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THE THIRD BOUNDARY VALUE PROBLEM FOR THE SYSTEM OF EQUATIONS OF NON-EQUILIBRIUM SORPTION

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Abstract. In this paper, we investigate the system of equations modeling the process of non-equilibrium sorption. In particular, such systems are used in mathematical modeling of the production process of the useful component by the method of borehole underground leaching. The theorem of existence and uniqueness of the solution of the third boundary value problem in the multidimensional case in Hölder classes of functions is proved. The obtained maximum principle plays an important role in the proof of the theorem. The uniqueness of the solution is a consequence of this principle. The existence of a solution to the problem is shown by Schauder's fixed point theorem of a completely continuous operator. The description of the corresponding operator is given. Estimates are obtained to ensure the continuity of the constructed operator, and it is shown that the operator maps the original set of functions into itself at a small period of time. Then the estimates are given, allowing to continue the solution to any finite value of time.

Keywords: process of non-equilibrium sorption, third boundary value problem, global single-valued solvability.

Introduction

Almost all liquids existing in nature are solutions, i.e. mixtures of two or more substances (components). Filtration in porous media of liquids and gases containing associated (dissolved, suspended) solids is accompanied by diffusion of these substances and mass transfer between liquid (gas) and solid phases. The most common types of mass transfer are sorption and desorption, ion exchange, dissolution and crystallization, colmatation, sulfation and suffusion, paraffinization. The problems

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of equilibrium and non-equilibrium sorption are considered taking into account the peculiarities of physical and chemical interaction of solutions with formation rocks.

Let $m(x,t)$ be the porosity of the medium, $0 < m(x,t) \leq 1$; pore space is filled with the solution and solid phase precipitated from the solution; $c(x,t)$ is a mass concentration of a certain substance in liquid phase (per unit volume of solution); $s(x,t)$ is a mass concentration of the solid phase of the substance precipitated (per unit of pore volume).

Under equilibrium conditions, when the contact between the solution and the solid phase is maintained for a sufficiently long time, the ratio between the concentrations $c(x,t)$ in solution and $s(x,t)$ on the sorbent is determined by sorption isotherm. At low concentrations of the solution, the amount of absorption is determined by the linear relationship – Henry isotherm $s = \Gamma c$, where $\Gamma > 0$ is a certain constant depending on the physical and chemical properties of the medium (the Henry constant).

Equilibrium sorption equations can not always fully characterize the features of absorption and metabolism in a two-phase solution – solid phase system. In works [1]-[3] were proposed mathematical models for describing the processes of non-equilibrium sorption. The concentration of the solid phase $s(x,t)$ is associated with the concentration $c(x,t)$ in the liquid phase with the equation

$$\frac{\partial s}{\partial t} = \frac{1}{\tau}(\Gamma c - s), \tag{1}$$

where the positive constant τ is the characteristic relaxation time, Γ is the Henry's constant. The concentration $c(x,t)$ of the substance in solution satisfies the equation

$$m\frac{\partial c}{\partial t} = D\Delta c - \mathbf{v} \cdot \nabla c - \frac{\partial s}{\partial t}, \tag{2}$$

where $D(x,t) > 0$ is the diffusion coefficient, $\mathbf{v}\,(x,t)$ is the vector of the filtration rate, which are considered known functions of these arguments; Δ is the Laplace operator, ∇ is the gradient, $\mathbf{v} \cdot \nabla c$ denotes the scalar product of the vectors $\mathbf{v}$ and ∇c.

In [4] a system of the type (1)–(2) is used in modeling the uranium mining process by the method of underground leaching. This method is one of the most promising methods of uranium mining, as well as gold and a number of other rare and non-ferrous metals. With this method, the developers work on the deposit at the site of its occurrence with the purpose of transferring useful components to the solution and subsequently extracting them, usually through wells drilled from the surface to the location of the deposit. Underground leaching is more attractive, safer and more efficient than traditional mining methods, when developing poor deposits, as well as deep-lying deposits. Currently, about thirty percent of all uranium in Russia is extracted by underground leaching. This method is widely used in Kazakhstan, Uzbekistan and the United States, where almost all uranium is extracted in this way.

Analytical solutions in multi-dimensional case of the third boundary value problem for non-equilibrium solute transport were studied in [5].

In [6] the global unique solvability of the first initial-boundary value problem for system (1)–(2) is proved. In [7, 8], a difference approximation of the differential problem was formulated using an implicit scheme, a solution of the difference problem was constructed using the sweep method, and the results of numerical experiments were presented.

In the present paper we consider the third boundary-value problem for the system of equations (1)–(2), describing the process of non-equilibrium sorption.

Statement of the problem. Let Ω be a bounded domain of n-dimensional space $\mathbb{R}^n$ with a sufficiently smooth boundary $S = \partial\Omega$, $Q_T = \Omega \times (0, T)$, $T > 0$; $S_T = S \times (0, T)$ be the lateral surface of a cylinder Q_T. It is required to find the functions $c(x, t)$, $s(x, t)$, defined in domain Q_T satisfying in Q_T the equations (1), (2), when the initial conditions

$$c(x, 0) = c_0(x),\ x \in \Omega, \tag{3}$$

$$s(x, 0) = s_0(x),\ x \in \Omega, \tag{4}$$

and the third boundary value condition are fulfilled:

$$\frac{\partial c(x, t)}{\partial \mathbf{n}} + p(x, t)c(x, t) = 0,\ (x, t) \in S_T, \tag{5}$$

where $\mathbf{n}(x)$ is the unit vector of the internal normal to boundary S at the point x.

Formulation of main result.

The main result of the paper is the following

Theorem. *Let $0 < \alpha < 1$ be a certain number. The boundary S of the domain Ω belongs to the Hölder class $C^{2+\alpha}$, the coefficients m, D, $\mathbf{v}$ of the equation (2) belong to the Hölder class $C^{\alpha,\alpha/2}\left(\bar{Q}_T\right)$, functions $c_0(x) \in C^{2+\alpha}\left(\bar{\Omega}\right)$, $s_0(x) \in C^{\alpha}\left(\bar{\Omega}\right)$, $p(x, t) \in C^{1+\alpha,(1+\alpha)/2}\left(\bar{S}_T\right)$, $p(x, t) \le 0$ $\forall (x, t) \in S_T$, the compatibility conditions of the zero order are satisfied:*

$$\frac{\partial c_0(x)}{\partial \mathbf{n}} + p(x, 0)c_0(x) = 0,\ x \in S,$$

and the conditions $0 \le c_0(x) \le M$, $0 \le s_0(x) \le \Gamma M$, $x \in \Omega$ are fulfilled. Then the problem (1)–(5) has a unique classical solution $c(x, t) \in C^{2+\alpha,1+\alpha/2}\left(\bar{Q}_T\right)$, $s(x, t) \in C^{\alpha,1+\alpha/2}\left(\bar{Q}_T\right)$ and estimates $0 \le c(x, t) \le M$, $0 \le s(x, t) \le \Gamma M$, $(x, t) \in Q_T$ are valid.

Proof. First, estimates are obtained that represent the maximum principle

$$0 \le c(x, t) \le M,\ (x, t) \in Q_T, \tag{6}$$

$$0 \le s(x, t) \le \Gamma M,\ (x, t) \in Q_T. \tag{7}$$

From (1) and (3) we obtain the representation

$$s(x, t) = s_0(x)e^{-t/\tau} + \frac{\Gamma}{\tau}e^{-t/\tau}\int_0^t c(x, \theta)e^{\theta/\tau}\,d\theta. \tag{8}$$

Substituting (8) into (2), we obtain

$$m\frac{\partial c}{\partial t} - D\Delta c + \mathbf{v}\cdot\nabla c + \frac{\Gamma}{\tau}c = \frac{1}{\tau}s_0(x)e^{-t/\tau} + \frac{\Gamma}{\tau^2}e^{-t/\tau}\int_0^t c(x, \theta)e^{\theta/\tau}d\theta. \tag{9}$$

Suppose that the negative minimum $c_{\min} < 0$ of the function $c(x, t)$ is attained at some point (x_0, t_0) inside the domain Q_T. Then at this point $c_t \le 0$, $-\Delta c \le 0$, $\nabla c = 0$ and from (9) we obtain

$$\frac{\Gamma}{\tau}c_{\min} \ge \frac{1}{\tau}s_0(x_0)e^{-t_0/\tau} + \frac{\Gamma}{\tau^2}c_{\min}e^{-t_0/\tau}\int_0^{t_0} e^{\theta/\tau}d\theta,$$

$$\Gamma c_{\min} \ge s_0(x_0)e^{-t_0/\tau} + \Gamma c_{\min}e^{-t_0/\tau}(e^{t_0/\tau} - 1),$$

$$0 \geq s_0(x_0)e^{-t_0/\tau} - \Gamma c_{\min} e^{-t_0/\tau},$$

that is, they got a contradiction, because $s_0(x) \geq 0$ and $c_{\min} < 0$. Consequently, the negative minimum of the function $c(x,t)$ can not be achieved within the domain Q_T.

On the boundary S_T the minimum can not be achieved by condition (5) and the Zaremba-Giraud lemma.

Lemma. *Let* $Lu = \sum\limits_{i,j=1}^{n} a_{i,j}(x)u_{x_i x_j}(x) + \sum\limits_{i=1}^{n} b_i(x)u_{x_i}(x)$ *be an elliptic operator in a bounded domain* Ω*, with a sufficiently smooth boundary,* $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$*,* $Lu \leq 0$ *in* Ω *and let the function* $u(x)$ *reach a strict global minimum at the boundary point* $x_0 \in \partial\Omega$*. Then* $\frac{\partial u}{\partial \mathbf{n}}\big|_{x_0} > 0$*, where* $\mathbf{n}$ *is the internal normal to* $\partial\Omega$ *at the point* x_0.

This lemma for harmonic functions was proved by Zaremba [9], and in a more general formulation by Giraud [10].

In our case, we consider

$$Lc = D\Delta c - \mathbf{v}\cdot\nabla c = F(x,t) = \frac{\partial c}{\partial t} + \frac{\Gamma}{\tau}c - \frac{1}{\tau}s_0(x)e^{-t/\tau} - \frac{\Gamma}{\tau^2}e^{-t/\tau}\int_0^t c(x,\theta)e^{\theta/\tau}d\theta.$$

Suppose that the negative minimum $c_{\min} < 0$ of the function $c(x,t)$ is attained at some point (x_0,t_0) on the boundary S_T. Then

$$F(x_0,t_0) = \frac{\partial c}{\partial t}(x_0,t_0) + \frac{\Gamma}{\tau}c_{\min} - \frac{1}{\tau}s_0(x_0)e^{-t_0/\tau} - \frac{\Gamma}{\tau^2}e^{-t_0/\tau}\int_0^{t_0} c(x_0,\theta)e^{\theta/\tau}d\theta \leq$$

$$\leq \frac{\partial c}{\partial t}(x_0,t_0) + \frac{\Gamma}{\tau}c_{\min} - \frac{1}{\tau}s_0(x_0)e^{-t_0/\tau} - \frac{\Gamma}{\tau^2}c_{\min}e^{-t_0/\tau}\int_0^{t_0} e^{\theta/\tau}d\theta =$$

$$= \frac{\partial c}{\partial t}(x_0,t_0) + \frac{\Gamma}{\tau}c_{\min} - \frac{1}{\tau}s_0(x_0)e^{-t_0/\tau} - \frac{\Gamma}{\tau}c_{\min}e^{-t_0/\tau}(e^{t_0/\tau}-1) =$$

$$= \frac{\partial c}{\partial t}(x_0,t_0) - \frac{1}{\tau}s_0(x_0)e^{-t_0/\tau} + \frac{\Gamma}{\tau}c_{\min}e^{-t_0/\tau} < 0.$$

Hence $F(x,t_0) < 0$ in a neighborhood of a point x_0 and the Zaremba-Giraud lemma can be applied, that is $\frac{\partial c}{\partial \mathbf{n}}\big|_{(x_0,t_0)} > 0$. But this contradicts the corollary of boundary condition (5):

$$\left.\frac{\partial c}{\partial \mathbf{n}}\right|_{(x_0,t_0)} = -p(x_0,t_0)c_{\min} \leq 0.$$

Thus, the minimum of the function $c(x,t)$ is achieved at the lower boundary of the domain Q_T, that is at the initial time. At the initial time, the function $c_0(x)$ is nonnegative. Thus, we have proved that $c(x,t) \geq 0$, $(x,t) \in Q_T$.

Suppose now that within the domain Q_T a positive maximum $c_{\max} > M$ of the function $c(x,t)$ is attained, that is there exists a point $(x_1,t_1) \in Q_T : c(x_1,t_1) = c_{\max} > M$. At this point $c_t \geq 0$, $-\Delta c \geq 0$, $\nabla c = 0$, and from (9) we obtain the inequalities

$$\frac{\Gamma}{\tau}c_{\max} \leq \frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} + \frac{\Gamma}{\tau^2}c_{\max}e^{-t_1/\tau}\int_0^{t_1} e^{\theta/\tau}d\theta,$$

$$\Gamma c_{\max} \leq s_0(x_1)e^{-t_1/\tau} + \Gamma c_{\max}e^{-t_1/\tau}(e^{t_1/\tau}-1),$$

$$0 \leq s_0(x_1)e^{-t_1/\tau} - \Gamma c_{\max}e^{-t_1/\tau} = (s_0(x_1) - \Gamma c_{\max})\,e^{-t_1/\tau}.$$

Again we have a contradiction, because $s_0(x) \leq \Gamma M$, and $c_{\max} > M$.

The maximum of the function $c(x,t)$ can not be reached on the boundary S_T because of condition (5) and the Zaremba-Giraud lemma. Let's consider

$$Lc = D\Delta c - \mathbf{v}\cdot\nabla c = F(x,t) = \frac{\partial c}{\partial t} + \frac{\Gamma}{\tau}c - \frac{1}{\tau}s_0(x)e^{-t/\tau} - \frac{\Gamma}{\tau^2}e^{-t/\tau}\int_0^t c(x,\theta)e^{\theta/\tau}d\theta.$$

Suppose that the positive maximum $c_{\max} > M$ of the function $c(x,t)$ is reached at some point (x_1,t_1) on the boundary of the domain S_T. Then

$$\begin{aligned} F(x_1,t_1) &= \frac{\partial c}{\partial t}(x_1,t_1) + \frac{\Gamma}{\tau}c_{\max} - \frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} - \frac{\Gamma}{\tau^2}e^{-t_1/\tau}\int_0^{t_1} c(x_1,\theta)e^{\theta/\tau}d\theta \geq \\ &\geq \frac{\Gamma}{\tau}c_{\max} - \frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} - \frac{\Gamma}{\tau^2}c_{\max}e^{-t_1/\tau}\int_0^{t_1} e^{\theta/\tau}d\theta = \\ &= \frac{\Gamma}{\tau}c_{\max} - \frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} - \frac{\Gamma}{\tau}c_{\max}e^{-t_1/\tau}(e^{t_1/\tau}-1) = \\ &= -\frac{1}{\tau}s_0(x_1)e^{-t_1/\tau} + \frac{\Gamma}{\tau}c_{\max}e^{-t_1/\tau} = \\ &= \frac{1}{\tau}\left(\Gamma c_{\max} - s_0(x_1)\right)e^{-t_1/\tau} > \frac{1}{\tau}\left(\Gamma M - \Gamma M\right)e^{-t_1/\tau} = 0. \end{aligned}$$

Consequently, $F(x,t_1) > 0$ in the neighborhood of the point x_1 and one can apply the Zaremba-Giraud lemma for the case of a maximum, that is $\frac{\partial c}{\partial \mathbf{n}}\big|_{(x_1,t_1)} < 0$. But this contradicts the corollary of boundary condition (5):

$$\left.\frac{\partial c}{\partial \mathbf{n}}\right|_{(x_1,t_1)} = -p(x_1,t_1)\,c_{\max} \geq 0.$$

Thus, the maximum of the function $c(x,t)$ is achieved at the lower boundary of the domain Q_T, that is at the initial time. At the initial instant of time, the function $c_0(x) \leq M$. Therefore, $c(x,t) \leq M$, $(x,t) \in Q_T$. The estimate (6) is proved.

The estimate (7) follows from the representation (8) using (6). In fact, since $s_0(x) \geq 0$, $c(x,t) \geq 0$, it follows from (8) that $s(x,t) \geq 0$, $(x,t) \in Q_T$.

Since $s_0(x) \leq \Gamma M$, $c(x,t) \leq M$, then

$$s(x,t) \leq s_0(x)e^{-t/\tau} + \frac{\Gamma M}{\tau}e^{-t/\tau}\int_0^t e^{\theta/\tau}d\theta \leq \Gamma M e^{-t/\tau} + \Gamma M e^{-t/\tau}(e^{t/\tau}-1) = \Gamma M.$$

The estimate (7) is proved.

The uniqueness of the solution of problem (1)–(5) is a consequence of the estimates (6), (7).

The existence of a solution of problem (1)–(5) is proved with the help of Schauder's theorem on the fixed point of a completely continuous operator. Denote by V_{T_1} the next closed convex subset of $C^{2+\alpha,1+\alpha/2}(\bar{Q}_{T_1})$:

$$V_{T_1} = \Big\{ \tilde{c}(x,t) |\ \tilde{c}(x,0) = c_0(x), x \in \Omega;\ \frac{\partial \tilde{c}(x,t)}{\partial \mathbf{n}} + p(x,t)\tilde{c}(x,t) = 0, (x,t) \in S_{T_1};$$
$$||\tilde{c}||_{C^{2+\alpha,1+\alpha/2}(\bar{Q}_{T_1})} \leq K \Big\},$$

where K is some fixed positive number depending on the data of problem (1)–(5), which we will define later. By a given function $\tilde{c} \in V_{T_1}$ we find the function

$$\tilde{s}(x,t) = s_0(x)e^{-t/\tau} + \frac{\Gamma}{\tau}e^{-t/\tau}\int_0^t \tilde{c}(x,\theta)e^{\theta/\tau}d\theta. \tag{10}$$

Now to each function $\tilde{c} \in V_{T_1}$ we put the function $c = \Lambda(\tilde{c})$ as a solution of the problem

$$m\frac{\partial c}{\partial t} - D\Delta c + \mathbf{v}\cdot\nabla c + \frac{\Gamma}{\tau}c = \frac{1}{\tau}\tilde{s}, \tag{11}$$

$$c(x,0) = c_0(x),\ x \in \Omega;\ \frac{\partial c(x,t)}{\partial \mathbf{n}} + p(x,t)c(x,t) = 0,\ (x,t) \in S_{T_1}. \tag{12}$$

Let us prove that the operator Λ is completely continuous and, for sufficiently small T_1, maps the set V_{T_1} into itself.

Let us show that $\tilde{s} \in C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)$. It follows from (10) that

$$|\tilde{s}|^{(0)}_{Q_{T_1}} \equiv \max_{(x,t)\in\bar{Q}_{T_1}} |\tilde{s}(x,t)| \le |s_0|^{(0)}_{\Omega} + \Gamma|\tilde{c}|^{(0)}_{Q_{T_1}} \max_{t\in[0,T_1]}(1 - e^{-t/\tau}).$$

Hence, using the expansion of the function $e^{-t/\tau}$ in the Maclaurin series, it is easy to obtain (for $T_1 < \tau$) the estimate

$$|\tilde{s}|^{(0)}_{Q_{T_1}} \le |s_0|^{(0)}_{\Omega} + T_1\frac{\Gamma}{\tau}|\tilde{c}|^{(0)}_{Q_{T_1}}. \tag{13}$$

Similarly, from (10) follows the estimate

$$|\tilde{s}|^{(\alpha)}_{x,Q_{T_1}} \equiv \sup_{(x,t),(x',t)\in\bar{Q}_{T_1}} \frac{|\tilde{s}(x,t) - \tilde{s}(x',t)|}{|x-x'|^{\alpha}} \le$$

$$\le |s_0|^{(\alpha)}_{x,\Omega} + \Gamma|\tilde{c}|^{(\alpha)}_{x,Q_{T_1}} \max_{t\in[0,T_1]}(1 - e^{-t/\tau}) \le |s_0|^{(\alpha)}_{x,\Omega} + T_1\frac{\Gamma}{\tau}|\tilde{c}|^{(\alpha)}_{x,Q_{T_1}}, \tag{14}$$

that is the function $\tilde{s}$ satisfies the Hölder condition with respect to the space variable with exponent α.

The function $\tilde{s}$ satisfies the Hölder condition with respect to the variable t with any exponent $0 < \beta \le 1$ (even Lipschitz), since it has a bounded derivative with respect to time

$$\tilde{s}_t(x,t) = -\frac{1}{\tau}s_0(x)e^{-t/\tau} - \frac{\Gamma}{\tau^2}e^{-t/\tau}\int_0^t \tilde{c}(x,\theta)e^{\theta/\tau}d\theta + \frac{\Gamma}{\tau}\tilde{c}(x,t),$$

$$|\tilde{s}_t|^{(0)}_{Q_{T_1}} \le \frac{1}{\tau}|s_0|^{(0)}_{\Omega} + \frac{\Gamma}{\tau}|\tilde{c}|^{(0)}_{Q_{T_1}} \max_{t\in[0,T_1]}(1-e^{-t/\tau}) + \frac{\Gamma}{\tau}|\tilde{c}|^{(0)}_{Q_{T_1}} \le \frac{1}{\tau}|s_0|^{(0)}_{\Omega} + \frac{2\Gamma}{\tau}|\tilde{c}|^{(0)}_{Q_{T_1}}. \tag{15}$$

In particular, with $\beta = 1$ we have

$$\frac{|\tilde{s}(x,t) - \tilde{s}(x,t')|}{|t-t'|^{\alpha/2}|t-t'|^{1-\alpha/2}} \le |\tilde{s}_t|^{(0)}_{Q_{T_1}}.$$

This implies the inequality

$$|\tilde{s}|^{(\alpha/2)}_{t,Q_{T_1}} \equiv \sup_{(x,t),(x,t')\in\bar{Q}_{T_1}} \frac{|\tilde{s}(x,t) - \tilde{s}(x,t')|}{|t-t'|^{\alpha/2}} \le T_1^{1-\alpha/2}|\tilde{s}_t|^{(0)}_{Q_{T_1}}. \tag{16}$$

Estimates (13)–(16) prove that $\tilde{s} \in C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)$ and under the condition $T_1 < 1$ the next estimate holds

$$||\tilde{s}||_{C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)} \le C_1||s_0||_{C^{\alpha}(\bar{\Omega})} + T_1^{1-\alpha/2}C_2||\tilde{c}||_{C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)}, \tag{17}$$

where C_1, C_2 are some positive constants that do not depend on s_0, $\tilde{c}$. We will assume that C_1, C_2 depends on T, but does not depend on $T_1 < \min\{T, 1, \tau\}$.

Since

$$\tilde{s}_t = \frac{1}{\tau}(\Gamma \tilde{c} - \tilde{s}),$$

then $\tilde{s} \in C^{\alpha,1+\alpha/2}\left(\bar{Q}_{T_1}\right)$.

For a solution $c(x,t)$ of the problem (11), (12), the estimate [11, p. 365] is valid

$$||c||_{C^{2+\alpha,1+\alpha/2}\left(\bar{Q}_{T_1}\right)} \le C(||c_0||_{C^{2+\alpha}(\bar{\Omega})} + ||\tilde{s}||_{C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)}), \tag{18}$$

where C is a positive constant independent of c_0, $\tilde{s}$. We will assume that C depends on T, but does not depend on $T_1 < T$. Using (17), (18), we have

$$||c||_{C^{2+\alpha,1+\alpha/2}\left(\bar{Q}_{T_1}\right)} \le C_3(||c_0||_{C^{2+\alpha}(\bar{\Omega})} + ||s_0||_{C^{\alpha}(\bar{\Omega})})+ \tag{19}$$

$$+T_1^{1-\alpha/2} C_4 ||\tilde{c}||_{C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)}.$$

This implies that the operator $\Lambda : \tilde{c} \to c$ is completely continuous.

We choose the constant K, that appears in the definition of the set V_{T_1}, as outcome of the condition

$$K > C_3(||c_0||_{C^{2+\alpha}(\bar{\Omega})} + ||s_0||_{C^{\alpha}(\bar{\Omega})}).$$

For definiteness, we set

$$K = 2C_3(||c_0||_{C^{2+\alpha}(\bar{\Omega})} + ||s_0||_{C^{\alpha}(\bar{\Omega})}).$$

Then it follows from (19) that for sufficiently small T_1 the operator Λ maps the set V_{T_1} into itself.

By Schauder's theorem on the fixed point of a completely continuous operator, the set V_{T_1} contains a fixed point $\tilde{c}$, which together with its corresponding function $\tilde{s}$ from (10) is a solution of problem (1)–(5) on the time interval $[0, T_1]$.

The solution can be continued in k steps to $[T_k, T_{k+1}]$, $k = 1, 2, ...,$ and $T_{k+1} - T_k \ge \delta > 0$ and δ does not depend on the number k. This can be seen from the estimate (19)

$$C_3(||c_0||_{C^{2+\alpha}(\bar{\Omega})} + ||s_0||_{C^{\alpha}(\bar{\Omega})}) + T_1^{1-\alpha/2} C_4 ||\tilde{c}||_{C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)} < K =$$

$$= 2C_3(||c_0||_{C^{2+\alpha}(\bar{\Omega})} + ||s_0||_{C^{\alpha}(\bar{\Omega})}).$$

Because $||\tilde{c}||_{C^{\alpha,\alpha/2}\left(\bar{Q}_{T_1}\right)} \le K$, then it follows that as δ can be chosen

$$\delta^{1-\alpha/2} = \frac{K}{2C_4 K} = \frac{1}{2C_4},$$

not depending on the number k. Thus, the solution in a finite number of steps can be continued to any $0 < T < +\infty$. □

Conclusion.

In this paper, a global unique solvability of the third boundary value problem modeling the process of non-equilibrium sorption is proved.

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