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MMS-TYPE PROBLEMS FOR JOHNSON SCHEME

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Abstract. In the current work we consider the minimization problems for the number of nonzero or negative values of vectors from the first and second eigenspaces of the Johnson scheme respectively. The topic is a meeting point for generalizations of the Manikam-Miklós-Singhi conjecture and the minimum support problem for the eigenspaces of the Johnson graph, asymptotically solved in [16].

Keywords: eigenspace, equitable partition, MMS-conjecture, Johnson scheme, Eberlein polynomials.

1. Introduction

Let V be an eigenspace of a symmetric association scheme $(X, \{R_0, \dots, R_d\})$ of rank d. Following [3], given an eigenvector $v \in V$ denote by $X_+(v) = \{x \in X : v_x > 0\}$, $X_-(v) = \{x \in X : v_x < 0\}$, $X_0(v) = \{x \in X : v_x = 0\}$. For a finite set of n elements, a pair of its subsets of cardinality w are in ith relation, if their intersection is of size $w-i$. The w-element subsets of $\{1, \dots, n\}$ together with $w+1$ relations above define *the Johnson scheme* and the first relation defines the Johnson graph $J(n,w)$, $n \geq 2w$. The eigenvalues of the Johnson scheme are known as the values of the Eberlein polynomials $E_k(i,w,n) = \sum_{j=0}^{k}(-1)^j \binom{i}{j}\binom{w-i}{k-j}\binom{n-w-i}{k-j}$, $k, i \in \{0, 1, \dots w\}$. For this scheme (graph) by V_i, we denote the eigenspace corresponding to the eigenvalue $\lambda_i(n,w) = E_1(i,w,n) = (w-i)(n-w-i) - i$ for $i \in \{0, 1, \dots w\}$.

In the current correspondence, we consider the following two characteristics for the Johnson scheme $J(n,w)$:

$$m_i^-(n,w) = \min_{v : v \in V_i, X_0(v) = \emptyset} |X_-(v)|,$$

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$$m_i^0(n,w) = \min_{v: v\in V_i, v\not\equiv 0} |X_+(v)| + |X_-(v)|.$$

When $i=1$, the first number was suggested to be $\binom{n-1}{w-1}$, for $n \geq 4w$, which is known as the Manikam-Micklos-Singhi conjecture [12], [13]. There are recent works with quadratic [1] and linear improvements [15].

For a vector v by the support of this vector we mean the value $X_+(v) + X_-(v)$. The number $m_i^0(n,w)$ was shown to be equal to $2^i\binom{n-2i}{w-i}$ (along with the description of vectors attaining the bound) for sufficiently large n in [16]. In the paper we focus on the case when $i=1$ and show that for any n and w the minimum of the support of vectors from the first eigenspace of $J(n,w)$ is attained on the vectors from two classes having rather simple structure (see Section 3).

Bier and Delsarte [4] proposed to investigate the invariant $\min\limits_{v: v\in V_i, X_0(v)=\emptyset} |X_-(v)|$ for classical association schemes with further generalizations where v is from the direct sum of several eigenspaces. They obtained several bounds involving such well-known combinatorial concepts as coverings, completely regular codes, additive codes and designs. The current study is motivated by a recent progress in the area of completely regular codes and equitable partitions in Johnson graphs. In particular, the characterizations of equitable 2-partitions of $J(n,3)$ in [9] for odd n, completely regular codes in $J(n,w)$ with the eigenvalue λ_2 having nontrivial minimum distance were characterized by Martin in [14].

An eigenvector u of the antipodal Johnson graph $J(2w,w)$ corresponding to λ_i is such that its absolute values on the pairs of antipodal vertices are equal and signs are the same or opposite depending on the parity of i [6][p. 142-143]. So, in case of odd i we have that $m_i^-(2w,w) = \binom{2w-1}{w-1}$.

In [4], it was shown that

$$\frac{\binom{n}{w}}{|D|} \leq m_i^-(n,w) \leq |C|, \tag{1}$$

where C and D are codes (subsets of the vertices of $J(n,w)$), whose characteristic functions belong to $V_i \oplus V_0$ and $\bigoplus\limits_{j\in\{0,\ldots,w\}\setminus i} V_j$ respectively. Subtracting a constant vector from the characteristic function of C, we see that there is a two-valued eigenvector v from V_i such that $v_x \neq v_y$ iff $x \in C$, $y \notin C$. In other words, $(C,\overline{C})$ is an equitable 2-partition of $J(n,w)$. If there is a $(w-1)$-$(n,w,1)$-design C, then its size is the value for $m_w^-(n,w)$. Indeed, such a design produces an equitable 2-partition $(C,\overline{C})$ of $J(n,w)$ with eigenvalue $\lambda_w(n,w)$, see [14]. On the other hand the "anticode"D could be chosen to be the set $\{x : y \subset x\}$ where y is a $(w-1)$-element subset. The set D is a Delsarte clique in the Johnson graph, which is a completely regular code with eigenvalues $\lambda_0,\ldots,\lambda_{w-1}$; so, the characteristic function of D is orthogonal to V_w [5]. The smallest open case is $i=2$, $w=3$, because $m_1^-(n,2)$ was shown to be $\lceil n/2\rceil$ in [5]. Again, for $n = 1,3(mod\ 6)$, $w=3$, the best known "anticode"D from $V_0\oplus V_1\oplus V_3$ is a Steiner triple system. So from (1) we have that

$$n-2 \leq m_2^-(n,3). \tag{2}$$

The bound (2) could be tightened up to $2n-9$ by considering a modification of the weight distribution lower bound [11] with a generalization for arbitrary w, which we discuss in Section 4.1. The choice of C in (1) is generalized to be a part of an equitable partition with appropriate eigenvalue. This gives an upper bound in case of $J(n,3)$ for odd n and $i = 2$ (see Section 4.2), where no equitable 2-partitions exist [9]. For even n, the upper bound (1) from equitable 2-partitions of $J(n,3)$ is $n(n-2)/2$.

2. Definitions and Preliminaries

2.1. **Equitable partitions.** Let G be an undirected graph. An *equitable r-partition* with parts $C_1, \dots, C_r$ of the vertex set of G is called equitable if for any $i, j \in \{1, \dots, r\}$ a vertex from C_i has exactly A_{ij} neighbors in C_j. The matrix $A = (A_{ij})_{i,j\in\{1,\dots,r\}}$ is called the *quotient matrix*. An eigenvalue of the quotient matrix A is called *an eigenvalue* of the partition. Given an eigenvector u of A corresponding to an eigenvalue λ define u^G to be the vector indexed by the vertices of G such that $u^G_x = u_i$, if $x \in C_i$. The vector u^G is an eigenvector of the adjacency matrix of G corresponding to λ [7][§4.5]. In view of the said above, the upper bound in (1) is generalized as follows:

Proposition 1. *Let u be an eigenvector without zero entries of the quotient matrix of an equitable partition of the Johnson graph $J(n,w)$ with parts $C_1, \dots, C_r$. Then*

$$m_i^-(n,w) \le \sum_{j: u_j < 0} |C_j|.$$

2.2. **The first eigenspace of $J(n,w)$.** Consider the eigenvectors of the complete graph $K_n = J(n,1)$ with vertices indexed by integers from $\{1, \dots, n\}$. The graph has two eigenvalues: $n-1$ and -1. An eigenvector $a = (\alpha_1, \dots, \alpha_n)$ of the graph corresponding to the eigenvalue -1 could be characterized as a solution for the equation: $\alpha_1 + \dots + \alpha_n = 0$. We use the following isomorphism between the $\lambda_1(n,w)$-eigenspace of $J(n,w)$ and the $\lambda_1(n,1)$-eigenspace of $J(n,1)$ established by the inclusion mapping I (see [8]): the image $I(a)$ is such that $(I(a))_x = \sum\limits_{i\in x} \alpha_i$.

Consider the following two equitable 2-partitions of $J(n,w)$: $(\{x : 1 \in x\}, \{x : 1 \notin x\})$ and $(\{x : 2 \in x\}, \{x : 2 \notin x\})$ [14]. Denote by $v^{1,2}$ the difference of the eigenvectors of $J(n,w)$ arising from these partitions:

$$v^{1,2}_x = \begin{cases} 1, & 1 \in x \text{ and } 2 \notin x, \\ -1, & 1 \notin x \text{ and } 2 \in x, \\ 0, & \text{otherwise.} \end{cases}$$

In [16], it was shown that the minimum-support eigenvectors from the first eigenspace are exactly $v^{1,2}$ up to appropriate permutation of coordinate positions starting with large enough n (as well as a generalization of the result for any eigenspace). It is easy to see that $v^{1,2}$ is $I(e_1 - e_2)$, where e_1, e_2 are 1-st and 2-nd vectors of the standard basis.

In Section 3 we extend results from [16] in further details. We show that for any n the minimum-support eigenvector is either $v^{1,2}$ or $I(a)$, where a is a two-valued (-1)-eigenvector of $J(n,1)$.

3. Minimum support λ_1-eigenvectors

Theorem 1. *Let v be a λ_1-eigenvector of $J(n,w)$, $n \ge 2w$, $w \ge 2$, with minimum support. Then v is $I(e_1 - e_2)$ or $I(\sum_{i=1}^k e_i - \frac{k}{n-k}\sum_{i=k+1}^n e_i)$ for some*

$$k \in \{2, 3, \dots, n-2\}$$

such that $\frac{kw}{n} \in \mathbb{N}$ up to a permutation of coordinate positions and the multiplication by a scalar. In particular,

$$m_1^0(n,w) = \min\left(2\binom{n-2}{w-1}, \binom{n}{w} - \max_{k\in\{2,3,\dots,n-2\},\, \frac{kw}{n}\in\mathbb{N}} \binom{k}{\frac{kw}{n}}\binom{n-k}{\frac{(n-k)w}{n}}\right)$$

Proof. As it was mentioned above, every λ_1-eigenvector equals $I(a)$ for some vector $a = (\alpha_1,\dots,\alpha_n)$ such that $\alpha_1+\dots+\alpha_n = 0$. Our next goal is to determine values $\alpha_1,\dots,\alpha_n$ for which the support of $I(a)$ is minimal. Let $v = I(a)$ be a λ_1-eigenvector with minimum support. Since the vector $I(e_1-e_2)$ has the size of the support equal $2\binom{n-2}{w-1}$, we shall assume that the size of the support of $I(a)$ is not more than $2\binom{n-2}{w-1}$. Let us denote by m the size of $\{\alpha_1,\dots,\alpha_n\}$. There are two different cases:

$m \geq 3$. Without loss of generality we can assume that α_1, α_2 and α_3 are pairwise different. Take arbitrary subsets A_1, A_2 of the set $\{4,\dots,n\}$ of cardinalities $w-1$ and $w-2$ respectively. Clearly, there are at least 2 nonzero values among $I(a)_{i\cup A_1} = \alpha_i + \sum_{k\in A_1}\alpha_k$, $i = 1,2,3$ and at least 2 nonzero values among $I(a)_{\{i,j\}\cup A_2} = \alpha_i+\alpha_j+\sum_{k\in A_2}\alpha_k$, $i,j\in\{1,2,3\}$, $i\neq j$. So, the support of v is at least $2\binom{n-3}{w-1}+2\binom{n-3}{w-2} = 2\binom{n-2}{w-1}$. By hypothesis, the vector $I(a)$ has minimal size of the support; so, we conclude that $I(a)_x = 0$ for any w-subset x of $\{4,\dots,n\}$. In other words, $I'(a')$ is the zero vector, where a' is obtained from a by removing its first 3 entries, I' is the inclusion mapping from $J(n-3,1)$ to $J(n-3,w)$.

We have that $\sum_{i=4,\dots,n}\alpha_i = \frac{\sum_{x\subset\{4,\dots,n\},|x|=w} I'(a')_x}{\binom{n-4}{w-1}} = 0$. Therefore, the vector $a' = (\alpha_4,\dots,\alpha_n)$ belongs to $V_1(n-3,1)$ and is the zero vector because $I'(a')$ is the zero vector and I' is an isomorphism from $V_1(n-3,1)$ to $V_1(n-3,w)$.

From the above, there are exactly 2 nonzero values among $\alpha_1+\alpha_2$, $\alpha_1+\alpha_3$, $\alpha_2+\alpha_3$. Consequently, we can consider $\alpha_3 = 0$ and $\alpha_1 = -\alpha_2$, which means that v is equal to $cI(e_1-e_2)$ for some constant c.

$m = 2$. Without loss of generality we can take $\alpha_1 = \alpha_2 = \dots = \alpha_k = \hat{\alpha}$ and $\alpha_{k+1} = \alpha_{k+2} = \dots = \alpha_n = \hat{\beta}$ for some integer k such that $2 \leq k \leq n-2$. Using the equality $\alpha_1+\dots+\alpha_n = 0$, we have $\hat{\beta} = -\hat{\alpha}\frac{k}{n-k}$. Let us take an arbitrary vertex x of $J(n,w)$. It is easy to see that $I(a)_x = 0$ if and only if x has exactly $\frac{kw}{n}$ ones in the first k coordinate positions and $w - \frac{kw}{n}$ ones in the rest $n-k$ coordinate positions. Particularly, $\frac{kw}{n}$ must be an integer. Therefore, the support of v equals $\binom{n}{w} - \binom{k}{\frac{kw}{n}}\binom{n-k}{\frac{(n-k)w}{n}}$. Taking the minimum of this expression over all admissible k we obtain the statement of the theorem.

In the case $m = 1$, we automatically obtain the all-zero eigenvector v, which is not possible.

□

Theorem 1 reduces the problem of minimizing $m_1^0(n,w)$ to the comparison of two expressions containing binomial coefficients.

4. Bounds on $m_i^-(n,w)$

4.1. **A lower bound on $m_i^-(n,w)$.** Let v be an eigenvector of the Johnson graph $J(n,w)$ without zero entries; let x be a vertex such that v_x is negative and takes maximum absolute value over all negative entries of v. Consider the distance partition $(C_0,\dots,C_w)$ of the vertices of $J(n,w)$ with respect to the vertex x. It is well-known that the sum of the entries of v over C_k is expressed using the Eberlein polynomials and the value v_x:

$$\sum_{y\in C_k} v_y = v_x E_k(i,w,n).$$

Let $E_k(i,w,n)$ be non-negative. Then by the choice of v_x with the maximum absolute value we see that there are at least $|E_k(i,w,n)|$ negative values for v_y in C_k. Moreover, there are more than $|E_k(i,w,n)|$ negative v_y's not less then v_x, because there is at least one positive v_y in C_k, since obviously $|E_k(i,w,n)| < |C_k|$ for $k>0$. Thus we obtain the following bound.

Theorem 2. $m_i^-(n,w) \geq 1 + \sum\limits_{k>0:E_k(i,w,n)\geq 0} (|E_k(i,w,n)|+1)$.

The consideration for the proof above is similar to the one for the weight distribution bound on the number of nonzeros for the eigenvector of distance-regular graph, see [11]. The values of the Eberlein polynomials for $i=2$ and $w=3$ are as follows $E_0(2)=1$, $E_1(2)=n-7$, $E_2(2)=11-2n$, $E_3(2)=n-5$. Therefore, we have the bound below.

Corollary 1. $m_2^-(n,3) \geq 2n-9$.

4.2. **An upper bound on $m_2^-(n,3)$.** Let n be $2r$. The following construction could be found in [10] (see also [2]). Consider the complement of a perfect matching on vertices labeled with $\{1,\dots,2k\}$ to a complete bipartite graph. Then the triples of vertices are partitioned into three orbits C_1, C_2, C_3 with respect to the action of the automorphism group of the graph. The triples of C_1 consist of vertices belonging to the same part; the triples of C_2 induce a walk of length 2 in the graph; the triples of C_3 contain exactly one pair of adjacent vertices. Any two parts could be merged and result in equitable 2-partition of the triples, e.g. the Johnson graph $J(2r,3)$ [2]. In particular, the partition $(C_1' = C_1\cup C_2,\ C_2' = C_3)$ has the following quotient matrix:

$$\begin{pmatrix} 3(2r-5) & 6 \\ 4(r-2) & 2r-1 \end{pmatrix},$$

whose eigenvalues are $\lambda_0(n,3)$ and $\lambda_2(n,3)$. The cells of the partition are in $4(r-2)$ to 6 ratio; so, in view of Proposition 1, we see that

$$m_2^-(n,3) \leq n(n-2)/2.$$

Let n be $2r+1$. Consider the graph G with $2r+1$ vertices which is the union of an isolated vertex and the graph G' which is equal to a complete bipartite graph $K_{r,r}$ without a perfect matching. We have the following orbits of triples of vertices:

- C_1**:** the vertices of the triple are in one part of G';
- C_2**:** the vertices of the triple induce a walk of length 2 in G';
- C_3**:** the vertices of the triple belong to G' and contain only two adjacent vertices;
- C_4**:** two nonadjacent vertices belong to different parts of G' and the third one is isolated;

C_5**:** two vertices are in one part of G' and the third one is isolated;
C_6**:** two vertices are adjacent and the third one is isolated.

The equitable partition $(C_1,\ldots,C_6)$ of $J(2r+1,3)$ has the following quotient matrix:

$$\begin{pmatrix} 3(r-3) & 3(r-2) & 6 & 0 & 3 & 0 \\ r-2 & 5r-13 & 6 & 0 & 1 & 2 \\ r-2 & 3(r-2) & 2r-1 & 1 & 1 & 1 \\ 0 & 0 & 2(r-1) & 0 & 2(r-1) & 2(r-1) \\ r-2 & r-2 & 2 & 2 & 2(r-2) & 2(r-1) \\ 0 & 2(r-2) & 2 & 2 & 2(r-1) & 2(r-2) \end{pmatrix}.$$

The matrix has the eigenvector $(3,3,4-2r,2-2r,1,1)$ corresponding to eigenvalue $\lambda_2(2r+1,3)=2r-6$. By Proposition 1, we see that

$$m_2^-(n,3) \le |C_3|+|C_4| = 2r(r-1)+r = (n-1)(n-2)/2.$$

Thus we obtain

Theorem 3.

$$m_2^-(n,3) \le \begin{cases} n(n-2)/2, & \textit{if } n \textit{ is even;} \\ (n-1)(n-2)/2, & \textit{if } n \textit{ is odd.} \end{cases}$$

5. Conclusion

Theorem 1 reduces the problem of finding $m_1^0(n,w)$ to the determination which one of values

$$\left(\binom{n}{w} - \max_{k\in\{2,3,\ldots,n-2\},\,\frac{kw}{n}\in\mathbb{N}} \binom{k}{\frac{kw}{n}}\binom{n-k}{\frac{(n-k)w}{n}}\right) \quad \text{or} \quad 2\binom{n-2}{w-1}$$

is smaller. In [16], it was shown that the second one is the answer for all n starting from some value $n_0(w)$. We have compared these values for $6 \le n \le 600$ and $3 \le w \le \frac{n}{2}$ and consequently found corresponding $m_1^0(n,w)$. Based on these computational results, we state the following conjecture:

Conjecture 1. *For $w \ge 5$ and $n \ge 2w+1$ the following identity holds*

$$m_1^0(n,w) = 2\binom{n-2}{w-1}.$$

For $w<5$, we have found several curious examples:

(1) $m_1^0(6,2)=6$ is attained on the vector $v=I(e_1+e_2+e_3-e_4-e_5-e_6)$,
(2) $m_1^0(8,2)=12$ is attained on vectors $v=I(e_1+e_2+e_3+e_4-e_5-e_6-e_7-e_8)$ and $u=(e_1-e_2)$,
(3) $m_1^0(9,3)=39$ is attained on the vector $v=I(2e_1+2e_2+2e_3-e_4-e_5-e_6-e_7-e_8-e_9)$,
(4) $m_1^0(10,4)=110$ is attained on the vector $v=I(e_1+e_2+e_3+e_4+e_5-e_6-e_7-e_8-e_9-e_{10})$.

Let us notice that it is not hard to show using Theorem 1 and basic properties of binomial coefficients that

$$m_1^0(2w,w) = \binom{2w}{w} - 2\binom{2w-2}{w-1},$$

which is attained on the vector $I((w-1)(e_1+e_2)-\sum_{i=3}^{2w} e_i)$.

In Theorem 3, we described a construction providing a quadratic upper bound for the characteristic $m_2^-(n,3)$, $n \to \infty$. At the same time, Corollary 1 gives us a lower bound that is linear in n. The real behaviour of the growth rate of $m_2^-(n,3)$ remains to be an intriguing open problem.

The characteristic $\underset{v:v\in V_i, X_0(v)=\emptyset}{min} |X_-(v)|$, considered by Bier and Delsarte [4], requires that v does not have zero entries. It may be interesting in the future research to remove this condition and try to find this value in this case for classical association schemes.

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References

[1] N. Alon, H. Huang, and B. Sudakov, *Nonnegative k-sums, fractional covers, and probability of small deviations*, Journal of Combinatorial Theory, Series B , **102:3** (2012), 784–796. MR2900818

[2] S.V. Avgustinovich, I.Yu. Mogil'nykh, *Perfect colorings of the Johnson graphs J(8,3) and J(8,4) with two colors*, Journal of Applied and Industrial Mathematics, **5:1** (2011), 19–30. DOI 10.1134/S1990478911010030, translated from Diskretnyi Analiz i Issledovanie Operatsii, **17:2** (2010), 3–19. MR2682086

[3] T. Bier, *A distribution invariant for association schemes and strongly regular graphs*, Linear Algebra and its Applications, **57** (1984), 105–113. MR0729265

[4] T. Bier, P. Delsarte, *Some bounds for the distribution numbers of an association scheme*, European Journal of Combinatorics, **9:1** (1988), 1–5. MR0938815

[5] T. Bier, *Some distribution numbers of the triangular association scheme*, European Journal of Combinatorics, **9:1** (1988), 19–22. MR0938817

[6] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-regular graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Springer-Verlag, Berlin, 1989. MR1002568

[7] D.M. Cvetkovic, M. Doob, H. Sachs, *Spectra of graphs*, Academic Press, New York - London, 1980. MR0572262

[8] P. Delsarte, *An Algebraic Approach to the Association Schemes of Coding Theory*, Philips Res. Rep. Suppl. No. 10, 1973. MR0384310

[9] A.L. Gavrilyuk, S.V. Goryainov, *On perfect 2-colorings of Johnson graphs J(v,3)*, Journal of Combinatorial Designs, **21:6** (2013), 232–252. MR3150904

[10] C. Godsil, C. Praeger, *Completely transitive designs*, (2014) https://arxiv.org/abs/1405.2176.

[11] D.S. Krotov, I.Yu. Mogilnykh, V.N. Potapov, *To the theory of q-ary Steiner and other-type trades*, Discrete Mathematics, **339:3** (2016), 1150–1157. MR3433920

[12] N. Manickam and D. Miklós, *On the number of non-negative partial sums of a non-negative sum*, Proc. 7th Hung. Colloq., Eger/Hung. 1987, Colloq. Math. Soc. J?nos Bolyai, **52** (1988), 385–392. Zbl 0726.11014

[13] N. Manickam and N.M. Singhi, *First distribution invariants and EKR theorems*, Journal of Combinatorial Theory, Series A , **48:1** (1988), 91–103. MR0938860

[14] W.J. Martin, *Completely regular designs of strength one*, Journal of Algebraic Combinatorics, **3:2** (1994), 177–185. MR1268574

[15] A. Pokrovskiy, *A linear bound on the Manickam-Miklós-Singhi conjecture*, Journal of Combinatorial Theory, Series A, **133** (2015), 280–306. MR3325636

[16] K. Vorob'ev, I. Mogilnykh, A. Valyuzhenich, *Minimum supports of eigenfunctions of Johnson graphs*, Discrete Mathematics, **341:8** (2018), 2151–2158. MR3810266

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