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# MIRROR SYMMETRIES OF HYPERBOLIC TETRAHEDRAL MANIFOLDS

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Abstract. Let $\Lambda$ be the group generated by reflections in faces of a Coxeter tetrahedron in the hyperbolic space $\mathbb{H}^3$. A *tetrahedral manifold* is a hyperbolic manifold $\mathcal{M} = \mathbb{H}^3/\Gamma$ uniformized by a torsion free subgroup $\Gamma$ of the group $\Lambda$. By a *mirror symmetry* we mean an orientation reversing isometry of the manifold acting by reflection. The aim of the paper to investigate mirror symmetries of the tetrahedral manifolds.

**Keywords:** hyperbolic space, isometry group, automorphism group, hyperbolic manifolds.

## 1. Introduction

One of the first examples of tetrahedral manifolds was constructed by E. B. Vinberg [19] who proved that the well-known hyperbolic Seifert-Weber manifold [7] is tetrahedral. Further examples were given by L. A. Best [18]. The first non-orientable tetrahedral manifold appeared in the paper by N. K. Al-Jubouri [1]. The isometry group of the Al-Jubouri and the Seifert-Weber manifolds was found in the papers [4] and [15] respectively. Further generalization of tetrahedral manifolds known as Löbell type manifolds was done in series of papers ([20], [21], [3], [5], [6], [16], [2]). The arithmetic properties of the tetrahedral manifolds investigated in [14]. There also analogues of such manifolds in the high dimensional theory. Here, the construction known as small coverings was proposed by M. W. Davis and T. Januszkiewicz [8] and developed in [11]. In the present, the most effective way to construct the manifolds under consideration comes from the toric geometry [9], [10].

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## 2. Basic definitions and preliminary facts

Let $\mathbb{H}^3$ be the three dimensional hyperbolic space endowed by Riemannian metric of the constant sectional curvature -1. Let $I = Isom(\mathbb{H}^3)$ denote the group of all isometries of $\mathbb{H}^3$ and let $I^+ = Isom^+(\mathbb{H}^3)$ denote the subgroup of index 2 in $I$ consisting of the orientation preserving isometries. By a non-Euclidean crystallographic (NEC) group we shall mean a discrete subgroup $\Gamma$ of $I$ for which $\mathbb{H}^3/\Gamma$ is compact. If $\Gamma \cap (I - I^+) \neq \emptyset$ then we shall call $\Gamma$ a *proper* NEC group.

An important class of NEC groups is formed by the hyperbolic Coxeter groups and their subgroups of finite index. There are exactly nine Coxeter groups generated by reflections in the faces of a compact hyperbolic tetrahedron [17]. Following the Best [18] we will denote these groups $\Lambda_1, \Lambda_2, \ldots, \Lambda_9$. Also, let $\Delta_1, \Delta_2, \ldots, \Delta_9$ be they corresponding orientation preserving subgroup of order two, respectively.

Consider the group

$$\begin{aligned} \Delta &= \Delta[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3] \\ &= \langle a^{\lambda_1} = b^{\lambda_2} = c^{\lambda_3} = (bc^{-1})^{\mu_1} = (ca^{-1})^{\mu_2} = (ab^{-1})^{\mu_3} = 1 \rangle \end{aligned}$$

generated by rotations in the edges of compact Lanner tetrahedron

$$\mathcal{T} = \mathcal{T}[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3].$$

Let $\Gamma$ be a torsion free normal subgroup of $\Delta$. Then we shall call manifold $\mathcal{M} = \mathbb{H}^3/\Gamma$ a *$\mathcal{T}$-tetrahedral manifold.* Let also $\theta^*$ be the canonical epimorphism from $N(\Gamma)$ onto a finite group $G^* = N(\Gamma)/\Gamma$. The manifold $\mathcal{M}$ will be symmetric if there exist $\tau \in (I - I^+)$ such that $T = \theta^*(\tau)$ has order two. We shall say the symmetry $T$ is a *mirror symmetry* if $Fix(T)$ contains a closed surface. The manifold $\mathcal{M} = \mathbb{H}^3/\Gamma$ is called mirror symmetric in this case.

Let $\mathcal{M}$ be a closed three dimensional hyperbolic manifold. Then $\mathcal{M}$ can be represented as $\mathbb{H}^3/K$, where $K$ is a NEC group isomorphic to $\pi_1(\mathcal{M})$ and acting without fixed points in $H^3$. The later occurs if and only if $K$ is torsion free. In this case the group $Isom(\mathcal{M})$ of all isometries of $\mathcal{M}$ is isomorphic to $N(K)/K$, where is normalizer of $K$ in $I$. By the Mostow rigidity theorem the group $Isom(\mathcal{M})$ is always finite and isomorphic to $Out(K) = Aut(K)/Inn(K)$. The basic question which we discuss in the paper is whether tetrahedral manifolds are symmetric.

## 3. Basic lemma

Let $\mathcal{M}$ be a closed three dimensional hyperbolic manifold and $G$ be an automorphism group of $\mathcal{M}$. Then $\mathcal{M} = \mathbb{H}^3/K$, where $K$ is a NEC group acting without fixed points in $H^3$. Denote by $\Gamma$ the lifting of the group $G$ to the universal covering $\mathbb{H}^3$. Note that $\Gamma < N(K)$, where $N(K)$ is the normalizer of $K$ in $Isom(\mathbb{H}^3)$. Here, $K \lhd \Gamma$ and $\Gamma/K \cong G$.

Following traditions in the Riemann surface theory (see, for example, [13], [22]) we refer to the pair $(\Gamma,\mathbb{H}^3)$ as the *universal covering transformation group* of the transformation group $(G, \mathcal{M})$.

The following statement is a three-dimensional version of Lemma 4.1 in [22].

**Lemma 1.** *Let $\mathcal{M}$ be a closed orientable three dimensional hyperbolic manifold and $G$ be an automorphism group of $\mathcal{M}$. Denote by $(\Gamma, \mathbb{H}^3)$ the universal covering transformation group of $(G, \mathcal{M})$. Assume that $\mathcal{M} = \mathbb{H}^3/K$, where $K$ is a NEC group acting without fixed points in $\mathbb{H}^3$ and $\theta : \Gamma \to \Gamma/K = G$ be the canonical homomorphism. Let there exist a proper NEC group $\Gamma^*$ such that $(\Gamma^*)^+ = \Gamma$ and* *element $c \in (\Gamma^* - \Gamma)$ such that $c^2 \in K$. Suppose $G^*$ is a group which contains $G$ with index 2 and $\theta^* : \Gamma^* \to G^*$ is an epimorphism with $\theta^* |_\Gamma = \theta$. Then $\mathcal{M}$ is symmetric.*

*Proof.* $\Gamma^*$ contains an orientation reversing isometry $c$ such that $\Gamma^* = \Gamma + c\Gamma$. $\theta^*(c)$ is not the identity otherwise $G = G^*$. Since $c^2 \in K$ and $K = ker(\theta)$ we have $\theta^*(c)^2 = \theta^*(c^2) = \theta(c^2) = 1$. So $\theta^*(c)$ is an involution. As $\theta^* |_\Gamma = \theta$, we have $ker(\theta^*) \supseteq K$. Moreover, if $g \in ker(\theta^*), g \notin K$ then $g = ct$ where $t \in \Gamma$. Therefore $\theta^*(c) = \theta^*(g\,t^{-1}) = \theta^*(t^{-1}) \in G$ and again $G = G^*$. Hence $g \in K$ and so $ker(\theta^*) = K$. Thus $K \lhd \Gamma^*$ and $\theta^*(c)$ is a symmetry of $\mathcal{M}$.

□

3.1. **Mirror symmetric tetrahedral manifold.** Consider the group

$$\begin{aligned}\Lambda = \langle k^2 = l^2 = m^2 = n^2 &= (lk)^{\lambda_1} = (mk)^{\lambda_2} = (nk)^{\lambda_3} \\ &= (mn)^{\mu_1} = (nl)^{\mu_2} = (lm)^{\mu_3} = 1\rangle \end{aligned} \tag{1}$$

generated by reflections in the faces of compact Lanner tetrahedron

$$\mathcal{T} = \mathcal{T}[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3].$$

Let

$$\begin{aligned}\Delta &= \Delta[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3] \\ &= \langle a^{\lambda_1} = b^{\lambda_2} = c^{\lambda_3} = (bc^{-1})^{\mu_1} = (ca^{-1})^{\mu_2} = (ab^{-1})^{\mu_3} = 1\rangle\end{aligned}$$

be the subgroup of index two in $\Lambda$ generated by rotations $a = l\,k$, $b = m\,k$ and $c = n\,k$ in edges of the tetrahedron $\mathcal{T}$ (see, for example [14]). In what follows, we consider $\Lambda = \Delta^* = \langle k, \Delta\rangle$ is a natural proper $Z_2$-extension of the NEC group $\Delta$. A three dimensional hyperbolic manifold $\mathcal{M}$ is said to be *$\mathcal{T}$-tetrahedral manifold* if there exist an epimorphism $\theta : \Delta \to G$ onto a finite group $G$ with torsion free kernel such that $\mathcal{M} = \mathbb{H}^3/Ker(\theta)$.

**Theorem 1.** *Let $G$ be a group of automorphisms of a $\mathcal{T}$-tetrahedral manifold $M$ generated by $A$, $B$ and $C$ obeying*

$$A^{\lambda_1} = B^{\lambda_2} = C^{\lambda_3} = (BC^{-1})^{\mu_1} = (CA^{-1})^{\mu_2} = (AB^{-1})^{\mu_3} = 1$$

*(i.e. there is an epimorphism $\theta$ from $\Delta$ to $G$ defined by $a \to A$, $b \to B$, $c \to C$ and $\mathcal{M} = \mathbb{H}^3/Ker(\theta)$ is the quotient of the kernel). Then $\mathcal{M}$ is a mirror symmetric if and only if there is an automorphism $\alpha : G \to G$ obeying either*

$$\begin{array}{llll} (i) & \alpha(A) = A^{-1}, & \alpha(B) = B^{-1}, & \alpha(C) = C^{-1} \text{ or} \\ (ii) & \alpha(A) = BA^{-1}, & \alpha(B) = B, & \alpha(C) = CB^{-1}. \end{array}$$

**Remark.** The second possibility in the theorem is realized for tetrahedron $\mathcal{T}_1 = \mathcal{T}[2, 2, 3; 3, 5, 2]$ only. In this case, the rotation group $\Delta_1$ has two proper NEC $\mathbb{Z}_2$-extensions (see [12], Theorem 1). One is the canonical reflection group $\Lambda_3$, while the second is the group

$$R = \langle t, \Delta_1\rangle = \langle t^2 = a^2 = c^3 = (at)^4 = (ca^{-1})^5 = [t, ca^{-1}] = 1\rangle, \tag{2}$$

where $t$ is the reflection in the hyperbolic plane divided the tetrahedron $\mathcal{T}_1$ in two pieces each is congruent to tetrahedron $\mathcal{T}_3$.

*Proof.* Suppose that an automorphism $\alpha$ exists obeying (i). Assume that it is an outer automorphism. Then there exists a $\mathbb{Z}_2$-extension $G^*$ of $G$ and an element $T \in G^*$ obeying

$$T^2 = 1,\ TAT^{-1} = A^{-1},\ TBT^{-1} = B^{-1},\ TCT^{-1} = C^{-1}.$$

i.e.

$$\begin{array}{rlllll} T^2 &= (AT)^2 &= (BT)^2 &= (CT)^2 = A^{\lambda_1} = B^{\lambda_2} = C^{\lambda_3} \\ &= (BC^{-1})^{\mu_1} &= (CA^{-1})^{\mu_2} &= (AB^{-1})^{\mu_3} = 1. \end{array}$$

Then there is an epimorphism $\theta^* : \Delta^* = \Lambda \to G^*$ defined by

$$\theta(k) = T, \quad \theta(l) = AT, \quad \theta(m) = BT, \quad \theta(n) = CT$$

and

$$\theta((\Delta^*)^+) = \theta((\Delta) = G.$$

By Lemma 1, $\mathcal{M}$ is symmetric.

If $\alpha$ is an outer automorphism obeying $(ii)$ then there is a $\mathbb{Z}_2$-extension $G^*$ of $G$ and an element $T \in G^*$ which satisfies

$$T^2 = 1,\ TAT^{-1} = BA^{-1},\ TCT^{-1} = CB^{-1}.$$

In this case tetrahedron $\mathcal{T} = \mathcal{T}[2, 2, 3; 3, 5, 2]$.

Since $T^2 = 1$, it implies

$$T^2 = 1,\ (AT)^2 = BA^{-1},\ TCT^{-1} = CB^{-1}.$$

Hence

$$T^2 = A^2 = C^3 = (AT)^4 = (CA^{-1})^5 = [T, CA^{-1}] = 1$$

Let $\Delta^*$ be an NEC group

$$R = \langle t, \Delta_1 \rangle = \langle t^2 = a^2 = c^3 = (at)^4 = (ca^{-1})^5 = [t, ca^{-1}] = 1 \rangle.$$

Then there exists an epimorphism $\theta^* : \Delta^* = \Lambda \to G^*$ defined by $\theta(t) = T, \quad \theta(a) = A, \quad \theta(c) = C$.

By Lemma 1, $\mathcal{M}$ is symmetric.

If $\alpha$ is an inner automorphism then the element $T$ above lies in $G$. Let $G^* = \mathbb{Z}_2 \times G$, where $\mathbb{Z}_2 = \{V \mid V^2 = 1\}$. Then $G^* = \{(V, W), (1, W) : W \in G\}$ and $G$ can be identified with $\{(1, W) : W \in G\}$. Suppose that $\alpha$ obeys $(i)$. Then if we let

$$T_1 = (V, T),\ A_1 = (1, A),\ B_1 = (1, B),\ C_1 = (1, C)$$

then

$$\begin{array}{rlll} {T_1}^2 &= (A_1T_1)^2 &= (B_1T_1)^2 &= (C_1T_1)^2 \\ &= {A_1}^{\lambda_1} &= {B_1}^{\lambda_2} &= {C_1}^{\lambda_3} \\ &= (B_1{C_1}^{-1})^{\mu_1} &= (C_1{A_1}^{-1})^{\mu_2} &= (A_1{B_1}^{-1})^{\mu_3} = 1. \end{array}$$

Here $T_1 \notin G$, and we have reduced this case to the proceeding case. If $\alpha$ is an inner automorphism obeying $(ii)$ then we proceed similarly.

Converse. As $\mathcal{M}$ is symmetric, $G$ is contained in $G^*$ with index 2 and there is a proper NEC group $\Delta^*$ and an epimorphism $\theta^* : \Delta^* \to G^*$ such that $\mathbb{H}^3/ker\theta^* = \mathcal{M}$. Then $\Delta^*$ contains $\Delta[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3]$ with index 2 and $(\Delta^*)^+ = \Delta[\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3]$. By Theorem 1 from [12] the group $\Delta^*$ has presentation (1) or (2). If it has presentation (1) then $\theta^*(k)$ induces the required automorphism by conjugation and if it has presentation (2), $\theta^*(t)$ does.

□

## 4. Applications and Examples

**Example 1**

Consider the rotation group

$$\Delta_1 = \langle a^2 = b^2 = c^3 = (bc^{-1})^3 = (ca^{-1})^5 = (ab^{-1})^2 = 1\rangle,$$

of Lanner tetrahedron $\mathcal{T}_1 = \mathcal{T}[2,2,3;3,5,2]$. The natural $\mathbb{Z}_2$-extension of $\Delta_1$ is the reflection group

$$\begin{aligned}\Lambda_1 = \langle k^2 = l^2 = m^2 = n^2 &= (lk)^2 = (mk)^2 = (nk)^3 \\ &= (mn)^3 = (nl)^5 = (lm)^2 = 1\rangle.\end{aligned}$$

The generators of the groups are related by $a = l\,k,\ \ b = m\,k,\ \ c = n\,k$.

In this example, we construct a symmetric tetrahedral manifold $\mathcal{M} = \mathbb{H}^3/\Gamma$, $\Gamma \triangleleft \Delta_1$ whose symmetry $K$ is induced by action of the reflection $k$ on the universal covering $\mathbb{H}^3$.

To do this we consider the epimorphism $\theta^* : \Lambda_1 \to G^*$ onto a finite group of permutation $G^*$ generated by $K, L, M, N$ defined by

$$k \to K = (1\ 2)(3\ 4)(5\ 10)(6\ 9)(8\ 12)(7\ 11),$$

$$l \to L = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12),$$

$$m \to M = (1\ 2)(3\ 4)(5\ 10)(6\ 9)(8\ 11)(7\ 12),$$

$$n \to N = (1\ 10)(2\ 3)(4\ 5)(6\ 7)(8\ 9)(11\ 12).$$

The order of the group $G^*$ is equal to $1920 = 2^7 \cdot 3^1 \cdot 5^1$.

The restriction $\theta = \theta^*|_{\Delta_1}$ of the epimorphism $\theta^*$ on the rotation group $\Delta_1$ is given by

$$\begin{aligned} a = \ & l\,k \to A = (5\ 9)(6\ 10)(7\ 12)(8\ 11),\\ b = \ & m\,k \to B = (7\ 8)(11\ 12),\\ c = \ & n\,k \to C = (1\ 3\ 5)(2\ 10\ 4)(6\ 8\ 11)(7\ 12\ 9).\end{aligned}$$

In this case the group $G = \langle A, B, C\rangle$ coincides with the group $G^* = \langle K, L, M, N\rangle$.

Now we have to check the condition $(i)$ in 1. In this case the action of automorphism $\alpha$ on the group $G$ is defined as follows.

$$\begin{aligned}\alpha : A &\to K\,A\,K^{-1} = A^{-1},\\ B &\to K\,B\,K^{-1} = B^{-1},\\ C &\to K\,C\,K^{-1} = C^{-1}.\end{aligned}$$

The tetrahedral manifold $\mathcal{M} = \mathbb{H}^3/\Gamma$, where $\Gamma = Ker(\theta)$ gives a suitable example. The group $G^*$ is an automorphism group of $\mathcal{M}$. Then the mirror symmetry of $\mathcal{M}$ is given by the element $K \in G^*$.

Here we deal with the case $(i)$ in theorem 1. In this case, the action of automorphism $\alpha$ on the group $G$ is given by the conjugation of element $K$.

**Example 2** We have to show the example of symmetric tetrahedral manifold obtained by use of condition $(ii)$ in the theorem 1. Consider the rotation group

$$\Delta_1 = \langle a^2 = b^2 = c^3 = (bc^{-1})^3 = (ca^{-1})^5 = (ab^{-1})^2 = 1\rangle,$$

of Lanner tetrahedron $\mathcal{T}_1 = \mathcal{T}[2,2,3;3,5,2]$ as in previous example. Beside the $\Lambda_1$ there is exactly one proper NEC $\mathbb{Z}_2$-extension of $\Delta_1$. Namely, the group

$$R = \langle t, \Delta_1\rangle = \langle t^2 = a^2 = c^3 = (at)^4 = (ca^{-1})^5 = [t, ca^{-1}] = 1\rangle,$$

where $t$ is the reflection in the hyperbolic plane divided the tetrahedron $\mathcal{T}_1$ on two congruent pieces (see [12]). Here $b = (at)^2$ and $a$, $c$ are as above.

Denote by $G = \langle A, B, C\rangle$ the finite group generated by permutations

$$\begin{aligned} A &= (1\ 7)(6\ 10)(8\ 11)(9\ 12), \\ B &= (8\ 9)(11\ 12), \\ C &= (1\ 5\ 3)(2\ 4\ 6)(7\ 9\ 12)(8\ 11\ 10). \end{aligned}$$

Let

$$T = (1\ 6)(2\ 5)(3\ 4)(7\ 10)(8\ 9).$$

Direct calculation shows that

$$TAT^{-1} = AB, \quad TBT^{-1} = B, \quad TCT^{-1} = CB$$

and outer automorphism of $G$ induced by $T$ satisfies the condition $(ii)$ in the Theorem 1.

Let $G^* = \langle T, G\rangle$ is $\mathbb{Z}_2$-extension of $G$ by $T$. Note, that the orders of the group $G$ and $G^*$ are equal 1920 and 3840 respectively.

Consider the order preserving epimorphism $\theta^* : R \to G^*$ defined by

$$t \to T, \quad a \to A, \quad b \to B, \quad c \to C$$

and its restriction $\theta = (\theta^*)^+ : \Delta_1 \to G$.

Hence, by Theorem 1, the tetrahedral manifold $\mathcal{M} = \mathbb{H}^3/\Gamma$, where $\Gamma = Ker(\theta)$ is symmetric. Its mirror symmetry is given by $T$.

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