\xi} \in \mathbb{S}^2$ *and given a scalar spherical harmonic* $Y_{N\ell}(\boldsymbol{\xi})$ *the unnormalized vector spherical harmonics are the set*

$$\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) = \boldsymbol{\xi} Y_{N\ell}(\boldsymbol{\xi}), \quad N \in 0 \cup \mathbb{N}, \tag{36}$$

$$\mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} Y_{N\ell}(\boldsymbol{\xi}), \quad N \in \mathbb{N}, \tag{37}$$

$$\mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\xi}) = \boldsymbol{\xi} \times \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}}^{\perp} Y_{N\ell}(\boldsymbol{\xi}), \quad N \in \mathbb{N}. \tag{38}$$

The pure–spin VSHs form a complete set of orthogonal vector functions on the surface of a sphere $\mathbb{S}^2$ with the inner product of the $\mathbf{L}_2(\mathbb{S}^2)$ space, see [13, Theorem 5.2.7].

Clearly, $||\mathbf{y}_{N\ell}^{(1)}||_{\mathbf{L}_2(\mathcal{S}^2)} = 1$. To calculate the norms of vector functions $\mathbf{y}_{N\ell}^{(2)}$ and $\mathbf{y}_{N\ell}^{(3)}$, we can use (35). Therefore, the normalizing vector harmonics or orthonormal system of VSHs are

$$\mathbf{y}_{N\ell}^{(1)},\ \widetilde{\mathbf{y}}_{N\ell}^{(2)} = \mathbf{y}_{N\ell}^{(2)} / \sqrt{N(N+1)}\,,\ \widetilde{\mathbf{y}}_{N\ell}^{(3)} = \mathbf{y}_{N\ell}^{(3)} / \sqrt{N(N+1)}\,.$$

Each vector function $\mathbf{f} \in \mathbf{L}_2(\mathbb{S}^2)$ has the Fourier expansion

$$\mathbf{f}(\boldsymbol{\xi}) = f_{1,00}\mathbf{y}^{(1)}_{00}(\boldsymbol{\xi}) + \sum_{N=1}^{\infty}\sum_{\ell} f_{1,N\ell}\mathbf{y}^{(1)}_{N\ell}(\boldsymbol{\xi}) + f_{2,N\ell}\widetilde{\mathbf{y}}^{(2)}_{N\ell}(\boldsymbol{\xi}) + f_{3,N\ell}\widetilde{\mathbf{y}}^{(3)}_{N\ell}(\boldsymbol{\xi}),$$

$$||\mathbf{f}||^2_{\mathbf{L}_2(\mathbb{S}^2)} = |f_{1,00}|^2 + \sum_{N=1}^{\infty}\sum_{\ell} |f_{1,N\ell}|^2 + |f_{2,N\ell}|^2 + |f_{3,N\ell}|^2.$$

The hermitian inner products are then given by

$$(\mathbf{f}, \mathbf{h})_{\mathbf{L}_2(\mathbb{S}^2)} = f_{1,00}\overline{h}_{1,00} + \sum_{N=1}^{\infty}\sum_{\ell} f_{1,N\ell}\overline{h}_{1,N\ell} + f_{2,N\ell}\overline{h}_{2,N\ell} + f_{3,N\ell}\overline{h}_{3,N\ell}.$$

2.3.2. *Pure–orbit vector spherical harmonics.* An alternative orthogonal basis in the space $\mathbf{L}_2(\mathbb{S}^2)$ is the system of pure–orbit VSHs $\{\mathbf{h}^{(e)}_{00}, \mathbf{h}^{(e)}_{N\ell}, \mathbf{h}^{(i)}_{N\ell}, \mathbf{y}^{(3)}_{N\ell},\ |\ell| \le N\}_{N=1}^{\infty}$, where vector functions $\mathbf{h}^{(e)}_{N\ell}$ and $\mathbf{h}^{(i)}_{N\ell}$ defined by

$$\mathbf{h}^{(e)}_{00} = -\mathbf{y}^{(1)}_{00}, \tag{39}$$

$$\mathbf{h}^{(e)}_{N\ell} = -(N+1)\mathbf{y}^{(1)}_{N\ell} + \mathbf{y}^{(2)}_{N\ell}, \quad N \in \mathbb{N}, \tag{40}$$

$$\mathbf{h}^{(i)}_{N\ell} = N\mathbf{y}^{(1)}_{N\ell} + \mathbf{y}^{(2)}_{N\ell},\ \ N \in \mathbb{N}. \tag{41}$$

The pure–orbit vector spherical harmonics also has a nice properties, in particular, they are eigenfunctions for the vectorial Funk–Minkowski operator $\mathcal{F}$ in the space $\mathbf{L}_{2,even}(\mathbb{S}^2)$ and for vectorial Hilbert operator $\mathcal{S}$ in the space $\mathbf{L}_{2,odd}(\mathbb{S}^2)$, see Lemmas 1 and 2 in the section Proofs.

2.3.3. *Tangent vector fields and Helmholtz–Hodge decomposition.* Consider the tangent vector field $\mathbf{f} \in \mathbf{L}_{2,tan}(\mathbb{S}^2)$, it can be written uniquely as

$$\begin{aligned}\mathbf{f}(\boldsymbol{\theta}) &= \underbrace{\sum_{N=1}^{\infty}\sum_{\ell} f_{2,N\ell}\widetilde{\mathbf{y}}^{(2)}_{N\ell}(\boldsymbol{\theta})}_{\text{the curl-free component}} + \underbrace{\sum_{N=1}^{\infty}\sum_{\ell} f_{3,N\ell}\widetilde{\mathbf{y}}^{(3)}_{N\ell}(\boldsymbol{\theta})}_{\text{the divergence-free component}} \\ &= \sum_{N=1}^{\infty}\frac{1}{\sqrt{N(N+1)}}\sum_{\ell} f_{2,N\ell}\nabla Y_{N\ell}(\boldsymbol{\theta}) + f_{3,N\ell}\boldsymbol{\theta}\times\nabla Y_{N\ell}(\boldsymbol{\theta}).\end{aligned}$$

Then formally we have

$$\mathbf{f}(\boldsymbol{\theta}) = \nabla\sum_{N=1}^{\infty}\frac{1}{\sqrt{N(N+1)}}\sum_{\ell} f_{2,N\ell}Y_{N\ell}(\boldsymbol{\theta}) + \nabla^{\perp}\sum_{N=1}^{\infty}\frac{1}{\sqrt{N(N+1)}}\sum_{\ell} f_{3,N\ell}Y_{N\ell}(\boldsymbol{\theta}),$$

where according to (10) the velocity potential and stream functions are

$$u(\boldsymbol{\theta}) = \sum_{N=1}^{\infty}\frac{1}{\sqrt{N(N+1)}}\sum_{\ell} f_{2,N\ell}Y_{N\ell}(\boldsymbol{\theta}),$$

$$v(\boldsymbol{\theta}) = \sum_{N=1}^{\infty}\frac{1}{\sqrt{N(N+1)}}\sum_{\ell} f_{3,N\ell}Y_{N\ell}(\boldsymbol{\theta}).$$

Another evident approach consists in solving the Laplace–Beltrami equations on the sphere

$$\Delta_{\boldsymbol{\theta}} u(\boldsymbol{\theta}) = \mathrm{div}_{\boldsymbol{\theta}}\, \mathbf{f}(\boldsymbol{\theta}),$$
$$\Delta_{\boldsymbol{\theta}} v(\boldsymbol{\theta}) = \mathrm{curl}_{\boldsymbol{\theta}}\, \mathbf{f}(\boldsymbol{\theta}).$$

They can be solved in integral form, for example, involving Green's function with respect to the Laplace–Beltrami $\Delta_{\boldsymbol{\theta}}$, see [13, Theorem 4.6.9].

2.4. **Hilbertian Sobolev spaces on the sphere.**

2.4.1. *Sobolev scalar functions on* $\mathbb{S}^2$. The Sobolev space $H^s(\mathbb{S}^2)$ with a smoothness index $s \geq 0$ is defined by ([3, 24, 29, 32, 41])

$$H^s(\mathbb{S}^2) := \{u \in L_2(\mathbb{S}^2;\mathbb{C}) : \sum_{N=0}^{\infty} (1+N(N+1))^s \sum_{\ell} |u_{N\ell}|^2 < \infty\}.$$

In other words $u \in H^s(\mathbb{S}^2)$ if and only if $(I-\triangle)^{s/2} u \in L_2(\mathbb{S}^2)$. The space $H^s(\mathbb{S}^2)$ is a Hilbert space with the hermitian inner product

$$(u,v)_{H^s(\mathbb{S}^2)} = \sum_{N=0}^{\infty} (1+N(N+1))^s \sum_{\ell} u_{N\ell} \overline{v_{N\ell}}$$

and the induced norm

$$||u||^2_{H^s(\mathbb{S}^2)} = \sum_{N=0}^{\infty} (1+N(N+1))^s \sum_{\ell} |u_{N\ell}|^2 = ||(I-\triangle)^{s/2} u||^2_{L_2(\mathbb{S}^2)}.$$

Putting $s=0$ we obtain $H^0(\mathbb{S}^2) = L_2(\mathbb{S}^2)$. If $s=1$ then in addition to (18), (19) we have

$$\nabla_{\boldsymbol{\xi}} u(\boldsymbol{\xi}) = \sum_{N=0}^{\infty} \sum_{\ell} u_{N\ell} \nabla_{\boldsymbol{\xi}} Y_{N\ell}(\boldsymbol{\xi}), = \sum_{N=1}^{\infty} \sqrt{N(N+1)} \sum_{\ell} (u, Y_{N\ell})_{L_2(\mathbb{S}^2)} \widetilde{\mathbf{y}}^{(2)}_{N\ell},$$

$$||\nabla u||^2_{\mathbf{L}_2(\mathbb{S}^2)} = \sum_{N=0}^{\infty} N(N+1) \sum_{\ell} |u_{N\ell}|^2.$$

Thus we can define the Sobolev space $H^1(\mathbb{S}^2)$ as (see [29, p. 14])

$$H^1(\mathbb{S}^2) = \{u \in L_2(\mathbb{S}^2) : \nabla u \in \mathbf{L}_2(\mathbb{S}^2)\}$$

with its inner product and the finite Sobolev norm

$$(u,v)_{H^1(\mathbb{S}^2)} = (u,v)_{L_2(\mathbb{S}^2)} + (\nabla u, \nabla v)_{\mathbf{L}_2(\mathbb{S}^2)}, \quad ||u||^2_{H^1(\mathbb{S}^2)} = ||u||^2_{L_2(\mathbb{S}^2)} + ||\nabla u||^2_{\mathbf{L}_2(\mathbb{S}^2)},$$

where ∇ is the surface gradient on the sphere. Generally, if $s=m$ which is a positive integer, we can define the Sobolev norm via the following formula (∇-definition of Sobolev spaces)

$$||u||^2_{H^s(\mathbb{S}^2)} = (u,v)_{L_2(\mathbb{S}^2)} + \sum_{k=1}^{m} (\nabla^k u, \nabla^k v)_{\mathbf{L}_2(\mathbb{S}^2)}.$$

If we s consider a closed linear subspace $H^1(\mathbb{S}^2)/\mathbb{R}) \subset H^1(\mathbb{S}^2)$,

$$H^1(\mathbb{S}^2)/\mathbb{R} = \{u \in H^1(\mathbb{S}^2) : \int_{\mathbb{S}^2} u(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} = 0\},$$

then due to a Poincaré inequality for all $u \in H^1(\mathbb{S}^2)/\mathbb{R}$ we can define an equivalent norm for $H^1(\mathbb{S}^2)/\mathbb{R}$

$$||u||^2_{H^1(\mathbb{S}^2)/\mathbb{R}} = ||\nabla u||_{\mathbf{L}_2(\mathbb{S}^2)},$$

such that $H^1(\mathbb{S}^2)/\mathbb{R}$ becomes a Hilbert space with the inner product

$$(u,v)_{H^1(\mathbb{S}^2)/\mathbb{R}} = (\nabla u, \nabla v)_{\mathbf{L}_2(\mathbb{S}^2)}.$$

For more details on these spaces, we refer the reader to [3], [24, Theorems 4.12 and 6.12], [29, p. 41] .

2.4.2. *Sobolev tangent vector fields on* $\mathbb{S}^2$. For tangential vector fields we have the vectorial Sobolev space $\mathbf{H}^s_{tan}(\mathbb{S}^2)$, which is the set of all $f \in \mathbf{L}_{2,tan}(\mathbb{S}^2)$ such that

$$||\mathbf{f}||^2_{\mathbf{H}^s_{tan}(\mathbb{S}^2)} = \sum_{N=1}^{\infty} (1+N(N+1))^s \sum_{\ell} |f_{2,N\ell}|^2 + |f_{3,N\ell}|^2.$$

For the scale of Sobolev spaces $\mathbf{H}^s_{tan}(\mathbb{S}^2)$ there is a Helmholtz–Hodge decomposition ([4, Theorem 4.1])

$$\mathbf{H}^s_{tan}(\mathbb{S}^2) = \nabla\left(H^{s+1}(\mathbb{S}^2)/\mathbb{R}\right) \oplus \ker(\mathrm{div}) = \mathbf{H}^s_{tan,curl}(\mathbb{S}^2) + \mathbf{H}^s_{tan,div}(\mathbb{S}^2), \ s \geq 0.$$

Here we denote by $\mathbf{H}^s_{tan,div}(\mathbb{S}^2)$ and $\mathbf{H}^s_{tan,curl}(\mathbb{S}^2)$ the divergence-free and curl-free subspaces of $\mathbf{H}^s_{tan}(\mathbb{S}^2)$, respectively.

Another words vector field tangent to the sphere $\mathbf{f} \in \mathbf{H}^s_{tan}(\mathbb{S}^2)$ can be uniquely decomposed into surface curl–free and surface divergence–free components:

$$\mathbf{f} = \nabla u + \nabla^{\perp} v, \quad \int_{\mathbb{S}^2} u \, \mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{S}^2} v \, \mathrm{d}\boldsymbol{\xi} = 0,$$

where functions $u, v \in H^{s+1}(\mathbb{S}^2)/\mathbb{R}$. We can define its $\mathbf{H}^s$ norm, among other equivalent versions, as

$$||\mathbf{f}||^2_{\mathbf{H}^s(\mathbb{S}^2)} = ||u||^2_{H^{s+1}(\mathbb{S}^2)} + ||v||^2_{H^{s+1}(\mathbb{S}^2)}.$$

2.5. **Fourier multiplier and spherical convolution operators.**

2.5.1. *Fourier multiplier operators.* Here we define Fourier multiplication operators.

Definition 7. *The operator $\Lambda : L_2(\mathbb{S}^2) \to L_2(\mathbb{S}^2)$ is called the Fourier multiplier operator with corresponding sequence of multipliers $\{\lambda_N\}_{N=0}^{\infty}$ if operator Λ acts on a function $u \in L_2(\mathbb{S}^2)$ by the formula*

$$\{\Lambda u\}(\boldsymbol{\xi}) \equiv \Lambda_{\boldsymbol{\xi}} u = \sum_{N=0} \lambda_N \sum_{\ell} u_{N\ell} Y_{N\ell}(\boldsymbol{\xi}),$$

where $u_{N\ell}$ denote the Fourier coefficients of u with respect to the spherical harmonics,

$$u(\boldsymbol{\xi}) = \sum_{N=0} \sum_{\ell} u_{N\ell} Y_{N\ell}(\boldsymbol{\xi}).$$

The sequence of multipliers $\{\lambda_N\}_{N=0}^{\infty}$ gives complete information about properties of operator Λ, especially the behavior and asymptotics of multipliers at infinity. It is not hard to see that a multiplier operator on $L_2(\mathbb{S}^2)$ is bounded if and only if its sequence of multipliers is bounded. The works of many authors are devoted to the study of such operators, see [5, 39, 44].

2.5.2. *Spherical convolution operators.* An important example of the multiplier operator will be a spherical convolution operator.

Definition 8. *The spherical convolution $K * u$ of $K \in L_2(-1,1)$ with a function $u \in L_2(\mathbb{S}^2)$ is defined as*

$$(K * u)(\boldsymbol{\xi}) = \int_{\mathbb{S}^2} K(\boldsymbol{\xi} \boldsymbol{.} \boldsymbol{\eta}) u(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta},\ \boldsymbol{\xi} \in S^2,$$

$\mathrm{d}\boldsymbol{\eta}$ *is the rotation invariant measure, normalized so that* $\int_{\mathbb{S}^2} \mathrm{d}\boldsymbol{\eta} = 4\pi$ *— the surface area of* $\mathbb{S}^2$*. We recall that* $\boldsymbol{\eta} \boldsymbol{.} \boldsymbol{\xi}$ *is the usual pointwise inner product.*

By the Funk—Hecke formula in Theorem 3 we have the sequence of multipliers $\{\lambda_N\}_{N=0}^{\infty}$

$$\{K * Y_{N\ell}\}(\boldsymbol{\xi}) = 2\pi Y_{N\ell}(\boldsymbol{\xi}) \int_{-1}^{1} K(x) P_N(x)\, dx = \lambda_N Y_{N\ell}(\boldsymbol{\xi}).$$

2.5.3. *Funk's inversion formula for the F–M transform.* In [9] Funk showed that Funk–Minkowski- transform (1) is the Fourier multiplier operator with multiplicators $\lambda_{2j} = P_{2j}(0)$,

$$\{\mathcal{F} Y_{N\ell}\}(\boldsymbol{\xi}) = P_{2j}(0) Y_{2j,\ell}(\boldsymbol{\xi}),$$

and asymptotics $\lambda_{2j} = P_{2j}(0) \sim (2j+1)^{-1/2}$ if $j \to \infty$ ([1]). Hence any even function $f_{even} \in C^{\infty}(\mathbb{S}^2)$ can be reconstructed explicitly from its Funk–Minkowski transform by the formula

$$f_{even}(\boldsymbol{\xi}) = \sum_{j=0}^{\infty} \sum_{\ell} f_{2j,\ell} Y_{2j,\ell}(\boldsymbol{\xi}) = \sum_{j=0}^{\infty} \sum_{\ell} \frac{(\mathcal{F} f_{even}, Y_{2j,\ell})_{L_2(S^2)}}{P_{2j}(0)} Y_{2j,\ell}(\boldsymbol{\xi}),$$

where

$$(\mathcal{F} f_{even}, Y_{2j,\ell})_{L_2(\mathbb{S}^2)} = P_{2j}(0) f_{2j,\ell}\,.$$

The following mapping property of the Funk–Minkowski transform between Sobolev spaces was shown by R. S. Strichartz in [51, Lemma 4.3] : operator

$$\mathcal{F} : H^s_{even}(\mathbb{S}^2) \to H^{s+1/2}_{even}(\mathbb{S}^2),\ \ s \geq 0$$

is continuous and bijective, see also [16, 32].

2.5.4. *The spherical convolution operator $\mathcal{S}$.* Now consider the spherical convolution operator $\mathcal{S}$, which defined by formula (5), we repeat it

$$\{\mathcal{S} v\}(\boldsymbol{\xi}) \equiv \mathcal{S}_{\boldsymbol{\xi}} v = \frac{1}{4\pi} \{x^{-1} * u\}(\boldsymbol{\xi}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} \frac{v(\boldsymbol{\eta})}{\boldsymbol{\xi} \boldsymbol{.} \boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{\eta},\ \ \boldsymbol{\xi} \in \mathbb{S}^2.$$

The operator $\mathcal{S}$ does not exist as an absolutely convergent integral and should be understood in the principal value sense, see [41, 45],

$$\{\mathcal{S} v\}(\boldsymbol{\xi}) = \lim_{\varepsilon \to 0} \frac{1}{4\pi} \int_{|\boldsymbol{\xi} \boldsymbol{.} \boldsymbol{\eta}| > \varepsilon} \frac{v(\boldsymbol{\eta})}{\boldsymbol{\xi} \boldsymbol{.} \boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{\eta} = p.v. \frac{1}{4\pi} \int_{\mathbb{S}^2} \frac{v(\boldsymbol{\eta})}{\boldsymbol{\xi} \boldsymbol{.} \boldsymbol{\eta}}\, \mathrm{d}\boldsymbol{\eta}.$$

The operator $\mathcal{S}$ is considered as operator from $L_2(\mathbb{S}^2)$ into $L_2(\mathbb{S}^2)$ and can be regarded as the spherical analogue of the Hilbert transform, [41]. Evidently, that for even spherical harmonics $\{\mathcal{S} Y_{2j,\ell}\}(\boldsymbol{\xi}) = 0$, so we can consider this operator only on the subspace of odd SHs, $L_{2,odd}(\mathbb{S}^2)$.

Proposition 1 ([21, 41]). *The spherical analogue of the Hilbert transform (5)*

$$\mathcal{S} : L_{2,odd}(\mathbb{S}^2) \to L_{2,odd}(\mathbb{S}^2)$$

is a compact operator and a multiplier operator on $L_{2,odd}(\mathbb{S}^2)$ *with corresponding sequence of Fourier-Laplace multipliers* $\left\{\frac{1}{C^{(3/2)}_{N-1}(0)} = \frac{1}{NP_{N-1}(0)},\ N = 2j+1\right\}_{j=0}^{\infty}$,

$$\{\mathcal{S}Y_{N\ell}\}(\boldsymbol{\xi}) = \frac{1}{C^{(3/2)}_{N-1}(0)}Y_{N\ell}(\boldsymbol{\xi}) = \frac{1}{NP_{N-1}(0)}Y_{N\ell}(\boldsymbol{\xi}),\ N = 2j+1 \tag{42}$$

and asymptotics

$$\frac{1}{C^{(3/2)}_{2j}(0)} = \frac{1}{(2j+1)P_{2j}(0)} \sim \frac{1}{\sqrt{2j+1}} \ \text{if} \ \ j \to \infty. \tag{43}$$

The operator $\mathcal{S}$, as well as the operator $\mathcal{F}$,

$$\mathcal{S} : H^s_{odd}(\mathbb{S}^2) \to H^{s+1/2}_{odd}(\mathbb{S}^2),\ s \ge 0$$

is continuous and bijective in the scale of Sobolev spaces $H^s_{odd}(\mathbb{S}^2)$, see [41, Proposition 3.2].

2.5.5. *Analytic family of fractional integrals and Funk–Minkowski transform.* We can write the F–M operator (1) in the form of spherical convolution operator as follows

$$\begin{aligned}
\{\mathcal{F}u\}(\boldsymbol{\xi}) &= \frac{1}{2\pi}\int_0^{2\pi} u\Big(\mathbf{e}_1(\boldsymbol{\xi})\cos\omega + \mathbf{e}_2(\boldsymbol{\xi})\sin\omega\Big)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-1}^{1}\delta(t)\int_0^{2\pi} u\Big(\mathbf{e}_1(\boldsymbol{\xi})\cos\omega\sqrt{1-t^2} + \mathbf{e}_2(\boldsymbol{\xi})\sqrt{1-t^2}\sin\omega\Big)\,\mathrm{d}\omega\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{\mathbb{S}^2}\delta(\boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\theta})u(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta} = \frac{1}{2\pi}\{\delta * u\}(\boldsymbol{\xi}),
\end{aligned}$$

where δ is the Dirac delta function.

The papers [23, 32] give a definition of the generalized Funk–Radon transform $S^{(j)}$ for $u \in C^\infty(\mathbb{S}^2)$ by

$$\{S^{(j)}u\}(\boldsymbol{\xi}) = \frac{1}{2\pi}\int_{\mathbb{S}^2}\delta^{(j)}(\boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\theta})u(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta},\quad j \in 0 \cup \mathbb{N}.$$

Here use the notation from [23, 32] and $\delta^{(j)}$ denotes the j-th derivative of the Dirac delta function and operator $S^{(0)}$ is the Funk–Minkowski transform $\mathcal{F}$.

The spherical Hilbert type operator $\mathcal{S}$ in (5) as well as operators $\mathcal{S}^{(j)}$ are the members of analytic family of fractional integrals $\{\mathcal{C}^\lambda, \widetilde{\mathcal{C}}^\lambda\}$ defined by

$$\{\mathcal{C}^\lambda f\}(\boldsymbol{\theta}) = \frac{\Gamma\left(-\frac{\lambda}{2}\right)}{2\pi\Gamma\left(\frac{1+\lambda}{2}\right)}\int_{\mathbb{S}^2} f(\boldsymbol{\sigma})|\boldsymbol{\theta}\boldsymbol{\cdot}\boldsymbol{\sigma}|^\lambda\, d\boldsymbol{\sigma}, \tag{44}$$

$$\{\widetilde{\mathcal{C}}^\lambda f\}(\boldsymbol{\theta}) = \frac{\Gamma\left(\frac{1-\lambda}{2}\right)}{2\pi\Gamma\left(1+\frac{\lambda}{2}\right)}\int_{\mathbb{S}^2} f(\boldsymbol{\sigma})|\boldsymbol{\theta}\boldsymbol{\cdot}\boldsymbol{\sigma}|^\lambda \mathrm{sgn}(\boldsymbol{\theta}\boldsymbol{\cdot}\boldsymbol{\sigma})\, d\boldsymbol{\sigma}, \tag{45}$$

see [35, 41]. The operators $\mathcal{C}^\lambda$ and $\widetilde{\mathcal{C}}^\lambda$ are called the λ-cosine transforms of f with even and odd kernel, respectively. If $f \in C^\infty(\mathbb{S}^2)$, they extend analytically to all $\lambda \in \mathbb{C}$ with the only poles $\lambda = 0, 2, 4, \ldots$ for $\mathcal{C}^\lambda$ and $\lambda = 1, 3, 5, \ldots$ for $\widetilde{C}^\lambda$.

The limit case $\lambda = -1$ corresponds to the Funk–Minkowski transform $\mathcal{F}$ and Hilbert spherical transform $\mathcal{S}$ (see [41, Lemma 3.4])

$$\mathcal{F} \sim \{\mathcal{C}^{-1} f\}(\boldsymbol{\theta}) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{S}^2} f(\boldsymbol{\eta}) \delta(\boldsymbol{\theta} \cdot \boldsymbol{\eta})\, d\boldsymbol{\eta},$$
$$\mathcal{S} \sim \{\widetilde{\mathcal{C}}^{-1} f\}(\boldsymbol{\theta}) = \frac{1}{2\pi^{3/2}} \int_{\mathbb{S}^2} \frac{f(\boldsymbol{\eta})}{\boldsymbol{\theta} \cdot \boldsymbol{\eta}}\, d\boldsymbol{\eta}.$$

The integral operator in the inverse formula (2) by V. Semyanisty also belongs to this family with $\lambda = -2$,

$$\{\mathcal{C}^{-2} f\}(\boldsymbol{\theta}) = \frac{-1}{4\pi^{3/2}} \int_{\mathbb{S}^2} f(\boldsymbol{\eta}) \frac{1}{(\boldsymbol{\theta} \cdot \boldsymbol{\eta})^2} d\boldsymbol{\eta}.$$

The corresponding operator $\widetilde{\mathcal{C}}^{-2}$ for $\mathcal{C}^{-2}$ is the generalized Funk–Radon transform

$$S^{(1)} \sim \{\widetilde{\mathcal{C}}^{-2} f\}(\boldsymbol{\theta}) = \frac{-1}{4\sqrt{\pi}} \int_{\mathbb{S}^2} f(\boldsymbol{\eta}) \delta'(\boldsymbol{\theta} \cdot \boldsymbol{\eta})\, d\boldsymbol{\eta}.$$

If for an analytic continuation we use formulas, see for example [14],

$$\frac{|x|^\lambda}{\Gamma\left(\frac{1+\lambda}{2}\right)}\Big|_{\lambda=-(2m+1)} = \frac{(-1)^m m!}{(2m)!} \delta^{(2m)}(x),\ m = 0, 1, 2, \dots\ , \tag{46}$$
$$\frac{|x|^\lambda \mathrm{sgn}(x)}{\Gamma\left(1+\frac{\lambda}{2}\right)}\Big|_{\lambda=-2m} = \frac{(-1)^m (m-1)!}{(2m-1)!} \delta^{(2m-1)}(x),\ m = 1, 2, 3, \dots\ , \tag{47}$$

then as the result, the following connection between $\mathcal{S}^{(2m)}$, $\mathcal{S}^{(2m+1)}$ and analytic family $\{\mathcal{C}^\lambda, \tilde{\mathcal{C}}^\lambda\}$ take place

$$\mathcal{S}^{(2m)} \sim \{\mathcal{C}^{-2m-1} f\}(\boldsymbol{\theta}) = \frac{(-1)^m \sqrt{\pi}}{2\pi 2^{2m}} \int_{\mathbb{S}^2} f(\boldsymbol{\sigma}) \delta^{(2m)}(\boldsymbol{\theta} \cdot \boldsymbol{\sigma})\, d\boldsymbol{\sigma},\ m = 0, 1, 2, \dots, \tag{48}$$

$$\mathcal{S}^{(2m+1)} \sim \{\widetilde{\mathcal{C}}^{-2m} f\}(\boldsymbol{\theta}) = \frac{(-1)^m \sqrt{\pi}}{2\pi 2^{2m-1}} \int_{\mathbb{S}^2} f(\boldsymbol{\sigma}) \delta^{(2m-1)}(\boldsymbol{\theta} \cdot \boldsymbol{\sigma})\, d\boldsymbol{\sigma},\ m = 1, 2, 3, \dots. \tag{49}$$

According to the general theory of analytic family $\{\mathcal{C}^\lambda, \widetilde{\mathcal{C}}^\lambda\}$ on the sphere $\mathbb{S}^2$, we can find inverse operators of $\mathcal{C}^\lambda$, $\widetilde{\mathcal{C}}^\lambda$ by the formulas (see [41, Proposition 3.1])

$$\mathcal{C}^\lambda \mathcal{C}^{-\lambda-3} f = \mathcal{C}^{-\lambda-3} \mathcal{C}^\lambda f = f,\ \text{where } \lambda, -\lambda - 3 \neq 0, 2, 4, \dots\ f \in C^\infty_{even}(\mathbb{S}^2),$$

and

$$\widetilde{\mathcal{C}}^\lambda \widetilde{\mathcal{C}}^{-\lambda-3} f = \widetilde{\mathcal{C}}^{-\lambda-3} \widetilde{\mathcal{C}}^\lambda f = f,\ \text{where } \lambda, -\lambda - 3 \neq 1, 3, 5, \dots\ ,\ f \in C^\infty_{odd}(\mathbb{S}^2).$$

In the particular case $\lambda = -1$ we have $\mathcal{F}^{-1} \sim \left(\mathcal{C}^{-1}\right)^{-1} = \mathcal{C}^{-2}$ and it is appropriate to formula (2) by V. Semyanisty, see also [40, Corollary 3.3]. If we apply (formally) the integration by parts formula (23) to (7), then we get

$$\begin{aligned}\frac{\boldsymbol{\theta}\,\boldsymbol{\cdot}}{4\pi}\int_{\mathbb{S}^2}\frac{\{\nabla\mathcal{F}f\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta}}\mathrm{d}\boldsymbol{\eta} &= -\frac{\boldsymbol{\theta}\,\boldsymbol{\cdot}}{4\pi}\int_{\mathbb{S}^2}\nabla\frac{1}{\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta}}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}+\frac{\boldsymbol{\theta}\,\boldsymbol{\cdot}}{2\pi}\int_{\mathbb{S}^2}\frac{\boldsymbol{\eta}\{\mathcal{F}f\}(\boldsymbol{\eta})}{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\boldsymbol{\theta}}\mathrm{d}\boldsymbol{\eta}\\
&= \frac{\boldsymbol{\theta}\,\boldsymbol{\cdot}}{4\pi}\int_{\mathbb{S}^2}\frac{\boldsymbol{\theta}-(\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta})\boldsymbol{\eta}}{(\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta})^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}+\frac{1}{2\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}\\
&= \frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{1-(\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta})^2}{(\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta})^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}+\frac{1}{2\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}\\
&= \frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{1}{(\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta})^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}+\frac{1}{4\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}.\end{aligned}$$

Thus, this formal calculations show that formula (7) corresponds to formula (2) and serves as its regularization

$$-\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{1}{(\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta})^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}=\frac{1}{4\pi}\int_{\mathbb{S}^2}\{\mathcal{F}f\}(\boldsymbol{\eta})\mathrm{d}\boldsymbol{\eta}-p.v.\frac{1}{4\pi}\int_{\mathbb{S}^2}\frac{\boldsymbol{\theta}\,\boldsymbol{\cdot}\{\nabla\mathcal{F}f\}(\boldsymbol{\eta})}{\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\boldsymbol{\eta}}\mathrm{d}\boldsymbol{\eta}.$$

3. Proofs

In this section we present the proofs of Theorems 1, 2, which will be based on Lemmas 1 and 2. In vector case, as in the scalar case, the vectorial Funk–Minkowski transform $\mathcal{F}:\mathbf{L}_{2,even}(\mathbb{S}^2)\to\mathbf{L}_{2,even}(\mathbb{S}^2)$ and vectorial Hilbert type spherical transform $\mathcal{S}:\mathbf{L}_{2,odd}(\mathbb{S}^2)\to\mathbf{L}_{2,odd}(\mathbb{S}^2)$ are multiplier operators and relevant mapping properties between Sobolev spaces are valid. The accurate formulations are given below.

Lemma 1. *Vectorial Funk–Minkowski transform $\mathcal{F}:\mathbf{L}_{2,even}(\mathbb{S}^2)\to\mathbf{L}_{2,even}(\mathbb{S}^2)$ is a multiplier operator*

$$\mathcal{F}\mathbf{h}^{(i)}_{N\ell}=P_{N-1}(0)\mathbf{h}^{(i)}_{N\ell},\ N=2j+1, \tag{50}$$

$$\mathcal{F}\mathbf{y}^{(3)}_{N\ell}=P_{N}(0)\mathbf{y}^{(3)}_{N\ell},\quad N=2j, \tag{51}$$

$$\mathcal{F}\mathbf{h}^{(e)}_{N\ell}=P_{N+1}(0)\mathbf{h}^{(e)}_{N\ell},\ N=2j+1, \tag{52}$$

where $\mathbf{h}^{(i)}_{N\ell},\mathbf{y}^{(3)}_{N\ell},\mathbf{h}^{(e)}_{N\ell}$ are pure–orbit vector spherical harmonics (39)–(41). We have that in the scale of Sobolev spaces operator $\mathcal{F}:\mathbf{H}^{s}_{even}(\mathbb{S}^2)\to\mathbf{H}^{s+1/2}_{even}(\mathbb{S}^2)$, $s\ge 0$ is continuous and bijective.

If we choose as a basis pure–spin vector spherical harmonics, then following formulas take place

$$\mathcal{F}\mathbf{y}^{(1)}_{N\ell}=P_{N-1}(0)\frac{\mathbf{y}^{(2)}_{N\ell}}{N+1}\,,\ N=2j+1, \tag{53}$$

$$\mathcal{F}\mathbf{y}^{(2)}_{N\ell}=P_{N-1}(0)\Big(N\mathbf{y}^{(1)}_{N\ell}+\frac{\mathbf{y}^{(2)}_{N\ell}}{N+1}\Big),\ N=2j+1\,, \tag{54}$$

$$\mathcal{F}\mathbf{y}^{(3)}_{N\ell}=P_{N}(0)\mathbf{y}^{(3)}_{N\ell}\,,\ N=2j\,. \tag{55}$$

Similar statements are valid for the operator $\mathcal{S}$

Lemma 2. *Vectorial spherical convolution transform* $\mathcal{S} : \mathbf{L}_{2,odd}(\mathbb{S}^2) \to \mathbf{L}_{2,odd}(\mathbb{S}^2)$ *is a multiplier operator*

$$\mathcal{S}\mathbf{h}_{00}^{(e)} = \mathbf{h}_{00}^{(e)}, \tag{56}$$

$$\mathcal{S}\mathbf{h}_{N\ell}^{(e)} = \frac{\mathbf{h}_{N\ell}^{(e)}}{(N+1)P_N(0)}, \ N = 2j, \tag{57}$$

$$\mathcal{S}\mathbf{h}_{N\ell}^{(i)} = -\frac{1}{(N+1)P_N(0)}\frac{N+1}{N}\mathbf{h}_{N\ell}^{(i)}, \ N = 2j, \tag{58}$$

$$\mathcal{S}\mathbf{y}_{N\ell}^{(3)} = \frac{\mathbf{y}_{N\ell}^{(3)}}{NP_{N-1}(0)}, \ N = 2j+1\,, \tag{59}$$

where $\mathbf{h}_{N\ell}^{(i)}, \mathbf{y}_{N\ell}^{(3)}, \mathbf{h}_{N\ell}^{(e)}$ *are pure–orbit vector spherical harmonics (39)–(41). In the scale of Sobolev spaces operator* $\mathcal{S} : \mathbf{H}^s_{odd}(\mathbb{S}^2) \to \mathbf{H}^{s+1/2}_{odd}(\mathbb{S}^2),\ s \geq 0$ *is continuous and bijective.*

The images of pure–spin spherical harmonics under the action of operator $\mathcal{S}$ *are listed below*

$$\mathcal{S}\mathbf{y}_{00}^{(1)} = \mathbf{y}_{00}^{(1)}, \tag{60}$$

$$\mathcal{S}\mathbf{y}_{N\ell}^{(1)} = \frac{-1}{P_N(0)}\frac{\mathbf{y}_{N\ell}^{(2)}}{N(N+1)}, \ N = 2j, \tag{61}$$

$$\mathcal{S}\mathbf{y}_{N\ell}^{(2)} = \frac{-1}{P_N(0)}\Big(\mathbf{y}_{N\ell}^{(1)} + \frac{\mathbf{y}_{N\ell}^{(2)}}{N(N+1)}\Big), \ N = 2j, \tag{62}$$

$$\mathcal{S}\mathbf{y}_{N\ell}^{(3)} = \frac{1}{P_{N-1}(0)}\frac{\mathbf{y}_{N\ell}^{(3)}}{N}, \ N = 2j+1\,. \tag{63}$$

Proof of Lemma 1. The pure–orbit VSHs are expressed through scalar spherical harmonics with the help of three term relations, see for example [8],

$$\begin{aligned}\mathbf{h}_{N\ell}^{(i)} &= N\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi}) \\ &= \alpha_1 Y_{N-1,\ell-1}(\boldsymbol{\xi})\begin{pmatrix}1\\ \mathrm{i}\\ 0\end{pmatrix} + \beta_1 Y_{N-1,\ell}(\boldsymbol{\xi})\begin{pmatrix}0\\0\\1\end{pmatrix} + \gamma_1 Y_{N-1,\ell+1}(\boldsymbol{\xi})\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix},\end{aligned} \tag{64}$$

$$\begin{aligned}\mathbf{h}_{N\ell}^{(e)} &= -(N+1)\mathbf{y}_{N\ell}^{(1)}(\boldsymbol{\xi}) + \mathbf{y}_{N\ell}^{(2)}(\boldsymbol{\xi}) \\ &= \alpha_2 Y_{N+1,\ell-1}(\boldsymbol{\xi})\begin{pmatrix}1\\ \mathrm{i}\\ 0\end{pmatrix} + \beta_2 Y_{N+1,\ell}(\boldsymbol{\xi})\begin{pmatrix}0\\0\\1\end{pmatrix} + \gamma_2 Y_{N+1,\ell+1}(\boldsymbol{\xi})\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix},\end{aligned} \tag{65}$$

$$\mathbf{y}_{N\ell}^{(3)}(\boldsymbol{\xi}) = \alpha_3 Y_{N,\ell-1}(\boldsymbol{\xi})\begin{pmatrix}1\\ \mathrm{i}\\ 0\end{pmatrix} + \beta_3 Y_{N\ell}(\boldsymbol{\xi})\begin{pmatrix}0\\0\\1\end{pmatrix} + \gamma_3 Y_{N,\ell+1}(\boldsymbol{\xi})\begin{pmatrix}1\\-\mathrm{i}\\0\end{pmatrix}, \tag{66}$$

where $\alpha_i, \beta_i, \gamma_i$ $(i = 1,2,3)$ some coefficients. The values of this coefficients are unimportant here, but their accurate expressions can be found in [8].

By applying the operator $\mathcal{F}$ to these three term relations we immediately obtain: for $N = 2j+1$

$$\mathcal{F}\mathbf{h}_{N\ell}^{(i)} = P_{N-1}(0)\mathbf{h}_{N\ell}^{(i)}, \quad \mathcal{F}\mathbf{h}_{N\ell}^{(e)} = P_{N+1}(0)\mathbf{h}_{N\ell}^{(e)} \tag{67}$$

and for $N = 2j$

$$\mathcal{F}\mathbf{y}^{(3)}_{N\ell} = P_N(0)\mathbf{y}^{(3)}_{N\ell}.$$

Because the multipliers have asymptotics $P_{2j}(0) \sim \frac{1}{\sqrt{2j+1}}$ as j goes to infinity, we have that $\mathcal{F} : \mathbf{H}^s_{even}(\mathbb{S}^2) \to \mathbf{H}^{s+1/2}_{even}(\mathbb{S}^2)$ is a continuous operator in the scale of Sobolev spaces, as in the scalar case.

The two equations (67) can be written as

$$\begin{cases} N\mathcal{F}\mathbf{y}^{(1)}_{N\ell} + \mathcal{F}\mathbf{y}^{(2)}_{N\ell} & = NP_{N-1}(0)\mathbf{y}^{(1)}_{N\ell} + P_{N-1}(0)\mathbf{y}^{(2)}_{N\ell} \\ -(N+1)\mathcal{F}\mathbf{y}^{(1)}_{N\ell} + \mathcal{F}\mathbf{y}^{(2)}_{N\ell} & = -(N+1)P_{N+1}(0)\mathbf{y}^{(1)}_{N\ell} + P_{N+1}(0)\mathbf{y}^{(2)}_{N\ell}. \end{cases}$$

We need to solve this system with respect to $\mathcal{F}\mathbf{y}^{(1)}_{N\ell}$ and $\mathcal{F}\mathbf{y}^{(2)}_{N\ell}$. Subtracting the second from the first equation, we obtain

$$\begin{aligned} &(2N+1)\mathcal{F}\mathbf{y}^{(1)}_{N\ell} = (NP_{N-1}(0) + (N+1)P_{N+1}(0))\mathbf{y}^{(1)}_{N\ell} + (P_{N-1}(0) - P_{N+1}(0))\mathbf{y}^{(2)}_{N\ell} \\ &= P_{N-1}(0)\left(N - (N+1)\frac{N}{N+1}\right)\mathbf{y}^{(1)}_{N\ell} + P_{N-1}(0)\left(1 + \frac{N}{N+1}\right)\mathbf{y}^{(2)}_{N\ell} \\ &= P_{N-1}(0)\frac{2N+1}{N+1}\mathbf{y}^{(2)}_{N\ell}. \end{aligned}$$

Here we used the formula (14), $(N+1)P_{N+1}(0) = -NP_{N-1}(0)$, thus we have

$$\mathcal{F}\mathbf{y}^{(1)}_{N\ell} = P_{N-1}(0)\frac{\mathbf{y}^{(2)}_{N\ell}}{N+1}, \quad \mathcal{F}\mathbf{y}^{(2)}_{N\ell} = P_{N-1}(0)\Big(N\mathbf{y}^{(1)}_{N\ell} + \frac{\mathbf{y}^{(2)}_{N\ell}}{N+1}\Big), \ N = 2j+1.$$

Proof of Lemma 2. By applying operator $\mathcal{S}$ to three term relations, as well as in the previous case, we obtain: for $N = 2j$

$$\begin{aligned} \mathcal{S}\mathbf{h}^{(i)}_{N\ell} &= \frac{\mathbf{h}^{(i)}_{N\ell}}{(N-1)P_{N-2}(0)} = -\frac{1}{(N+1)P_N(0)}\frac{N+1}{N}\mathbf{h}^{(i)}_{N\ell}, \\ \mathcal{S}\mathbf{h}^{(e)}_{N\ell} &= \frac{\mathbf{h}^{(e)}_{N\ell}}{(N+1)P_N(0)} \end{aligned}$$

and for $N = 2j+1$

$$\mathcal{S}\mathbf{y}^{(3)}_{N\ell} = \frac{\mathbf{y}^{(3)}_{N\ell}}{NP_{N-1}(0)}.$$

In the first formula we used equality $(N-1)P_{N-2}(0) = -NP_N(0)$.

Continuity of the operator $\mathcal{S}$ in the scale $\mathbf{H}^s_{odd}(\mathbb{S}^2)$ follows from asymptotic behavior $\frac{1}{(2j+1)P_{2j}(0)} \sim \frac{1}{\sqrt{2j+1}}$ if $j \to \infty$.

The first two equations above are equivalent to the system

$$\begin{cases} N\mathcal{S}\mathbf{y}^{(1)}_{N\ell} + \mathcal{S}\mathbf{y}^{(2)}_{N\ell} & = -\frac{\mathbf{y}^{(1)}_{N\ell}}{P_N(0)} - \frac{\mathbf{y}^{(2)}_{N\ell}}{NP_N(0)} \\ -(N+1)\mathcal{S}\mathbf{y}^{(1)}_{N\ell} + \mathcal{S}\mathbf{y}^{(2)}_{N\ell} & = -\frac{\mathbf{y}^{(1)}_{N\ell}}{P_N(0)} + \frac{\mathbf{y}^{(2)}_{N\ell}}{(N+1)P_N(0)}. \end{cases}$$

Solving this system, we obtain the desired

$$\mathcal{S}\mathbf{y}^{(1)}_{N\ell} = \frac{-\mathbf{y}^{(2)}_{N\ell}}{P_N(0)N(N+1)}, \quad \mathcal{S}\mathbf{y}^{(2)}_{N\ell} = \frac{-1}{P_N(0)}\Big(\mathbf{y}^{(1)}_{N\ell} + \frac{\mathbf{y}^{(2)}_{N\ell}}{N(N+1)}\Big).$$

3.1. **Proof Theorem 1.** We recall some of the basic properties that are implied in our proof. The Funk–Minkowski transform is even, $\{\mathcal{F}f\}(-\boldsymbol{\xi}) = \{\mathcal{F}f\}(\boldsymbol{\xi})$, and $\mathcal{F}f_{odd} = 0$, but spherical transform $\mathcal{S}$ is odd, $\{\mathcal{S}f\}(-\boldsymbol{\xi}) = -\{\mathcal{S}f\}(\boldsymbol{\xi})$, and $\mathcal{S}f_{even} = 0$. It is obviously that the surface gradient ∇, scalar (dot) product $\boldsymbol{\eta}\,\boldsymbol{\cdot}$ and vector (cross) product $\boldsymbol{\eta}\times$ change the parity. We also recall the parity rules for scalar and vector spherical harmonics : $Y_{N\ell}(-\boldsymbol{\xi}) = (-1)^N Y_{N\ell}(\boldsymbol{\xi})$, $\mathbf{y}^{(1)}_{N\ell}(-\boldsymbol{\xi}) = (-1)^{N+1}\mathbf{y}^{(1)}_{N\ell}(\boldsymbol{\xi})$, $\mathbf{y}^{(2)}_{N\ell}(-\boldsymbol{\xi}) = (-1)^{N+1}\mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\xi})$, $\mathbf{y}^{(3)}_{N\ell}(-\boldsymbol{\xi}) = (-1)^{N}\mathbf{y}^{(3)}_{N\ell}(\boldsymbol{\xi})$.

Now we can proceed to our formula (6) and without loss of generality, we assume that $f(\boldsymbol{\theta}) \in H^1(\mathbb{S}^2)/\mathbb{R}$, then we have

$$
\begin{aligned}
f(\boldsymbol{\theta}) &= \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{(\boldsymbol{\eta}+\boldsymbol{\theta})\,\boldsymbol{\cdot}\Big\{\Big[\mathcal{F},\nabla\Big]f\Big\}(\boldsymbol{\eta})}{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} \\
&= \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{\boldsymbol{\eta}\,\boldsymbol{\cdot}\Big\{\Big[\mathcal{F},\nabla\Big]f\Big\}(\boldsymbol{\eta}) + \boldsymbol{\theta}\,\boldsymbol{\cdot}\Big\{\Big[\mathcal{F},\nabla\Big]f\Big\}(\boldsymbol{\eta})}{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} \\
&= \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\eta}}\nabla f - \overbrace{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\nabla\mathcal{F}_{\boldsymbol{\eta}}f}^{=0} - \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\nabla\mathcal{F}_{\boldsymbol{\eta}}f}{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} + \frac{\boldsymbol{\theta}\boldsymbol{\cdot}}{4\pi}\int_{\mathbb{S}^2} \frac{\overbrace{\mathcal{F}_{\boldsymbol{\eta}}\nabla f}^{eveen}}{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} \\
&= \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\eta}}\nabla f - \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\nabla\mathcal{F}_{\boldsymbol{\eta}}f}{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} = \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\eta}}\nabla f - \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\nabla\mathcal{F}_{\boldsymbol{\eta}}f.
\end{aligned}
$$

It is clear that $\ker(\mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\eta}}\nabla) = H^s_{even}(\mathbb{S}^2)$ and $\ker(\boldsymbol{\theta}\boldsymbol{\cdot}\mathcal{S}_{\boldsymbol{\theta}}\nabla\mathcal{F}_{\boldsymbol{\eta}}) = H^s_{odd}(\mathbb{S}^2)$. From the Lemmas 1, 2 we have $\mathcal{S}\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\eta}}\nabla : H^s(\mathbb{S}^2) \to H^s(\mathbb{S}^2)$ if $s \geq 1$, that looks on the diagram

$$
H^s(\mathbb{S}^2) \xrightarrow{\ \nabla\ } \mathbf{H}^{s-1}_{tan}(\mathbb{S}^2) \xrightarrow{\boldsymbol{\eta}\boldsymbol{\cdot}\mathcal{F}_{\boldsymbol{\eta}}} H^{s-1/2}(\mathbb{S}^2) \xrightarrow{\ \mathcal{S}\ } H^s(\mathbb{S}^2).
$$

Similarly, $\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\mathcal{S}_{\boldsymbol{\theta}}\nabla\mathcal{F} : H^s(\mathbb{S}^2) \to H^s(\mathbb{S}^2)$ if $s \geq 1/2$, which is also confirmed by the diagram

$$
H^s(\mathbb{S}^2) \xrightarrow{\ \mathcal{F}\ } H^{s+1/2}(\mathbb{S}^2) \xrightarrow{\ \nabla\ } \mathbf{H}^{s-1/2}_{tan}(\mathbb{S}^2) \xrightarrow{\boldsymbol{\theta}\boldsymbol{\cdot}\mathcal{S}_{\boldsymbol{\theta}}} H^s(\mathbb{S}^2).
$$

For further calculations we take specifically $f = Y_{N\ell}$, then from (54) and (55) we have

$$
\text{for } N = 2j : \mathcal{F}_{\boldsymbol{\eta}}\nabla Y_{N\ell} = 0, \quad \nabla\mathcal{F}_{\boldsymbol{\eta}}Y_{N\ell} = P_N(0)\mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\eta}),
$$

$$
\text{for } N = 2j+1 : \mathcal{F}_{\boldsymbol{\eta}}\nabla Y_{N\ell} = P_{N-1}(0)\Big(N\mathbf{y}^{(1)}_{N\ell}(\boldsymbol{\eta}) + \frac{\mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\eta})}{N+1}\Big), \quad \nabla\mathcal{F}_{\boldsymbol{\eta}}Y_{N\ell} = 0.
$$

Consequently

$$
\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\eta}}\nabla Y_{N\ell} - \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\nabla\mathcal{F}_{\boldsymbol{\eta}}Y_{N\ell} = \begin{cases} -P_N(0)\boldsymbol{\theta}\,\boldsymbol{\cdot}\,\mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\eta}), & N = 2j \\ P_{N-1}(0)NY_{N\ell}(\boldsymbol{\eta}), & N = 2j+1 \end{cases}
$$

and finally, using formulas (42) and (62), we get

$$
\frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\mathcal{F}_{\boldsymbol{\eta}}\nabla Y_{N\ell} - \boldsymbol{\theta}\,\boldsymbol{\cdot}\,\nabla\mathcal{F}_{\boldsymbol{\eta}}Y_{N\ell}}{\boldsymbol{\eta}\,\boldsymbol{\cdot}\,\boldsymbol{\theta}}\,\mathrm{d}\boldsymbol{\eta} = \begin{cases} Y_{N\ell}(\boldsymbol{\theta}), & N = 2j \\ Y_{N\ell}(\boldsymbol{\theta}), & N = 2j+1\,. \end{cases}
$$

So we proved that, if $s \geq 1$ then operator $\mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta}\boldsymbol{\cdot}\mathcal{F}_{\boldsymbol{\eta}}\nabla - \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\theta}\boldsymbol{\cdot}\nabla\mathcal{F}_{\boldsymbol{\eta}} : H^s(\mathbb{S}^2) \to H^s(\mathbb{S}^2)$ is identical operator.

3.2. **Proof Theorem 2.** We have already mentioned two approaches to the solution of Helmholtz–Hodge decomposition problem for $\mathbf{f} \in \mathbf{L}_{2,tan}(\mathbb{S}^2)$. Now we proof formulas (11) and (12) in Theorem 2 for the velocity potential u and stream functions v of the Helmholtz–Hodge decomposition (10),

$$\begin{aligned} u(\boldsymbol{\theta}) &= \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{F}_{\boldsymbol{\eta}}\mathbf{f} - \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{S}_{\boldsymbol{\eta}}\mathbf{f}, \\ v(\boldsymbol{\theta}) &= \boldsymbol{\theta} \,\boldsymbol{.}\, \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta} \times \mathcal{F}_{\boldsymbol{\eta}}\mathbf{f} - \boldsymbol{\theta} \,\boldsymbol{.}\, \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta} \times \mathcal{S}_{\boldsymbol{\eta}}\mathbf{f}. \end{aligned}$$

For the proof it suffices to verify these formulas on the basis elements $f = \mathbf{y}^{(2)}_{N\ell}$ and $f = \mathbf{y}^{(3)}_{N\ell}$. Applying the scalar and cross products to the the formulas (54)–(55) and (62)–(63), we obtain

$$\text{for } N = 2j : \boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{F}_{\boldsymbol{\eta}}\mathbf{y}^{(3)}_{N\ell} = 0,\ \boldsymbol{\eta} \times \mathcal{F}_{\boldsymbol{\eta}}\mathbf{y}^{(3)}_{N\ell} = -P_N(0)\mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\eta}),$$

$$\text{for } N = 2j+1 : \boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{F}_{\boldsymbol{\eta}}\mathbf{y}^{(2)}_{N\ell} = P_{N-1}(0)NY_{N\ell}(\boldsymbol{\eta}),\ \boldsymbol{\eta} \times \mathcal{F}_{\boldsymbol{\eta}}\mathbf{y}^{(2)}_{N\ell} = P_{N-1}(0)\frac{\mathbf{y}^{(3)}_{N\ell}(\boldsymbol{\eta})}{N+1},$$

$$\text{for } N = 2j : \boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{S}_{\boldsymbol{\eta}}\mathbf{y}^{(2)}_{N\ell} = \frac{-1}{P_N(0)}Y_{N\ell}(\boldsymbol{\eta}),\quad \boldsymbol{\eta} \times \mathcal{S}_{\boldsymbol{\eta}}\mathbf{y}^{(2)}_{N\ell} = \frac{-1}{P_N(0)}\frac{\mathbf{y}^{(3)}_{N\ell}(\boldsymbol{\eta})}{N(N+1)},$$

$$\text{for } N = 2j+1 : \boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{S}_{\boldsymbol{\eta}}\mathbf{y}^{(3)}_{N\ell} = 0,\ \boldsymbol{\eta} \times \mathcal{S}_{\boldsymbol{\eta}}\mathbf{y}^{(3)}_{N\ell} = \frac{-1}{P_{N-1}(0)}\frac{\mathbf{y}^{(2)}_{N\ell}(\boldsymbol{\eta})}{N}.$$

Based on the above, we get

$$\begin{aligned} \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{F}_{\boldsymbol{\eta}}\mathbf{y}^{(2)}_{N\ell} - \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{S}_{\boldsymbol{\eta}}\mathbf{y}^{(2)}_{N\ell} &= \begin{cases} Y_{N\ell}(\boldsymbol{\theta}), & N = 2j \\ Y_{N\ell}(\boldsymbol{\theta}), & N = 2j+1, \end{cases} \\ \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{F}_{\boldsymbol{\eta}}\mathbf{y}^{(3)}_{N\ell} - \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta} \,\boldsymbol{.}\, \mathcal{S}_{\boldsymbol{\eta}}\mathbf{y}^{(3)}_{N\ell} &= \begin{cases} 0, & N = 2j \\ 0, & N = 2j+1, \end{cases} \\ \boldsymbol{\theta} \,\boldsymbol{.}\, \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta} \times \mathcal{F}_{\boldsymbol{\eta}}\mathbf{y}^{(2)}_{N\ell} - \boldsymbol{\theta} \,\boldsymbol{.}\, \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta} \times \mathcal{S}_{\boldsymbol{\eta}}\mathbf{y}^{(2)}_{N\ell} &= \begin{cases} 0, & N = 2j \\ 0 & N = 2j+1, \end{cases} \\ \boldsymbol{\theta} \,\boldsymbol{.}\, \mathcal{S}_{\boldsymbol{\theta}}\boldsymbol{\eta} \times \mathcal{F}_{\boldsymbol{\eta}}\mathbf{y}^{(3)}_{N\ell} - \boldsymbol{\theta} \,\boldsymbol{.}\, \mathcal{F}_{\boldsymbol{\theta}}\boldsymbol{\eta} \times \mathcal{S}_{\boldsymbol{\eta}}\mathbf{y}^{(3)}_{N\ell} &= \begin{cases} Y_{N\ell}(\boldsymbol{\theta}), & N = 2j \\ Y_{N\ell}(\boldsymbol{\theta}), & N = 2j+1. \end{cases} \end{aligned}$$

4. Conclusion

This paper is devoted to the study of Funk–Minkowski transform $\mathcal{F}$ and Hilbert type spherical convolution $\mathcal{S}$. We provide inversion formulas for two F–M transforms $\mathcal{F}f$ and $\mathcal{F}\nabla f$. In this case both even and odd parts of the function f are determined. Also, the formulas for decomposition of a tangent vector field on the sphere into divergence–free and curl–free parts with the participation of operators $\mathcal{F}$ and $\mathcal{S}$ are derived. In the process of obtaining and proving all formulas, the spherical multipliers approach is used.

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