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COMPUTATION OF THE CENTRALIZER DIMENSION OF GENERALIZED BAUMSLAG-SOLITAR GROUPS

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Abstract. A finitely generated group G acting on a tree so that all vertex and edge stabilizers are infinite cyclic groups is called a generalized Baumslag-Solitar group (GBS group). The centralizer dimension of a group G is the maximal length of a descending chain of centralizers. In this paper we complete a description of centralizers for unimodular GBS groups. This allows us to find the centralizer dimension of all GBS groups and to establish a way to compute it.

Keywords: centralizer of set of elements, centralizer dimension, generalized Baumslag–Solitar group, Baumslag–Solitar group.

1. Introduction

A finitely generated group G acting on a tree so that all vertex and edge stabilizers are infinite cyclic groups is called a *generalized Baumslag–Solitar group* (GBS group). By the Bass–Serre Theorem, G is representable as $\pi_1(\mathbb{A})$, the fundamental group of a graph of groups $\mathbb{A}$ [1] whose vertex and edge groups are infinite cyclic.

GBS groups are important examples of JSJ decompositions. JSJ decompositions appeared first in 3-dimensional topology with the theory of the characteristic submanifold by Jaco-Shalen and Johannson. These topological ideas were carried over to group theory by Kropholler for some Poincaré duality groups of dimension at least 3, and by Sela for torsion-free hyperbolic groups. In this group-theoretical context, one has a finitely generated group G and a class of subgroups $\mathcal{A}$ (such as cyclic groups, abelian groups, etc.), and one tries to understand splittings (i.e. graph of groups decompositions) of G over groups in $\mathcal{A}$ (see [2] for details).

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With each GBS group G, we can associate a labeled graph $\mathbb{A}$, a particular case of a graph of groups. Such a labeled graph corresponds to an action of G on a tree and defines a presentation of G (more details on labeled graphs and their properties are given in [3]). The necessary definitions and properties are given in section 1.

As was observed by Robinson in [4], the GBS groups occupy central positions in combinatorial group theory due to the following properties: noncyclic GBS groups are exactly those finitely generated groups of cohomological dimension 2 having a commensurable cyclic subgroup; GBS groups are coherent (each finitely generated subgroup admits a finite presentation).

Let G be a group and let M be a subset of G. Denote by $C(M)$ the centralizer of M in G:

$$C(M) = \left\{g \in G | g^{-1}mg = m, \text{ for all } m \in M\right\}.$$

Suppose that a group G has a strictly descending chain of centralizers $C_1 \supset C_2 \supset \cdots \supset C_d$ of length d, i.e., a chain containing exactly d elements, but G does not have such a chain of length $d+1$. Then *the centralizer dimension $cdim(G)$* equals d. If there is no such number d then we put $cdim(G) = \infty$. More complete information on the centralizer dimensions of groups can be found in [5]. It is noticed in [6] that $cdim(G)$ coincides with the centralizer lattice height.

In [13], Proposition 4.1 G. Levitt proved that a GBS group G is unimodular (i.e. such GBS groups G that $\Delta(G) = \{\pm 1\}$) if and only if G contains a subgroup of finite index isomorphic to $F_n \times \mathbb{Z}$. Therefore, the results of [5] imply that unimodular GBS groups have finite centralizer dimension. However, such an approach gave no exact estimates, no examples and no way to calculate the centralizer dimension of GBS groups.

In [12] centralizers of sets of elements and centralizer dimension were described for GBS groups presented by labeled trees. In section 3 we describe the centralizers of elements for all GBS groups with $\Delta(G) = \{1\}$.

Theorem 1 *Let G be a GBS group, $\Delta(G) = \{1\}$. If $g \in G$ is not elliptic element then*

$$C_G(g) = u^{-1} \cdot (\langle r \rangle \times Z(\pi_1(\mathbb{B}_a))) \cdot u$$

for a suitable vertex element a and $u, r \in G$.
If $g \in G$ is elliptic element then

$$C_G(g) = v^{-1} \cdot \pi_1(\mathbb{B}_b) \cdot v$$

for a suitable vertex element b and $u \in G$.

In section 4 the description of centralizers of sets of elements in the case $\Delta(G) = \{1\}$ established.

Theorem 2 *Let G be a GBS group, $\Delta(G) = \{1\}$. If M is the finite set of elements from G, then $C_G(M)$ can be one of the three types:*

$$u^{-1} \cdot (\langle r \rangle \times Z(\pi_1(\mathbb{B}_a))) \cdot u,$$

$$v^{-1} \cdot \pi_1(\mathbb{B}_b) \cdot v,$$

$$w^{-1} \cdot Z(\pi_1(\mathbb{B}_c)) \cdot w,$$

for a suitable $u, v, w, r \in G$ and Z-maximal subgraphs B_a, B_b, and B_c.

In sections 6 and 7 we provide a description of centralizer dimension for unimodular GBS groups. Considering the results of section 2, theorems 3 and 4 complete description of centralizer dimension for all GBS groups.

Theorem 3 *Given a reduced labeled graph $\mathbb{A}$ such that $\pi_1(\mathbb{A})$ is non-abelian group, $\Delta(\pi_1(\mathbb{A})) = \{1\}$ and $b_1(A) = n$. Then $cdim(\pi_1(\mathbb{A}))$ is odd and*

$$3 \leqslant cdim(\pi_1(\mathbb{A})) \leqslant 2 \cdot |E(A)| + 1.$$

Moreover, for every odd $k, 3 \leqslant k \leqslant 2 \cdot m + 1$ there exists a labeled graph $\mathbb{B}_{m,n}$ with m edges such that $b_1(B_{m,n}) = n \leqslant m$, $\Delta(\pi_1(\mathbb{B}_{m,n})) = \{1\}$ and $cdim(\pi_1(\mathbb{B}_{m,n})) = k$.

Theorem 4 *Given a reduced labeled graph $\mathbb{A}$ such that $\pi_1(\mathbb{A})$ is non-abelian group, $\Delta(\pi_1(\mathbb{A})) = \{\pm 1\}$ and $b_1(A) = n$. Then $cdim(\pi_1(\mathbb{A}))$ is odd and*

$$3 \leqslant cdim(\pi_1(\mathbb{A})) \leqslant 2 \cdot |E(A)| + 3.$$

Moreover, for every odd $k, 3 \leqslant k \leqslant 2 \cdot m + 3$ there exists a labeled graph $\mathbb{B}_{m,n}$ with m edges such that $1 \leqslant b_1(B_{m,n}) = n \leqslant m$, $\Delta(\pi_1(\mathbb{B}_{m,n})) = \{\pm 1\}$ and $cdim(\pi_1(\mathbb{B}_{m,n})) = k$.

Since the proofs are constructive, we do not just describe the centralizer dimension for GBS groups, but also establish a way to compute it.

Remark 5 *Given a labeled graph $\mathbb{A}$. There is an algorithm to compute $cdim(\pi_1(\mathbb{A}))$.*

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2. Preliminaries

A graph A is the vertex set $V(A)$, the edge set $E(A)$, the mappings $\alpha, \omega\colon E(A) \to V(A)$, are sending an edge to its beginning and end, and an inversion $^{-}\colon E(A) \to E(A)$ such that $\alpha(\overline{e}) = \omega(e), \omega(\overline{e}) = \alpha(e)$, $\overline{\overline{e}} = e$, $\overline{e} \neq e$. *An edge path* is a sequence of edges $p = (e_1, e_2, \ldots, e_k)$ such that $\alpha(e_{i+1}) = \omega(e_i)$ for $i = 1, 2, \ldots, k-1$.

If A is a tree then for every two vertices a and b there exists a unique shortest path with beginning a and end b. We will refer to this path as *geodesic* and denote it by $a - b$.

Given a GBS group G, we can present the corresponding graph of groups by a labeled graph $\mathbb{A} = (A, \lambda)$, where A is a finite connected graph and $\lambda\colon E(A) \to \mathbb{Z} \setminus \{0\}$ labels the edges of A. The label λ_e of an edge e with the origin v defines an embedding $\alpha_e\colon e \to v^{\lambda_e}$ of the cyclic edge group $\langle e \rangle$ into the cyclic vertex group $\langle v \rangle$ (for more details see [3])

The fundamental group $\pi_1(\mathbb{A})$ of a labeled graph $\mathbb{A} = (A, \lambda)$ is defined by generators and defining relations. Denote by $\overline{A}$ the graph obtained from A by identifying e and $\overline{e}$. The maximal subtree T in $\overline{A}$ defines the presentation of the group $\pi_1(\mathbb{A})$

$$\left\langle \begin{array}{ll} g_v, v \in V(\overline{A}), & | \quad g_{\alpha(e)}^{\lambda(e)} = g_{\omega(e)}^{\lambda(\overline{e})}, e \in E(T), \\ t_e, e \in E(\overline{A}) \setminus E(T) & | \quad t_e^{-1} g_{\alpha(e)}^{\lambda(e)} t_e = g_{\omega(e)}^{\lambda(\overline{e})}, e \in E(\overline{A}) \setminus E(T) \end{array} \right\rangle.$$

For different maximal subtrees, the corresponding presentations define isomorphic groups. Denote the number $|E(\overline{A}) \setminus E(T)|$ of generators of second type by $b_1(A)$. That is a first Betti number of graph A. If A is a tree then $\pi_1(\mathbb{A})$ admits the presentation

$$\left\langle g_v, v \in V(\overline{A}), | g_{\alpha(e)}^{\lambda(e)} = g_{\omega(e)}^{\lambda(\overline{e})}, e \in E(T) \right\rangle$$

In what follows, we for convenience denote by v the vertex of the graph as well as the corresponding generator g_v of the fundamental group. To each connected subgraph B of a graph A, there naturally corresponds the labeled graph $\mathbb{B}$, where the natural homomorphism $\pi_1(\mathbb{B}) \to \pi_1(\mathbb{A})$ is an embedding.

A group is said to be *Hopfian* if any homomorphism of the group onto itself has trivial kernel, i.e. is an automorphism. Baumslag and Solitar [8] came up with a series of examples of two-generator one-relator non-Hopfian groups. In particular, such are the Baumslag–Solitar groups

$$BS(p,q) = \langle x, y | xy^p x^{-1} = y^q \rangle$$

where p and q are coprime integers, $p, q \neq 1$.

If a labeled graph $\mathbb{B}$ consists of one vertex and two inverse loops with labels p and q, then $\pi_1(\mathbb{B}) \cong BS(p,q)$. Therefore, every Baumslag–Solitar group is a generalized Baumslag–Solitar group.

It is sometimes useful to regard a GBS-group as a group obtained as follows: start with the group $\mathbb{Z}$, perform consecutive amalgamated products in accordance with the labels on the maximal subtree; finally, apply several times the construction of the HNN-extension (the number of times is equal to the number of the edges outside the maximal tree). In this approach, the standard theory of amalgamated products and HNN-extensions is applicable to the full extent. In particular, GBS-groups admit a normal form of an element and have no torsion.

We say that word w in generators of group $\pi_1(\mathbb{A})$ is *reduced*, if it is reduced as a word of HNN-extension with respect to all generators of second type. In other words, the word w *is reduced* if it can not be written using a smaller number of generators of second type. The word w is called *cyclically reduced*, if all cyclic permutations (in the usual sense of HNN-extensions) of w are reduced.

Given the generator of second type t and reduced word w, the number of occurrences of symbols t, t^{-1} in w is called *t-length* of w and denoted by $|w|_t$. If the value of t is clear from the context, then we write $|w|$ and call this number the *length* of the word w. Such a notation is well-defined because $\pi_1(\mathbb{A})$ is an HNN-extension with a stable letter t (here any generator of second type can be taken). We will also use the right normal form (see [9]), considering the group $\pi_1(\mathbb{A})$ as an HNN-extension.

In accordance with [7], call an element *elliptic* if it is conjugate to an element of $\langle a \rangle$ for some $a \in V(A)$; otherwise, the element is called *hyperbolic*. An elliptic element is called a *vertex* element if it belongs to $\langle a \rangle$ for some a $a \in V(A)$. The subgroup generated by all vertex elements is denoted by E. It coincides with $\pi_1(\mathbb{T})$. Given $g \in E$ denote by S_g (see [12]) the minimal subtree of the tree T such that $g \in \pi_1(\mathbb{S}_g)$.

If two labeled graphs $\mathbb{A}$ and $\mathbb{B}$ define isomorphic GBS groups $\pi_1(\mathbb{A}) \cong \pi_1(\mathbb{B})$ and $\pi_1(\mathbb{A})$ is not isomorphic to $\mathbb{Z}$ and $\mathbb{Z}^2$ or to the Klein bottle group $\langle a, b | a^{-1}ba = b^{-1} \rangle$, then there exists a finite sequence of expansions and collapses (Fig. 1) joining $\mathbb{A}$ and $\mathbb{B}$ [7] (in Fig. 1, to each edge there correspond two integers $\lambda(e), \lambda(\overline{e})$). A labeled graph is called *reduced* if it does not admit collapses (this means that the labeled graph does not contain an edge with distinct endpoints and labels ± 1).

Given a reduced labeled graph $\mathbb{A}$ and B, C some subgraphs of the graph A. It is easy to prove that inclusion $\pi_1(\mathbb{B}) \subset \pi_1(\mathbb{C})$ holds if and only if $B \subset C$.

Unless otherwise specified, we assume further that $\mathbb{A}$ is a reduced labeled graph and $G = \pi_1(\mathbb{A})$ is the corresponding GBS group.

Define the modular homomorphism $\Delta\colon G \to \mathbb{Q}^*$. Given $g \in G$, choose an arbitrary nontrivial elliptic element $a \in G$. Then, for some integers m and n not equal to 0, we have $g^{-1}a^m g = a^n$. In this case, we put $\Delta(g) = \frac{m}{n}$. It is not hard to prove that Δ is well-defined. The modular homomorphism plays an important role

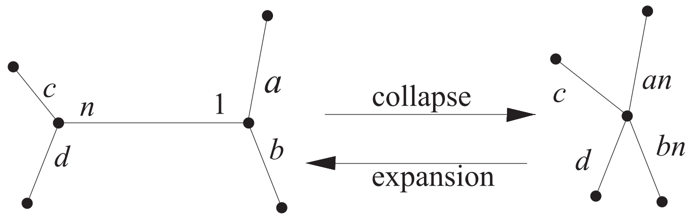


FIG. 1. Expansion and collapse.

in study of GBS groups. If A is a tree then $\Delta(\pi_1(\mathbb{A})) = \{1\}$ (details and proofs can be found, for example, in [7]).

3. Centralizer dimension: case $\Delta(G) \not\subseteq \{\pm 1\}$

If $\Delta(G) \not\subseteq \{\pm 1\}$ then either $\frac{p}{q} \in \Delta(G)$ for coprime integers $p, q \notin \{0, 1, -1\}$ and in this case $cdim(G) = \infty$ [11] or $\Delta(G)$ is generated by $\langle n \rangle$ as subgroup of $\mathbb{Q}^*$ under multiplication.

Lemma 3.1 *Given a GBS group G such that $\Delta(G) = \langle n \rangle$, $n \neq \pm 1$. If there exist vertex element a, $t \in G$ and $k \geqslant 2$ such that $t^{-1} \cdot a^k \cdot t = a^{k \cdot n}$ and for all $|l| < k$ word $t^{-1} \cdot a^l \cdot t$ is reduced in G then $cdim(G) = \infty$.*

PROOF. Denote by a_i element $t^{-i} \cdot a \cdot t^i$. Then $C_G(a^{k \cdot n^s}) \supseteq \langle a, a_1, \ldots, a_s \rangle$, but $a_{s+1} \notin C_G(a^{k \cdot n^s})$. Otherwise

$$t^{-s-1} \cdot a^{-1} \cdot t^{s+1} \cdot a^{-k \cdot n^s} \cdot t^{-s-1} \cdot a \cdot t^{s+1} \cdot a^{k \cdot n^s} = 1,$$

but, using the condition of the lemma s times, we get

$$t^{-s-1} \cdot a^{-1} \cdot t \cdot a^{-k} \cdot t^{-1} \cdot a \cdot t^{s+1} \cdot a^{k \cdot n^s} = 1.$$

This equation is impossible since the left-hand side is reduced and, consequently, is not equal to 1 by the Britton lemma [9]. The lemma is proved.

Theorem 3.2 *Given a GBS group G such that $\Delta(G) = \langle n \rangle$, $n \neq \pm 1$. If $G \ncong BS(1, n)$ then $cdim(G) = \infty$. The centralizer dimension of $BS(1, n)$ is equal to* 3.

PROOF. At first, suppose that $G \ncong BS(1, n)$ and G is represented by a reduced labeled graph $\mathbb{A}$.

If A has more than one vertex then the conditions of lemma 3.1 are satisfied. As t, we need to take any word in generators of the second type such that $\Delta(t) = n$, and as a a suitable vertex of the graph A. Such a vertex exists, otherwise for all $v \in V(A)$ we have $t^{-1} \cdot v \cdot t = v^n$. In this case, all the vertex elements belong to the vertex group $\langle w \rangle$, where w is the beginning of the edge corresponding to the first stable letter of t. Therefore $\mathbb{A}$ can not be reduced; a contradiction.

If A has a single vertex a and the number of edges of the graph A is greater than one, then we denote two distinct edges and the corresponding generators of the second type by t, r. We can assume that $\Delta(t) = m = n^p \neq \pm 1, \Delta(r) = s = n^d$. Denote by $b = a^q$ the minimal power of a such that $t^{-1}bt = b^m$. Then $C_G(b^{m^k \cdot s}) \supseteq C_G(b^{m^l \cdot s})$, for $k \geqslant l$. Moreover, $t^{-k} \cdot r^{-1} \cdot t^k \cdot a \cdot t^{-k} \cdot r \cdot t^k$ belongs to $C_G(b^{m^k \cdot s})$ and does not belong to $C_G(b^{m^l \cdot s})$.

If A has a single vertex a and a single edge t, but $G \ncong BS(1, n)$, then $G \cong BS(k, kn), k \neq \pm 1$. In this case we can apply lemma 3.1.

Now we need to prove that $cdim(BS(1,n)) = 3$. Note that the group

$$BS(1,n) = \langle a, t | t^{-1} \cdot a \cdot t = a^n \rangle$$

is isomorphic to a subgroup M of the group T of upper triangular nondegenerate rational 2×2 matrices with 1 at the place (1,1). The isomorphism φ is given on generators

$$\varphi\colon a \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b \to \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$

Therefore $cdim(BS(1,n)) = cdim(M) \leqslant cdim(T)$ by lemma 2.2 from [5]. Since every matrix in T is given by a pair of rational numbers (one of which is not equal to 0) and the centralizer of the set of elements of T is given by a system of linear equations, we get $cdim(T) \leqslant 3$ (the maximum number of embedded subspaces). Therefore $cdim(BS(1,n)) \leqslant 3$. Since $BS(1,n)$ is non-abelian and the centralizer dimension can not be equal to 2 [6], then $cdim(BS(1,n)) = 3$. The Theorem is proved.

4. Centralizers of elements: case $\Delta(G) = \{1\}$

Given a labeled graph $\mathbb{A}$ such that $\Delta(\mathbb{A}) = \{1\}$. The group $\pi_1(\mathbb{A})$ is given by a presentation depending on the choice of the maximal subtree T in the graph A. However, the groups $\pi_1(\mathbb{A}, T_1)$ and $\pi_1(\mathbb{A}, T_2)$ are isomorphic, see, for example, [10].

If B is a connected subgraph of the graph A, then we can choose the maximal subtrees T_A and T_B of the graphs A and B such that $T_B \subseteq T_A$. Such a pair of maximal subtrees is called *coherent*.

Remark 4.1 *If B is a connected subgraph of the graph A and $T_B \subseteq T_A$, then the homomorphism $\pi_1(\mathbb{B}, T_B) \to \pi_1(\mathbb{A}, T_A)$ identical on generators is an embedding.*

PROOF. Induction on the number of edges outside B, using the classical embedding results for a free product with amalgamation and HNN-extensions (see, for example, [9]). The remark is proved.

Definition 4.2 Let $a = v^k, v \in V(A)$ be a vertex element from $\pi_1(\mathbb{A})$, define a subgraph B_a of the graph A by induction:

Base: $V(B_a) = \{v\}, E(B_a) = \emptyset$.

Induction step: Suppose we already have a graph B_a. Consider a set of edges $\{e \in E(A) | \alpha(e) \in V(B_a)\} \setminus E(B_a)$ and denote it by U. Then for any $e \in U$ there exist an element $r \in \pi_1(\mathbb{B}_a)$ and an integer $l(e)$ such that $a = v^k = r^{-1} \cdot \alpha(e)^{l(e)} \cdot r$. If $\lambda_e | l(e)$ for some $e \in U$ then we attach e and $\overline{e}$ to the graph B_a and go to the next induction step. Otherwise, the induction stops.

Note that if we add an edge e then $\alpha(e)^{l(e)} = \omega(e)^{l(e) \cdot \frac{\lambda_{\overline{e}}}{\lambda_e}}$ for $e \in E(T_A)$ or $\alpha(e)^{l(e)} = t^{\epsilon} \cdot \omega(e)^{l(e) \cdot \frac{\lambda_{\overline{e}}}{\lambda_e}} \cdot t^{-\epsilon}$ for $e \notin E(T_A)$. In addition, we note that if for some vertex w we get that on the one hand $a = r_1^{-1} \cdot w^m \cdot r_1$ and on the other hand $a = r_2^{-1} \cdot w^n \cdot r_2$, then $m = n$ because $\Delta(G) = 1$. Therefore definition 4.2 is well-defined.

Proposition 4.3 *Suppose that $u, v \in V(A)$ and the maximal subtrees of the graphs A and B_{v^k} are coherent. If $g^{-1} \cdot v^k \cdot g = u^l$ then $u \in V(B_{v^k})$ and $g \in C_G(v^k)$.*

PROOF. Induction on the number of the stable letters in the reduced form of g.

Base of induction. If $g \in E$ then by corollary 4 [12], the equality $g^{-1} \cdot v^k \cdot g = u^l$ is possible only if $v^k \in Z(\pi_1(\mathbb{S}_g))$. Then $g^{-1} \cdot v^k \cdot g = v^k = u^l$; therefore, by Lemma 1 and Proposition 2 [12], v^k belongs to the intersection of the vertex groups of

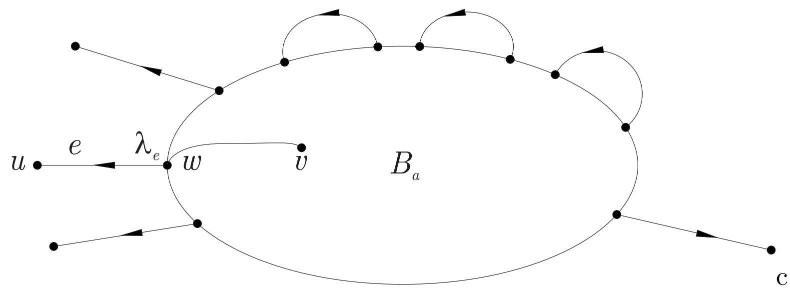


FIG. 2. Construction of a graph B_a.

the geodesic path joining u and v in T_A. In this case, this path belongs to B_a by definition 4.2.

Induction step. Suppose that $g = a_0 \cdot t^{\varepsilon_1} \cdot \dots \cdot t^{\varepsilon_n} \cdot a_n$, $a_0 \in E$ and t is the first stable letter in the reduced form of g. Then we have

$$a_n^{-1} \cdot t^{-\varepsilon_n} \cdot \dots \cdot t^{-\varepsilon_1} \cdot a_0^{-1} \cdot v^k \cdot a_0 \cdot t^{\varepsilon_1} \cdot \dots \cdot t^{\varepsilon_n} \cdot a_n = u^l.$$

Let e be an edge corresponding to the generator t. Denote $\alpha(e)$ by w. Then $a_0^{-1} \cdot v^k \cdot a_0 = w^m$, otherwise the left-hand side is reduced. By the induction hypothesis, $w \in V(B_a)$. By the definition of B_a, we obtain the equality $a = v^k = w^n$, then $m = n$ because $\Delta(G) = \{1\}$. Therefore $a_0 \in C_G(v^k)$.

On the left side of the equation t-reductions must continue, therefore $t^{-\varepsilon_1} \cdot v^k \cdot t^{\varepsilon} = u^l, e \in E(B_a)$ and $t^{-\varepsilon_1} \cdot v^k \cdot t^{\varepsilon_1} = \omega(e)^s$. Consequently $\omega(e) \in V(B_a)$ and since the maximal subtrees are coherent, we have $v^k = \omega(e)^p$, $p = s$ and $t \in C_G(v^k)$. Then the induction hypothesis finishes the proof of the proposition. The proposition is proved.

Proposition 4.4 *If a is a vertex element and the maximal subtrees T_{B_a} and T_A are coherent then $C_G(a) = \pi_1(\mathbb{B}_a) \leqslant \pi_1(\mathbb{A})$.*

PROOF. At first we will prove that $\pi_1(\mathbb{B}_a) \subseteq C_G(a)$. The generator w of the first type of $\pi_1(\mathbb{B}_a)$ is corresponding to some vertex of the graph B_a, therefore $w^k = a$ and $[w, a] = 1$. Let $t \in \pi_1(\mathbb{B}_a)$ be a generator of the second type, then, using coherency of the maximal subtrees, we have that t is a generator of the second type in the group $\pi_1(\mathbb{A})$. Since $\alpha(e), \omega(e) \in T_{B_a}$, then $a = \alpha(e)^k = \omega(e)^l$ and $t^{-1} \cdot \alpha(e)^m \cdot t = \omega(e)^n$. From $e \in B_a$ follows $m|k$. In this case $t^{-1} \cdot a \cdot t = t^{-1} \cdot \alpha(e)^k \cdot t = t^{-1} \cdot \alpha(e)^{m \cdot s} \cdot t = \omega(e)^{n \cdot s} = t^{-1} \cdot \omega(e)^l \cdot t$. Therefore $l = n \cdot s$ and $t \in C_G(a)$.

Now we need to prove that $C_G(a) \subseteq \pi_1(\mathbb{B}_a)$. Induction on the number of stable letters in reduced form of $g \in C_G(a)$.

Base of induction. If $g \in E$ then we can assume that A is a tree. In this case $C_G(a) = \pi_1(\mathbb{B}_a)$ by Lemma 13 [12].

Induction step. If element $g = a_0 \cdot t^{\varepsilon_1} \cdot \dots \cdot t^{\varepsilon_n} \cdot a_n \in C_G(a)$ is reduced, $g, a_0 \in E$ and t is the first stable letter, then

$$a_n^{-1} \cdot t^{-\varepsilon_n} \cdot \dots \cdot t^{-\varepsilon_1} \cdot a_0^{-1} \cdot v^k \cdot a_0 \cdot t^{\varepsilon_1} \cdot \dots \cdot t^{\varepsilon_n} \cdot a_n = v^k.$$

Arguing as in the proof of Proposition 4.3, we get $a_0^{-1} \cdot v^k \cdot a_0 = u^l$, where $u = \alpha(e)$ and e is the edge corresponding to the generator t. Then, by Proposition 4.3, using coherency of the maximal subtrees, we get $u \in V(B_a)$ and $a_0 \in C_G(a)$. By the induction hypothesis $a_0 \in \pi_1(\mathbb{B}_a)$. Further, as in Proposition 4.3 $t^{-\varepsilon} \cdot v^k \cdot t^{\varepsilon} = \omega(e)^s$ and $e \in E(B_a)$. Therefore $t \in \pi_1(\mathbb{B}_a)$. The proposition is proved.

Corollary 4.5 *The center of $\pi_1(\mathbb{B}_a)$ contains element a.*

Remark 4.6 *The center of $\pi_1(\mathbb{B}_{v^k})$ coincides with $\langle v^m \rangle$ for some $m|k$. Moreover, $B_{v^m} = B_{v^k}$.*

PROOF. By Proposition 2 [12] and corollary 4.5 we have

$$\langle v^k \rangle \leqslant Z(\pi_1(\mathbb{B}_{v^k})) \leqslant Z(\pi_1(\mathbb{T}_{B_{v^k}})) = \bigcap_{w \in V(B_{v^k})} \langle w \rangle \leqslant \langle v \rangle.$$

Therefore $Z(\pi_1(\mathbb{B}_{v^k})) = \langle v^m \rangle$, $m|k$ and, consequently, $B_{v^m} \subseteq B_{v^k}$ by definition. Considering the generators of the first and the second types of the group $\pi_1(\mathbb{B}_{v^k})$ and using $v^m \in Z(\pi_1(\mathbb{B}_{v^k}))$, it is possible as in the proof of Proposition 4.4, prove that $\pi_1(\mathbb{B}_{v^k}) \leqslant \pi_1(\mathbb{B}_{v^m})$. Then from the fact that $\mathbb{A}$ is reduced it follows that $B_{v^k} \subseteq B_{v^m}$. Therefore $B_{v^k} = B_{v^m}$. The remark is proved.

Remark 4.6. allows us to assume that for every subgraph B_a we can choose a vertex element c such that $Z(\pi_1(\mathbb{B}_a)) = Z(\pi_1(\mathbb{B}_c)) = \langle c \rangle$. It is clear from the definition that B_c is the maximal subgraph with given center. Therefore, we will always assume that $Z(\pi_1(\mathbb{B}_c)) = \langle c \rangle$. The subgraphs B_c is called the *Z-maximal subgraphs*. The latter well agreed with Proposition 5 and Remark 6 about Z-maximal subtrees [12].

If $g \in G$ is not conjugate to an element of E, then we can choose a cyclic permutation g_0 of g such that $C_E(g_0) = Z(\pi_1(\mathbb{B}_a))$ for a suitable vertex element a.

Lemma 4.7 *Suppose that g is not conjugate to an element of E, then we can choose a g_0 – cyclic permutation of g such that $C_E(g_0) = Z(\pi_1(\mathbb{B}_a))$ for a suitable vertex element a.*

PROOF. Since g is not conjugate to an element of E, there exists a reduced cyclic permutation g_0 of the element g such that

$$g_0 = a_0 \cdot t^{\varepsilon_1} \cdot a_1 \cdot \dots \cdot a_{n-1} \cdot t^{\varepsilon_n}.$$

If $h \in C_E(g_0)$ then

$$a_0 \cdot t^{\varepsilon_1} \cdot a_1 \cdot \dots \cdot a_{n-1} \cdot t^{\varepsilon_n} \cdot h \cdot t^{-\varepsilon_n} \cdot a_{n-1}^{-1} \cdot \dots \cdot a_1^{-1} \cdot t^{-\varepsilon_1} \cdot a_0^{-1} \cdot h^{-1} = 1.$$

Therefore $h \in \langle v \rangle$, where v is the beginning of the edge corresponding to the generator t. Thus $C_E(g_0) = \langle v^k \rangle$. By Proposition 4.4 we get $g_0 \in C_G(v^k) = \pi_1(\mathbb{B}_{v^k})$. On the other hand, by Remark 4.6 for a suitable $m|k$ we have $\langle v^m \rangle \cong Z(\pi_1(\mathbb{B}_{v^k})) = Z(\pi_1(\mathbb{B}_{v^m}))$ and $g_0 \in \pi_1(\mathbb{B}_{v^m})$. Therefore $v^m \in C_E(g_0) = \langle v^k \rangle$ and, consequently, $k = m$ and $C_E(g_0) = Z(\pi_1(\mathbb{B}_{v^m}))$. The lemma is proved.

Lemma 4.8 *If $g \in E$ is not elliptic element then*

$$C_G(g) = C_E(g) = w^{-1} \cdot (\langle r \rangle \times Z(\pi_1(\mathbb{T}_a))) \cdot w.$$

PROOF. If $h \in C_G(g)$ then $g \in C_E(h)$ and if h is not conjugate to an element of E, then by Lemma 4.7 g is elliptic; a contradiction. Therefore an element $h \in C_G(g)$ conjugate to an element of E and has the form $h = u^{-1} \cdot a \cdot u$. Moreover, if $u \notin E$ then the equality

$$u^{-1} \cdot a^{-1} \cdot u \cdot g \cdot u^{-1} \cdot a \cdot u = g$$

implies that g is an elliptic element. This contradicts the hypothesis of the lemma, therefore $u \in E$. Thus, $C_G(g) \subseteq E$ and $C_G(g) = C_E(g)$. Now the conclusion of the lemma follows from Corollary 4 and Corollary 12 [12]. The lemma is proved.

Lemma 4.9 *Let G be a GBS group, $\Delta(G) = \{1\}$, elements $g, h \in G$ are not conjugate to elements of E and $[g, h] = 1$. Then there exist reduced element $w \in G$ and cyclically reduced element $r \in G$ such that $g = w \cdot g_1 \cdot w^{-1}$, $h = w \cdot h_1 \cdot w^{-1}$ and $g_1 = r^k \cdot a$, $h_1 = r^l \cdot b$, where $a, b \in C_E(r)$.*

PROOF. If g and h have no common stable letters, then we can conjugate them so that

$$h = b_0 \cdot t^{\delta_1} \cdot b_1 \cdot \dots \cdot b_{m-1} \cdot t^{\delta_m}, g = a_0$$

and elements $a_0, b_0, b_1, \dots, b_{m-1}$ have no stable letter t. Then $[g, h] = 1$ if and only if

$$b_0 \cdot t^{\delta_1} \cdot b_1 \cdot \dots \cdot b_{m-1} \cdot t^{\delta_m} \cdot a_0 \cdot t^{-\delta_m} \cdot b_{m-1}^{-1} \cdot \dots \cdot b_1^{-1} \cdot t^{-\delta_1} \cdot b_0^{-1} = a_0.$$

Therefore $g = a_0$ is a vertex element and belongs to the centralizer $C_E(h)$. In this case we can take $g = h^0 \cdot a_0$ and $h = h^1 \cdot 1$.

Suppose that g and h have common stable letter t. Conjugating g and h simultaneously, if necessary, we can obtain

$$h = b_0 \cdot t^{\delta_1} \cdot b_1 \cdot \dots \cdot b_{m-1} \cdot t^{\delta_m}, g = a_0 \cdot t^{\varepsilon_1} \cdot a_1 \cdot \dots \cdot a_{n-1} \cdot t^{\varepsilon_n} \cdot a_n,$$

where g is cyclically reduced. It follows from $[g, h] = 1$ that

$$a_0 \cdot t^{\varepsilon_1} \cdot a_1 \dots a_{n-1} \cdot t^{\varepsilon_n} \cdot a_n \cdot b_0 \cdot t^{\delta_1} \cdot b_1 \dots b_{m-1} \cdot t^{\delta_m} =$$

$$= b_0 \cdot t^{\delta_1} \cdot b_1 \dots b_{m-1} \cdot t^{\delta_m} a_0 \cdot t^{\varepsilon_1} \cdot a_1 \dots a_{n-1} \cdot t^{\varepsilon_n} \cdot a_n.$$

Case 1. There are no stable letter t in both sides after t-reductions. Then $g \cdot h = a \in H, m = n$ and $\varepsilon_i = -\delta_{n+1-i}$. Moreover, $[g, h] = [a \cdot h^{-1}, h] = [h^{-1}, a] = 1$ and it follows from the beginning of the proof that $a \in C_E(h)$. We need to take $h = h, g = h^{-1} \cdot a$ to prove the lemma.

Case 2. There are some t-reductions, but they are not complete. In this case $t^{\delta_m} \cdot a_0 \cdot t^{\varepsilon_1}$ is possible to reduce, therefore a_0 is a vertex element and $\delta_m = -\varepsilon_1$. It follows from the fact that t-reductions are not complete, that there are some t reductions in $g \cdot h \cdot g^{-1} \cdot h^{-1} = 1$ in subword $h \cdot g^{-1}$. Therefore $t^{\delta_m} \cdot a_n^{-1} \cdot t^{-\varepsilon_n}$ is possible to reduce, a_n is a vertex element and $\delta_m = \varepsilon_n = -\varepsilon_1$. Thus, $\varepsilon_1 = -\varepsilon_n$ and

$$t^{\varepsilon_n} \cdot a_n \cdot a_0 \cdot t^{\varepsilon_1} = t^{\delta_m} \cdot a_n \cdot t^{-\varepsilon_n} \cdot t^{\varepsilon_n} \cdot a_0 \cdot t^{\varepsilon_1}$$

is possible to reduce, this contradicts the cyclic reducibility of g.

Case 3. There are no t-reductions. Then in the left-hand side of the equality $h^{-1} \cdot g^{-1} \cdot h \cdot g = 1$ there are $n + m$ t-reductions in the middle. Assume that $|g|_t \geqslant |h|_t$, then $g = g_1 \cdot g_2$ is a reduced form such that $|h^{-1} \cdot g_1|_t = 0$. Therefore $g_1 = h \cdot a$ and $g = h \cdot a \cdot g_2 = h \cdot g_3$, where $g_3 = a \cdot g_2$ and there are no t-reductions in $h \cdot g_3$. Moreover, the equality $[h, g] = 1$ holds if and only if $[g_3, h] = 1$. Arguing in this way, using the induction on the number $\min\{|g|_t, |h|_t\}$, we can assume that $g_3 = w \cdot g_3' \cdot w^{-1}, h = w \cdot h' \cdot w^{-1}$ and $g_3' = r^k \cdot a, h' = r^l \cdot b$, where $a, b \in C_E(r)$. In this case we have $g = h \cdot g_3 = w \cdot h' \cdot g_3' \cdot w^{-1}$, $h' \cdot g_3' = r^l \cdot b \cdot r^k \cdot a = r^{k+l} \cdot b \cdot a$ and $b \cdot a \in C_E(r)$. The lemma is proved.

Proof of the theorem 1 now follows from Lemmas 4.9, 4.8 and proposition 4.4.

5. Centralizers of sets: case $\Delta(G) = \{1\}$

The following lemma is well-known and can be proved using Theorem 2.1 [10, p. 51] or Lemma 1.1 [14, p. 79].

Lemma 5.1 *The product $b \cdot a$ of two elliptic elements a, b is an elliptic element if and only if a, b stabilize same vertex.*

Lemma 5.2 *If g_1 and g_2 are not elliptic elements, then $C_G(g_1, g_2)$ either coincides with $C_G(g_1) = C_G(g_2)$, or is conjugate to $Z(\pi_1(\mathbb{B}_c))$ for a suitable element c.*

PROOF. By Lemmas 4.7 – 4.9 we have $C_G(g_i) = w_i^{-1} \cdot (\langle r_i \rangle \times \langle a_i \rangle) \cdot w_i$, where $\langle a_i \rangle = Z(\pi_1(\mathbb{B}_{a_i})) = C_E(r_i)$ and we can assume, that r_i can not be represented as $h^k \cdot u, u \in C_E(h), |k| \geqslant 2$. Then the element of $C_G(g_1, g_2)$ have to be represented as

$$w_1^{-1} \cdot r_1^{k_1} \cdot a_1^{l_1} \cdot w_1 = w_2^{-1} \cdot r_2^{k_2} \cdot a_2^{l_2} \cdot w_2,$$

or equivalently

$$r_1^{k_1} \cdot a_1^{l_1} = w^{-1} \cdot r_2^{k_2} \cdot a_2^{l_2} \cdot w,$$

for $w = w_2 \cdot w_1^{-1}$.

If $r_1 \notin E$ then $r_2 \notin E$. Up to conjugacy we can assume that r_1 is a cyclically reduced and ends with t^ε. By the Theorem 2.8 (see [9]) an element $r_1^{k_1} \cdot a_1^{l_1}$ can be obtained from $r_2^{k_2} \cdot a_2^{l_2}$ by a cyclic permutation, ends with t^ε up to conjugacy by a vertex element h such that $t^\varepsilon \cdot h \cdot t^{-\varepsilon}$ is reducible. Therefore

$$r_1^{k_1} \cdot a_1^{l_1} = r_3^{k_2} \cdot a_2^{l_2},$$

where r_3 is a cyclic permutation of r_2 conjugate by h. Thus, $C_E(r_3) = C_E(r_2)$.

If $|r_1| = |r_3|$ then $|k_1| = |k_2|$. Assume that $k_1 = k_2 > 0$, then comparing right normal forms we get $r_1 = r_3 \cdot u, u \in E$.

If $k_1 = k_2 = 1$ then $\langle a_1 \rangle = C_E(r_3 \cdot u), \langle a_2 \rangle = C_E(r_3)$. Moreover, $a_1 \in C_E(u)$ and $a_1 \in C_E(r_3) = C_E(r_2)$. It follows from the symmetry of notation that $a_2 \in C_E(r_1)$ and, consequently, $C_E(r_1) = C_E(r_3)$

If $k_1 = k_2 > 1$ then $u \cdot (r_3 \cdot u)^{k_1 - 1} \cdot a_1^{l_1} = r_3^{k_2 - 1} \cdot a_2^{l_2}$. Therefore $r_3^{-1} \cdot u \cdot r_3 = v$ and by Proposition 4.3 $u = v \in C_E(r_3)$. Furthermore, arguing as in the previous case, we obtain $C_G(r_1) = C_G(r_3)$.

If $|r_1| > |r_3| \neq 0$ then $r_1 = r_3^m \cdot r_3'$, where $r_3 = r_3' \cdot r_3''$. In this case $(r_3^m \cdot r_3')^{k_1} \cdot a_1^{l_1} = r_3^{k_2} \cdot a_2^{l_2}$ and $r_3' \cdot (r_3^m \cdot r_3')^{k_1 - 1} \cdot a_1^{l_1} = r_3^{k_2 - m} \cdot a_2^{l_2}$. Therefore, comparing the initial segments of the left and right sides of equality, we get $r_3 = r_3' \cdot r_3'' = r_3'' \cdot r_3' \cdot u$. Moreover, for $k_1 > 2$ we have

$$r_3' \cdot u \cdot (r_3'' \cdot r_3' \cdot u)^{m-1} \cdot r_3' \cdot (r_3^m \cdot r_3')^{k_1 - 2} \cdot a_1^{l_1} = (r_3' \cdot r_3'')^{k_2 - m - 1} \cdot a_2^{l_2}.$$

Again, comparing the initial segments of the left and right sides of equality we get $u \cdot r_3'' \cdot r_3' = r_3'' \cdot r_3' \cdot v$. Since $u, v \in E$, it follows from the last equality that u and v are elliptic. By Proposition 4.3 we have $u = v \in C_E(r_1), C_E(r_3)$. Therefore, by reducing $u = v$ if necessary, we can assume that $u = 1$. By Lemma 4.9 we get $r_3' = r^{k_0} \cdot w^{l_0}$ and $r_3'' = r^{k_3} \cdot w^{l_3}$; a contradiction.

If $|r_1| = |r_3| = 0$ then all elements belongs to the subgroup E and we can apply the Lemma 14 from [12] and the Lemma 4.8, the last case appears. The lemma is proved.

Thus, there are three types of centralizers:

$$u^{-1} \cdot (\langle r \rangle \times Z(\pi_1(\mathbb{B}_a))) \cdot u, \tag{1}$$

$$v^{-1} \cdot \pi_1(\mathbb{B}_b) \cdot v, \tag{2}$$

$$w^{-1} \cdot Z(\pi_1(\mathbb{B}_c)) \cdot w, \tag{3}$$

for suitable Z-maximal subgraphs B_a, B_b, and B_c.

Lemma 5.3 *Given a non-elliptic $g_1 \in G$ and an elliptic $g_2 \in G$. Then $C_G(g_1, g_2)$ either coincides with $C_G(g_1)$ of type (1), or is conjugate to $Z(\pi_1(\mathbb{B}_c))$ for an element c such that $\mathbb{B}_c \supseteq \mathbb{B}_a$.*

PROOF. By Proposition 4.4 and Lemmas 4.8, 4.9 we can assume that

$$C_G(g_1) = w_1^{-1} \cdot (\langle r_1 \rangle \times Z(\pi_1(\mathbb{B}_a))) \cdot w_1, C_G(g_2) = w_2^{-1} \cdot \pi_1(\mathbb{B}_b) \cdot w_2.$$

If $g \in C_G(g_1, g_2)$ then $w_2 \cdot g \cdot w_2^{-1} \in \pi_1(\mathbb{B}_b)$ on the other hand

$$w_2 \cdot g \cdot w_2^{-1} = w_2 \cdot w_1^{-1} \cdot r_1^k \cdot a^l \cdot w_1 \cdot w_2^{-1}.$$

After reducing the right-hand side of the last equality, we obtain $w \cdot r^k \cdot a^l \cdot w^{-1}$. There w is obtained from $w_2 \cdot w_1^{-1}$ first by reductions in $w_2 \cdot w_1^{-1}$, and then, if necessary, by reductions with $r_1^k \cdot a^l$. Since r_1 is a cyclically reduced, then we can assume that r is obtained from r_1 by cyclic permutation so that r ends with t^ε (this can be achieved by selecting w).

Let $k \neq 0$. Reduced element $w^{-1} \cdot r^k \cdot a^l \cdot w$ belongs to $\pi_1(\mathbb{B}_b)$ only if $r \in \pi_1(\mathbb{B}_b)$. By Lemma 4.7 we get

$$\langle b \rangle = Z(\pi_1(\mathbb{B}_b)) \subseteq C_E(r) = Z(\pi_1(\mathbb{B}_a)) = \langle a \rangle,$$

therefore $b = a^k$ and, consequently, $B_b \supseteq B_a$ (otherwise we can find $v \in V(B_a) \setminus V(B_b)$ such that $v^p = a^q$, but there are no such non-zero integers m, n that $v^n = b^m$. On the other hand, b centralizes $v \in B_a$ and we have a contradiction with Lemma 1 [12]). Therefore $C_G(g_1) \subseteq C_G(g_2)$.

If $k = 0$ then for a suitable $l \neq 0$ (we can assume that l is minimal) $w^{-1} \cdot a^l \cdot w \in \pi_1(\mathbb{B}_b)$. Suppose that after reductions element $w^{-1} \cdot a^l \cdot w$ takes a form $w_0^{-1} \cdot a^l \cdot w_0 \in \pi_1(\mathbb{B}_b)$, then $w_0 \in \pi_1(\mathbb{B}_b)$ and $a^l = d^k$ for a suitable element $d \in V(\mathbb{B}_b)$.

Prove that in this case $C_G(g_1, g_2)$ is conjugate to $Z(\pi_1(\mathbb{B}_c))$. If $g \in C_G(g_1, g_2)$ then we have shown that g is conjugate to a^l. If $\langle c \rangle = Z(\pi_1(\mathbb{B}_{a^l}))$ then by Remark 4.6 we get $c = a^k, k|l$. On the other hand, $\mathbb{B}_{a^k} \supseteq \mathbb{B}_a$. We need to take $c = a^k$ to finish the proof. The lemma is proved.

If B_1, B_2 are two intersecting subgraphs of the graph A, then it is some times impossible to choose coherent maximal subtrees of A and B_1, B_2. In this case, we need to elaborate the embedding of $\pi_1(\mathbb{B}_1)$ and $\pi_1(\mathbb{B}_2)$ in $\pi_1(\mathbb{A})$. To do this, consider the equivalent definition of the fundamental group of the graph of groups [10]:

Definition 5.4 *Given a labeled graph $\mathbb{A}$, define a group*

$$F(\mathbb{A}) = (*_{v \in V(A)} \langle v \rangle * F(t_e, e \in E(A)))/N,$$

where N is a normal closure of $t_e^{-1} \cdot (\alpha(e))^{\lambda_e} \cdot t_e \cdot (\omega(e))^{\lambda_{\overline{e}}}, t_e \cdot t_{\overline{e}}, e \in E(A)$

Definition 5.5 *Given a vertex $P \in V(A)$, then $\pi_1(\mathbb{A}, P)$ is a subgroup of $F(A)$ consisting of the elements of the form $g_0 \cdot t_{e_1} \cdot g_1 \cdot t_{e_2} \cdot \dots \cdot t_{e_n} \cdot g_n$, where $e_1, e_2, \dots, e_n$ is a closed path in A with endpoints P, $g_0 \in \langle P \rangle, g_i \in G_{\omega(e_i)}$.*

In terms of definition 5.4 and 5.5 for each $v \in V(A)$ we denote the element $t_{e_1} \cdot t_{e_2} \cdot \dots \cdot t_{e_n}$ by γ_v, where $e_1, e_2, \dots, e_n$ is the geodesic path in T_A that connects P and v, $\gamma_P = 1$.

Remark 5.6 (Theorem 16.5 [10]**)** *A map from the set of generators of $\pi_1(\mathbb{A}, T_A)$ to $\pi_1(\mathbb{A}, P)$ given by the rules*

$$v \to \gamma_v \cdot v \cdot \gamma_v^{-1}, v \in V(A), t_e \to \gamma_{\alpha(e)} \cdot t_e \cdot \gamma_{\omega(e)}^{-1}, e \in E(A)$$

can be extended to isomorphism $\varphi_P\colon \pi_1(\mathbb{A}, T_A) \to \pi_1(\mathbb{A}, P)$.

If B_1 and B_2 are two intersecting subgraphs of the graph A, then we can take $P \in B_1 \cap B_2$. Then groups $\pi_1(\mathbb{B}_1, P)$ and $\pi_1(\mathbb{B}_2, P)$ can be naturally embedded in $\pi_1(\mathbb{A}, P)$. Therefore we can denote by $\pi_1(\mathbb{B}_1)$ and $\pi_1(\mathbb{B}_2)$ the images of $\pi_1(\mathbb{B}_1, P)$ and $\pi_1(\mathbb{B}_2, P)$ in $\pi_1(\mathbb{A}, T_A)$ under φ_P.

Remark 5.7 *It is clear from Remark 5.6 that the isomorphism* φ_P *depends on the choice of* P. *It is easy to understand that* φ_P *can be obtained from* φ_Q *by the conjugation by* $t_{e_1} \cdot t_{e_2} \cdot \dots \cdot t_{e_m}$, *where* $e_1, e_2, \dots, e_m$ *is a geodesic path in* T_A *joining* P *and* Q. *Therefore* $\pi_1(\mathbb{B}_1)$ *and* $\pi_1(\mathbb{B}_2)$ *are defined up to the conjugation.*

Remark 5.8 *If* B_1, B_2 *are two non-intersecting subgraphs of the graph* A, *then it is possible to choose coherent maximal subtree* T_A, T_{B_1} *and* T_{B_2}. *In this case* $\pi_1(\mathbb{B}_1)$ *and* $\pi_1(\mathbb{B}_2)$ *can be defined as the images under the trivial embedding.*

Lemma 5.9 *If* $P \in B_a \cap B_b$ *then* $\pi_1(\mathbb{B}_a, P) \cap \pi_1(\mathbb{B}_b, P) = \pi_1(\mathbb{B}, P)$, *where* B *is a connected component of* $B_a \cap B_b$ *containing the vertex* P.

PROOF. Follows from the definition 5.5 directly.

Lemma 5.10 *Given elliptic elements* g_1, g_2. *Suppose that* g_1 *is conjugate to the vertex element* a, g_2 *is conjugate to the vertex element* b. *Then* $C_G(g_1, g_2)$ *is either conjugate to* $\pi_1(\mathbb{B})$, *where* B *is a suitable connected component* $B_a \cap B_b$, *or conjugate to* $Z(\pi_1(\mathbb{B}_c))$ *for a suitable vertex element* c.

PROOF. If $g \in C_G(g_1, g_2)$ then it follows from proposition 4.4 that $g \in \pi_1(\mathbb{B}_a) \cap w^{-1} \cdot \pi_1(\mathbb{B}_b) \cdot w$ up to conjugacy. Write a reduced form of w: $w = r_1 \cdot r_2 \cdot r_3$ such that $r_1 \in \pi_1(\mathbb{B}_b)$, $r_3 \in \pi_1(\mathbb{B}_a)$ and r_2 is a word of the minimal length such that $r_2 \notin \pi_1(\mathbb{B}_b) \cap \pi_1(\mathbb{B}_a)$.

If $r_2 = 1$ and $B_a \cap B_b \neq \emptyset$ then by Lemma 5.9 $C_G(g_1, g_2)$ is conjugate to $\pi_1(\mathbb{B}_a) \cap \pi_1(\mathbb{B}_b) = \pi_1(\mathbb{B})$, where B is a suitable connected component $B_a \cap B_b$.

If $r_2 = 1$ and $B_a \cap B_b = \emptyset$ then $\pi_1(\mathbb{B}_a)$ and $\pi_1(\mathbb{B}_b)$ are defined by Remark 5.8 as natural subgroups of $\pi_1(\mathbb{A})$. Moreover, $g \in \pi_1(\mathbb{B}_a) \cap \pi_1(\mathbb{B}_b)$ only if $g = h_1 = h_2$, for $h_1 \in \pi_1(\mathbb{B}_a), h_2 \in \pi_1(\mathbb{B}_b)$. This equality does not hold if h_1 and h_2 are non-elliptic. To prove this we can find a subgroup G_1 of the group G such that G_1 is an amalgamated product of A and B with amalgamated subgroup $C \cong \mathbb{Z}$ and $\pi_1(\mathbb{B}_a) \hookrightarrow A, \pi_1(\mathbb{B}_b) \hookrightarrow B$. The element $g \in \pi_1(\mathbb{B}_a) \cap \pi_1(\mathbb{B}_b)$ have to belong to the vertex group C.

Therefore we can assume that $h_i = s_i^{-1} \cdot v_i^{k_i} \cdot s_i, i = 1, 2$. Then by Proposition 4.3 this equality implies $v_2 \in V(B_{v_1^{k_1}})$ and $v_1^{k_1} = v_2^{k_2}$. If $\langle v_1^{k_1} \rangle \subset Z(\pi_1(\mathbb{B}_{v_1^{k_1}})) = \langle v_1^{l_1} \rangle$, then $v_1^{l_1} \in \pi_1(\mathbb{B}_a) \cap \pi_1(\mathbb{B}_b)$. On the other hand, this intersection is generated by $v_1^{k_1}$; a contradiction. Hence the subgroup $\langle v_1^{k_1} \rangle = Z(\pi_1(\mathbb{B}_{v_1^{k_1}}))$ is conjugate to $\pi_1(\mathbb{B}_a) \cap \pi_1(\mathbb{B}_b)$ and $v_1^{k_1} = c$.

If $r_2 \neq 1$ then $g = h_1 = r_2^{-1} \cdot h_2 \cdot r_2$. In this case the proof is similar. The lemma is proved.

Lemma 5.11 *Given a non-elliptic element* g, *and* $C_G(g)$ *is of the type (1). Then the intersection* $C_G(g) \cap w^{-1} \cdot Z(\pi_1(\mathbb{B}_c)) \cdot w = H$ *is conjugate to* $Z(\pi_1(\mathbb{B}_b))$ *for a suitable* $B_b \supseteq B_c, B_b \supseteq B_a$.

PROOF. By Lemmas 4.7 – 4.9 the element $h \in H$ up to conjugacy belongs to $\langle r \rangle \times \langle a \rangle \cap w^{-1} \cdot Z(\pi_1((B)_c)) \cdot w = \langle r \rangle \times \langle a \rangle \cap w^{-1} \cdot \langle c \rangle \cdot w$. Hence $h = r^k \cdot a^l \in w^{-1} \cdot \langle c \rangle \cdot w$. Therefore h is an elliptic element and $k = 0$. We get $h = a^l = w^{-1} \cdot c \cdot w$. By Proposition 4.3 $a \in V(B_{c^m}), c \in V(B_{a^l})$. Thus, $c^m = a^l \in C_E(w)$. Let m, l be a minimal pair satisfying this equation (this is a well-defined condition, since all such pairs are proportional), denote $c^m = a^l$ by b. We can assume that $Z(\pi_1(\mathbb{B}_b)) = \langle b \rangle$, otherwise we can find $k|l, k < l$ as in Remark 4.6 such that $a^k \in Z(\pi_1(\mathbb{B}_b))$, in particular $a^k = c^s$, this contradicts minimality. The lemma is proved.

Lemma 5.12 *The intersection* $w_1^{-1} \cdot Z(\pi_1(\mathbb{B}_a)) \cdot w_1 \cap w_2^{-1} \cdot Z(\pi_1(\mathbb{B}_b)) \cdot w_2$ *is conjugate to* $Z(\pi_1(\mathbb{B}_c))$ *for a suitable* $B_c \supseteq B_a, B_c \supseteq B_b$.

PROOF. As in Lemma 5.11.

Lemma 5.13 *Given an elliptic element* g*, then the intersection* $C_G(g) \cap w^{-1} \cdot Z(\pi_1(\mathbb{B}_c)) \cdot w = H$ *is conjugate to* $Z(\pi_1(\mathbb{B}_a))$ *for a suitable* $B_a \supseteq B_c$ *such that* $B_a \cap B_b \neq \emptyset$.

PROOF. The element $h \in H$ up to conjugacy belongs to $\pi_1(\mathbb{B}) \cap w_1^{-1} \cdot Z(\pi_1(\mathbb{B}_c)) \cdot w_1$. Therefore h is an elliptic element of $\pi_1(\mathbb{B})$. Using conjugation of h, if necessary, we get $h = v^k = w_2^{-1} \cdot c^l \cdot w_2$ for a suitable $v \in V(B)$ and a minimal pair k, l. Arguing as in the proof of Lemma 5.11, one can understand that $h = v^k = c^l \in C_G(w_2)$ and $\langle v^k \rangle = Z(\pi_1(\mathbb{B}_{v^k}))$. If we take $a = v^k = c^l$ then we get $B_a \supseteq B_c$ and $v \in V(B_a \cap B)$. The lemma is proved.

In this section we described the intersection of centralizers and proved the theorem 2.

6. Embeddings of centralizers: case $\Delta(G) = \{1\}$

Proposition 6.1 *Given non-elliptic elements* $g_1, g_2 \in G$*, elliptic elements* $h_1, h_2 \in G$*, Z-maximal subgraphs* B_{c_1}*,* B_{c_2}*. If* $C_G(g_i) = w_i^{-1} \cdot (\langle r_i \rangle \times Z(\pi_1(\mathbb{B}_{a_i}))) \cdot w_i$*,* $C_G(h_i) = u_i^{-1} \cdot \pi_1(\mathbb{B}_{b_i}) \cdot u_i$*,* $v_i^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_i})) \cdot v_i, i = 1, 2$ *are the centralizers of type (1), (2) and (3) respectively, then*
1. $C_G(g_1) \not\supset C_G(g_2)$*,*
2. If $C_G(g_1) \supset C_G(h_1)$ *then* $B_{b_1} = \{b_1\}$*,*
3. If $C_G(g_1) \supset v_1^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_1})) \cdot v_1$ *then* $B_{c_1} \supseteq B_{a_1}$*,*
4. If $C_G(h_1) \supset C_G(g_1)$ *then* $r_1 \in \pi_1(\mathbb{B}_{b_1})$*,* $B_{a_1} \subseteq B_{b_1}$*,*
5. If $C_G(h_1) \supset C_G(h_2)$ *then* $B_{b_1} \supset B_{b_2}$*,*
6. If $C_G(h_1) \supset v_1^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_1})) \cdot v_1$ *then* $B_{b_1} \cap B_{c_1} \neq \emptyset$*,*
7. $v_1^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_1})) \cdot v_1 \not\supset C_G(g_1)$*,*
8. $v_1^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_1})) \cdot v_1 \not\supset C_G(h_1)$*,*
9. If $v_1^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_1})) \cdot v_1 \supset v_2^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_2})) \cdot v_2$ *then* $B_{c_2} \supset B_{c_1}$*.*

PROOF. 1. If $C_G(g_1) \supset C_G(g_2)$, then $C_G(g_2) = C_G(g_1, g_2)$ and by Lemma 5.2 $C_G(g_2)$ is either coincide with $C_G(g_1) = C_G(g_2)$, this is impossible because of strict inclusion, or a cyclic group. The latter is impossible since by Lemmas 4.8 and 4.9 $C_G(g_2) \cong \mathbb{Z} \times \mathbb{Z}$.
2. If $C_G(g_1) \supset C_G(h_1)$, then $C_G(h_1) = C_G(g_1, h)$ and by Lemma 5.3 $C_G(h_1)$ is either coincide with $C_G(g_1)$, this is impossible because of strict inclusion, or a cyclic group. Since $\mathbb{A}$ is a reduced labeled graph, then the latter is possible only if $B_{b_1} = \{b_1\}$.
3. If $C_G(g_1) \supset v_1^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_1})) \cdot v_1$ then $v_1^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_1})) \cdot v_1 = v_1^{-1} \cdot Z(\pi_1(\mathbb{B}_{c_1})) \cdot v_1 \cap C_G(g_1)$ and by Lemma 5.11 the latter is conjugate to $Z(\pi_1(\mathbb{B}_d))$ and $B_d \supseteq B_{c_1}, B_d \supseteq B_{a_1}$. Thus, the center of $\pi_1(\mathbb{B}_{c_1})$ is conjugate to $\pi_1(\mathbb{B}_d)$, therefore the element c_1 is conjugate to the element d and by Proposition 4.3 we get $c_1 = d$. Finally, we get $B_{c_1} \supseteq B_{a_1}$.
4. If $C_G(h_1) \supset C_G(g_1)$ then $C_G(g_1) = C_G(g_1, h_1)$ and by Lemma 5.3 the latter is either coincide with $C_G(g_1)$, or conjugate to $Z(\pi_1(\mathbb{B}_d))$. In the first case $r_1 \in \pi_1(\mathbb{B}_{b_1})$

and $C_E(r_1) = \langle a_1\rangle = Z(\pi_1(\mathbb{B}_{a_1})) \supseteq Z(\pi_1(\mathbb{B}_{b_1}))$. Therefore $B_{b_1} \supseteq B_{a_1}$. In the second case $C_G(g_1) = C_G(g_1, h_1)$ is a cyclic group. But by Lemmas 4.8 and 4.9 $C_G(g_1) \cong \mathbb{Z}\times\mathbb{Z}$; a contradiction.
5. If $C_G(h_1) \supset C_G(h_2)$ then $C_G(h_2) = C_G(h_1, h_2)$ and by Lemma 5.10 the latter is either conjugate to $\pi_1(\mathbb{B})$, or a cyclic group. In the first case B is a connected component of $B_{b_1} \cap B_{b_2}$, then $B = B_{b_2}$ and $B_{b_1} \supset B = B_{b_2}$. In the second case $B_{b_2} = \{b_2\} \subset B_{b_1}$.
6. If $C_G(h_1) \supset v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1$ then $v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1 = v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1 \cap C_G(h_1)$ and by Lemma 5.13 the latter is conjugate to $Z(\pi_1(\mathbb{B}_d))$, where $B_d \supseteq B_{c_1}, B_d \cap B_{b_1} \neq \emptyset$. Hence $Z(\pi_1(\mathbb{B}_d))$ is conjugate to $Z(\pi_1(\mathbb{B}_{c_1}))$. Therefore $c_1 = d$ and $B_{b_1} \cap B_{c_1} \neq \emptyset$.
7. If $v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1 \supset C_G(g_1)$ then $C_G(g_1) = C_G(g_1) \cap v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1$ and by Lemma 5.11 $C_G(g_1)$ is a cyclic group. The latter is impossible because by Lemmas 4.8 and 4.9 $C_G(g_1) \cong \mathbb{Z}\times\mathbb{Z}$.
8. If $v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1 \supset C_G(h_1)$ then $C_G(h_1) = C_G(h_1) \cap v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1$ and by Lemma 5.13 $C_G(h_1)$ is a cyclic group. Therefore $B_{b_1} = \{b_1\}, b_1 \in V(A)$ and $Z(\pi_1(\mathbb{B}_{c_1}))$ is conjugate to $\langle b_1\rangle$. Hence $c_1 = b_1$ by Proposition 4.3 and $v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1 = C_G(h_1)$, this is impossible because of strict inclusion.
9. If $v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1 \supset v_2^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_2}))\cdot v_2$ then $v_2^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_2}))\cdot v_2 = v_1^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_1}))\cdot v_1 \cap v_2^{-1}\cdot Z(\pi_1(\mathbb{B}_{c_2}))\cdot v_2$ and by Lemma 5.12 the latter is conjugate to $Z(\pi_1(\mathbb{B}_c))$. Therefore $c_2 = c$ and $Z(\pi_1(\mathbb{B}_{c_1})) \supset Z(\pi_1(\mathbb{B}_{c_2}))$. Thus, $B_{c_2} \supset B_{c_1}$. The proposition is proved.

Remark 6.2 *In the chain $C_G(A_1) \supset C_G(A_2) \supset \cdots \supset C_G(A_n)$, where A_i is a finite set of elements of G, the first $k, 0 \leqslant k \leqslant n$ centralizers have to be of type (2), then one can be of type (1) and the rest is of type (3).*

Remark 6.3 *Any strictly descending chain of centralizers of type (2) corresponds to a strictly descending chain of Z-maximal subgraphs*

$$\begin{array}{ccccccc} C_G(A_1) & \supset & C_G(A_2) & \supset & \dots & \supset & C_G(A_n) \\ B_{b_1} & \supset & B_{b_2} & \supset & \dots & \supset & B_{b_n}. \end{array}$$

Remark 6.4 *Any strictly descending chain of centralizers of type (3) corresponds to a strictly ascending chain of Z-maximal subgraphs*

$$\begin{array}{ccccccc} C_G(C_1) & \supset & C_G(C_2) & \supset & \dots & \supset & C_G(C_l) \\ B_{c_1} & \subset & B_{c_2} & \subset & \dots & \subset & B_{c_l}. \end{array}$$

7. Centralizer dimension: case $\Delta(G) = \{1\}$

PROOF of the theorem 3. By Remark 6.2 any centralizer chain has the form

$$(2) \supset (2) \supset \cdots \supset (2) \supset (1) \supset (3) \supset (3) \supset \cdots \supset (3)$$

or

$$(2) \supset (2) \supset \cdots \supset (2) \supset (3) \supset (3) \supset \cdots \supset (3),$$

where (1), (2) or (3) denotes the type of centralizer. The finiteness of such chains follows from Remarks 6.3 and 6.4.

Suppose that the maximal chain has the form $(2) \supset (2) \supset \cdots \supset (2) \supset (3) \supset (3) \supset \cdots \supset (3)$, then by Remark 6.3 centralizer subchain of type (2) corresponds to a maximal descending chain of Z-maximal subgraphs. Because of the maximality of the chain of subgraphs, the last Z-maximal subgraph in the chain consists of one vertex (because of $\mathbb{A}$ is reduced). Therefore, the last centralizer of type (2) has also type (3). Suppose that the longest chain of Z-maximal subgraphs consists of s graphs, then $s \leqslant |E(A)| + 1$. Similarly, by Remark 6.4, in the maximal chain of centralizers of type (3) there are s elements. Since one element is common, the centralizer dimension is equal to $2 \cdot s - 1 \leqslant 2 \cdot |E(A)| + 2 - 1$.

In the case when the maximal chain has the form $(2) \supset (2) \supset \cdots \supset (2) \supset (1) \supset (3) \supset (3) \supset \cdots \supset (3)$, we can act the same way. It only needs to be noted that a centralizer of type (1) is not contained in $\mathbb{Z}$ and not contain the vertex group, because by Proposition 6.1.3 $B_{c_1} \supseteq B_{a_1}$, $\langle a_1 \rangle = C_E(r)$ and r is not elliptic. Therefore $C_E(r)$ does not coincide with $\langle v \rangle, v \in V(A)$. Hence the maximal length of the centralizer subchains of type (2) and (3) is equal to $s-1$. Thus, $cdim(\pi_1(\mathbb{A})) = 2 \cdot (s-1) + 1 = 2 \cdot s - 1 \leqslant 2 \cdot |E(A)| + 1$.

It remains to give examples of labeled graphs $\mathbb{B}_{m,n}$. The idea is that the chain of Z-maximal subgraphs is need to have length $l, 2 \leqslant l \leqslant m+1$, then $k = 2 \cdot l - 1$.

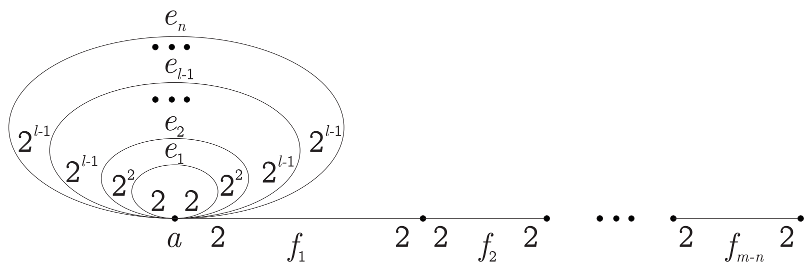


FIG. 3. Labeled graphs $\mathbb{B}_{m,n}$, for $l \leqslant n$.

In the case $l \leqslant n$ (see fig. 4) $T_i = \langle e_1, \ldots, e_i, f_1, \ldots, f_{n-m} \rangle$ are Z-maximal subgraphs forming the desired maximal chain and $Z(\pi_1(\mathbb{T}_i)) = \langle a^{2^i} \rangle, i = 1, 2, \ldots, l-1, n$.

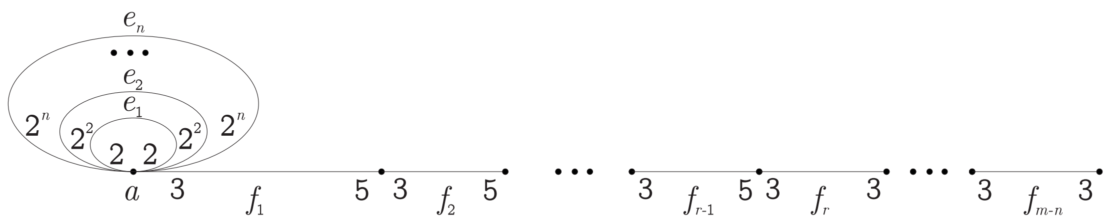


FIG. 4. Labeled graphs $\mathbb{B}_{m,n}$, for $m \geqslant l = n + r > n$.

In the case $m \geqslant l = n + r > n$ (see fig. 5) $T_i = \langle e_1, \ldots, e_i \rangle, S_j = \langle T_n, f_1, \ldots, f_j \rangle$ are Z-maximal subgraphs forming the desired maximal chain and $Z(\pi_1(\mathbb{T}_i)) = \langle a^{2^i} \rangle, i = 1, 2, \ldots, n, Z(\pi_1(\mathbb{S}_j)) = \langle a^{2^n \cdot 3^j} \rangle, j = 1, \ldots, r-1, m-n$. The Proposition is proved.

Using the labeled graphs constructed in the proof of Theorem 3, the following remark can be proved.

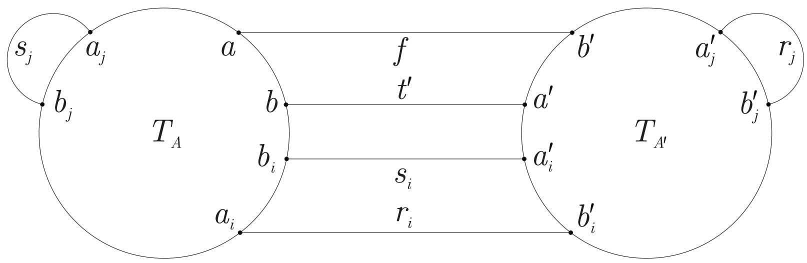
Remark 7.1 *Given a finite connected graph A. For any odd $k, 3 \leqslant k \leqslant 2|E(A)|+1$ we can choose labeling of the edges of A so that $cdim(\pi_1(\mathbb{A})) = k$.*

8. Centralizer dimension: case $\Delta(G) = \{\pm 1\}$

If $\Delta(G) = \{\pm 1\}$ then $Ker\Delta = G_1$ is a GBS group and a subgroup of index 2 of G. Denote by $\mathbb{A}_1$ the labeled graph as in fig. 6. Given a generator of second type t of $G \cong \pi_1(\mathbb{A})$ such that $\Delta(t) = -1$. Relations of second type in $\pi_1(\mathbb{A})$ have the form

$$t_i^{-1} \cdot b_i^{\alpha_i} \cdot t_i = a_i^{\beta_i}, i = 1, 2, \ldots, b_1(A) - 1, t^{-1} \cdot b^{\alpha} \cdot t = a^{\beta}.$$

Describe labeled graph $\mathbb{A}_1$. We can assume that $\Delta(t_i) = -1$ for $1 \leqslant i \leqslant k$, $\Delta(t_j) = 1$ for $k+1 \leqslant j \leqslant b_1(A)-1$. To each generator of the first type v of G there correspond two generators of the first type v, v' of $\pi_1(\mathbb{A}_1)$. To each edge $e \in E(T_A)$ there correspond two edges e, e' of the maximal subtree $T(A_1)$. Moreover, there is a special edge f with endpoints a and b' and the same labels as on the edge t in $T(A_1)$. There are no more edges in $T(A_1)$. To each edge $t_i, i = 1, 2, \ldots, b_1(A) - 1$ outside the maximal subtree in A there correspond two edges $s_i, r_i, i = 1, 2, \ldots, b_1(A) - 1$ outside the maximum subtree in A_1. The edges s_i have endpoints b_i and a_i'; the edges r_i have endpoints a_i and b_i' for $1 \leqslant i \leqslant k$, The edges s_j have endpoints b_j and a_j; the edges r_j have endpoints a_i' and b_i' for $k+1 \leqslant j \leqslant b_1(A) - 1$. Finally, there is one more edge t' outside maximal subtree of A_1 with endpoints b and a'.



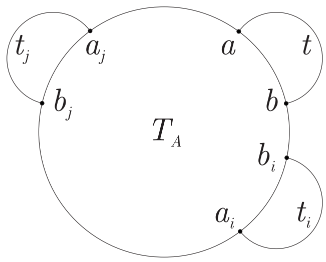


Fig. 5. Labeled graphs $\mathbb{A}_1$ and $\mathbb{A}$.

The labels on the corresponding edges are placed as in $\mathbb{A}$. Therefore relations of the first type in $\pi_1(\mathbb{A}_1)$ have form

$$u^{\lambda} = v^{\mu}, (u')^{\lambda} = (v')^{\mu},$$

for all relations of the first type of $u^\lambda = v^\mu$ in $\pi_1(\mathbb{A})$, plus the relation $a^\beta = (b')^\alpha$. The relations of the second type in $\pi_1(\mathbb{A}_1)$ have form

$$\begin{array}{rcccccl} s_i^{-1} & \cdot & (a_i')^{\beta_i} & \cdot & s_i & = & b_i^{\alpha_i}, \quad i = 1,2,\ldots,k, \\ r_i^{-1} & \cdot & (b_i')^{\alpha_i} & \cdot & r_i & = & a_i^{\beta_i}, \quad i = 1,2,\ldots,k, \\ s_j^{-1} & \cdot & b_j^{\alpha_j} & \cdot & s_j & = & a_j^{\beta_j}, \quad j = k+1,\ldots,b_1(A)-1 \\ r_j^{-1} & \cdot & (b_j')^{\alpha_j} & \cdot & r_j & = & (a_j')^{\beta_j}, \quad j = k+1,\ldots,b_1(A)-1, \\ (t')^{-1} & \cdot & b^\alpha & \cdot & t' & = & (a')^\beta. \end{array}$$

The constructed graph A_1 is a two-sheeted covering of A.

Proposition 8.1 *Group G_1 is isomorphic to $\pi_1(\mathbb{A}_1)$.*

PROOF. We construct a map φ on the set of generators $\pi_1(\mathbb{A}_1)$ in G by the rule

$$\varphi\colon \begin{array}{rcll} s_i & \to & t^{-1} \cdot t_i^{-1}, & i = 1,2,\ldots,k, \\ r_i & \to & t^{-1} \cdot t_i, & i = 1,2,\ldots,k, \\ s_j & \to & t_i, & j = k+1,\ldots,b_1(A)-1, \\ r_j & \to & t^{-1} \cdot t_j \cdot t, & j = k+1,\ldots,b_1(A)-1, \\ v & \to & v, & v \in V(T_A), \\ v' & \to & t^{-1} \cdot v \cdot t, & v \in V(T_A), \\ t' & \to & t^2. & \end{array}$$

This map extends to a homomorphism because the relations of $\pi_1(\mathbb{A}_1)$ pass to identity of G. Prove that $G_1 = Im\varphi$.

It is easy to see that $\Delta(\varphi(g)) = 1$ for any generator g of $\pi_1(\mathbb{A}_1)$, therefore $G_1 \supseteq Im\varphi$. In addition, it can be noted that $Im\varphi$ is a subgroup of index 2 in G with the coset representatives 1 and t. Therefore $G_1 = Im\varphi$.

Show that $Ker\varphi = \{id\}$. Suppose that reduced element g belongs to $Ker\varphi$. If $g \neq id$ then either $\varphi(g)$ belongs to E and reducible, or by Britton's Lemma [9] there is a subword of the form $t_i^{-1} \cdot a \cdot t_i = b$ in $\varphi(g)$. It is easy to understand that in both cases similar reduction should be in the word g of $\pi_1(\mathbb{A}_1)$, this contradicts reducibility. The proposition is proved.

Proposition 8.2 *The maximal chains of Z-maximal subgraphs in $\mathbb{A}_1$ have the same lengths as in $\mathbb{A}$.*

PROOF. At first we prove that if B_c is a Z-maximal subgraph of $\mathbb{A}_1$ and there is a pair $v, v' \in V(B_c)$, then $w \in V(B_c)$ if and only if $w' \in V(B_c)$.

By definition of B_c there are two integers k, k' such that $v^k = c = (v')^{k'}$. It follows from Proposition 8.1 that $k = k'$. If $w \in V(B_c)$ then $w^l = c = v^k = (v')^k$. Since $w^l = v^k$ then $(w')^l = (v')^k$. Therefore $w^l = c = (w')^l$. Hence $w' \in V(B_c)$. To prove the converse implication we can argue the same way.

Therefore each Z-maximal subgraph either has no pair v, v' of the vertices (consequently, it's vertex number is less then $|V(A)| + 1$), or contains $B \cup B'$ for a suitable Z-maximal subgraph B of the labeled graph $\mathbb{A}$. Therefore a maximal chain of Z-maximal subgraphs has the same length as in $\mathbb{A}$. The proposition is proved.

Corollary 8.3 *The centralizer dimension of $\pi_1(\mathbb{A}_1)$ is equal to the centralizer dimension of $\pi_1(\mathbb{A})$.*

Remark 8.4 *Given a GBS group G such that $\Delta(G) = \{\pm 1\}$. If $g \in G, \Delta(g) = -1$ then $C_G(g) = \langle r \rangle$, where $g = r^m, m$ is odd and an element r is not a power of some other element.*

PROOF. Arguing as in the proof of the Lemma 4.9 we can show that $C_G(g) = \langle r\rangle \times C_E(r)$. But $C_E(g) = \{1\}$ because of $C_E(g)$ consists of vertex elements and $\Delta(g) = -1$. The remark is proved.

PROOF of the theorem 4. The group $G = \pi_1(\mathbb{A})$ has a trivial center. Therefore there is a centralizer chain

$$G = C_G(1) \supset C_G(g_1) \supset \cdots \supset C_G(g_1, \ldots, g_{2s+1}) \supset \{1\} = C_G(t, g_1, \ldots, g_{2s+1})$$

of the length $2 \cdot s + 3$, where s is the length of the maximal chain of Z-maximal subgraphs in $\mathbb{A}_1$. Suppose that there exists a longer chain

$$G = C_G(1) \supset C_1 \supset \cdots \supset C_k \supset \{id\} = C_G(t, h_1, \ldots, h_r).$$

If $C_1, \ldots, C_l \not\subseteq G_1$ then there is $g \in C_i, 1 \leqslant i \leqslant l$ such that $\Delta(g) = -1$. Since $g \in C_i = C_G(g_1, \ldots, g_{i-1}, g)$ and $C_G(g) = \langle r\rangle$ then by remark 8.4 we have $g_i \in \langle r\rangle$. However, $C_G(g^m)$ is coincide either with $\langle r\rangle$ for odd m, or with $\langle r\rangle \times \langle a\rangle$ for even m (there $C_E(r^2)$ is generated by a). Therefore $l \leqslant 2$ and the length of the chain is less then 5.

Suppose that the length is equal to 4. If the length of the chain of Z-maximal subgraphs in $\mathbb{A}$ greater or equal to 2, then we know how to construct the centralizer chain of the length $2 \cdot 2 + 1$ and 4 is not maximal. Suppose that the length of the chain of Z-maximal subgraphs in $\mathbb{A}$ is equal to 1. Since $\mathbb{A}$ is a reduced labeled graph, we get $|V(A)| = 1$. Moreover, the labels equal to 1 or -1. Hence $\pi_1(\mathbb{A})$ contains subgroup H of index 2 isomorphic to $F_n \times \mathbb{Z}$. This subgroup H corresponds to two-sheeted covering (as in fig. 6). If $n \geqslant 2$ then $cdim(H) = 3$ and centralizer chain constructed at the beginning of the proof has the length equals to 5. If $n = 1$ then H is an abelian group and $cdim(G)$ can not be equal to 4.

Therefore the length of the chain is not greater than 3. However, 3 is a minimal centralizer dimension of non-abelian group, this contradicts to the minimality. Therefore $l = 0$.

Examples can be constructed as in fig. 4 and 5, changing a sign of one suitable label. The theorem is proved.

PROOF of the remark 5. At first we need to compute $\Delta(\pi_1(\mathbb{A}))$. If $\Delta(\pi_1(\mathbb{A})) \not\subseteq \{\pm 1\}$ then by Theorem 3.2 either $cdim(\pi_1(\mathbb{A})) = \infty$, or $\pi_1(\mathbb{A}) \cong BS(1, n)$, $cdim(\pi_1(\mathbb{A})) = 3$.

If $\Delta(\pi_1(\mathbb{A})) \subseteq \{\pm 1\}$ then, starting from an arbitrary vertex, one can find a chain of Z-maximal subgraphs. Then, as in Theorems 3 and 4, we can compute $cdim(\pi_1(\mathbb{A}))$. The remark is proved.

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