Greedy cycles in the star graphs

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Abstract. We apply the greedy approach to construct greedy cycles in Star graphs $S_n$, $n \geq 3$, defined as Cayley graphs on the symmetric group $\text{Sym}_n$ with generating set $t = \{(1 \ i), 2 \leq i \leq n\}$ of transpositions. We define greedy sequences presented by distinct elements from $t$, and prove that any greedy sequence of length $k$, $2 \leq k \leq n - 1$, forms a greedy cycle of length $2 \cdot 3^{k-1}$. Based on these greedy sequences we give a construction of a maximal set of independent greedy cycles in the Star graphs $S_n$ for any $n \geq 3$.

Keywords: Cayley graph; Star graph; greedy sequence; greedy cycle

1. Introduction

The Star graph $S_n = \text{Cay}(\text{Sym}_n, t)$, $n \geq 2$, is defined as the Cayley graph on the symmetric group $\text{Sym}_n$ of permutations $\tau = [\pi_1 \pi_2 \ldots \pi_i \ldots \pi_n]$ with the generating set $t = \{t_i \in \text{Sym}_n, 2 \leq i \leq n\}$ of all transpositions $t_i = (1 \ i)$ swapping the 1-st and $i$-th elements of a permutation $\tau$. It is a connected bipartite $(n - 1)$-regular graph of order $n!$ containing all even $\ell$-cycles, $6 \leq \ell \leq n!$, with the sole exception when $\ell = 4$ [2]. The hamiltonicity of this graph also follows from [7].

Greedy hamiltonian cycles were investigated in [6, 11] for constructing prefix–reversal Gray codes in the Pancake graphs $P_n = (\text{Sym}_n, PR)$ defined as Cayley graphs on $\text{Sym}_n$ with the generating set $PR = \{r_i \in \text{Sym}_n, 2 \leq i \leq n\}$ of all prefix–reversals $r_i$ inversing the order of any substring $[1, i]$ of a permutation when multiplied on the right. A greedy approach for constructing greedy cycles in $P_n$...
was initiated in [10]. Consider a sequence $GP = (r_{m_1}, r_{m_2}, \ldots, r_{m_k})$ of distinct $k \leq n - 1$ prefix–reversals $r_{m_j}, 2 \leq m_j \leq n$, from the set $PR$. A greedy cycle is formed by consecutive application of the leftmost suitable prefix–reversal from $GP$ which is called a greedy sequence of length $k$ in this setting. There are at least four different greedy sequences forming distinct greedy hamiltonian cycles in the Pancake graphs [6, 11, 12]. For example, greedy hamiltonian cycles formed by greedy sequences $(r_4, r_3, r_2)$ and $(r_4, r_2, r_3)$ in the Pancake graph $P_4$ are presented in Figure 1 (with dotted edges not belonging to the cycles).

The main goal of this paper is to investigate greedy sequences in the Star graphs with respect to the corresponding greedy cycles. Let us define a sequence

$$GS = (t_{m_1}, t_{m_2}, \ldots, t_{m_k})$$

of distinct $k \leq n - 1$ transpositions $t_{m_j}, 2 \leq m_j \leq n$, from the set $t$ in the Star graph $S_n$ for $n \geq 3$. A cycle $C$ is called a GS-greedy cycle if it is formed greedily by a sequence $GS$ which is called a greedy sequence of length $k$. A GS-greedy cycle without one edge is called a GS-greedy path. If a GS-greedy path has terminal vertices $\pi$ and $\tau$, then we write

$$\pi \xrightarrow{GS} \tau$$

taking into account all permutations within this path.

In this paper we prove the following main result.

**Theorem 1.** In the Star graph $S_n, n \geq 3$, any greedy sequence $GS$ of length $k$, where $2 \leq k \leq n - 1$, forms a GS-greedy cycle of length $2 \cdot 3^{k-1}$.

From this theorem we immediately have that there are no greedy hamiltonian cycles in the Star graphs. The following question naturally arises here: is it possible to get a cyclic covering of the Star graphs using their GS-greedy cycles from Theorem 1? Let us remind, that a two vertex–disjoint cycles are called independent. An independent set of cycles is a set of pairwise independent cycles in a graph. A set of independent cycles covering all vertices of a graph is called maximal. A maximal set of independent GS-greedy cycles in the Star graph is called GS-greedy cyclic
covering. Cycle coverings are interesting not only in their own right [1, 8], they are an important tool for the design of the traveling salesman problems [9] and for vehicle routing problems [3] aiming to get hamiltonian cycles in graphs.

Short cycles of the Star graphs were investigated in [6], where the characterization of 6-cycles and 8-cycles was given. In particular, 6-cycles are described by greedy sequences \((t_{m_1}, t_{m_2})\), where \(2 \leq m_1 < m_2 \leq n\). Moreover, a greedy cyclic covering can be found by a sequence \((t_{m_1}, t_{m_2})\) for given \(m_1\) and \(m_2\), \(2 \leq m_1 < m_2 \leq n\). Such a trivial greedy covering has \(\frac{n}{6}\) cycles of length 6. This is the only known greedy covering with cycles of fixed length. In the case of 8-cycles, there are only non-greedy cyclic coverings each of which has \(\frac{n}{8}\) cycles of length 8.

Theorem 1 gives a tool to get a GS-greedy cyclic covering with cycles of length \(2 \cdot 3^{k-1}\) for all \(2 \leq k \leq n - 1\). In this paper we construct such a maximal set of independent GS-greedy cycles in the Star graph \(S_n\), \(n \geq 3\). This is the first example of a greedy cyclic covering of the Star graph consisting cycles of non-fixed length.

The paper is organized as follows. In Section 2 we prove Theorem 1 and its corollaries. In Section 3 we construct a maximal set of independent GS-greedy cycles in the Star graph \(S_n\), \(n \geq 3\), that consists of greedy cycles of length \(2 \cdot 3^{k-1}\) for all \(2 \leq k \leq n - 1\). In proofs below we use hierarchical structure of \(S_n\) having \(n\) copies of induced subgraphs \(S_{n-1}(i)\), \(1 \leq i \leq n\), on \((n-1)!\) permutations of form \([\pi_1 \ldots \pi_{n-1} i]\) with the fixed last element [4].

2. Proof of Theorem 1

Proof. Since for any \(n \geq 3\) the graph \(S_n\) is vertex–transitive as any Cayley graph and edge–transitive [4], without loss of generality we prove the theorem by induction for a given sequence

\[
GS_m = (t_2, t_3, \ldots, t_m), \ 3 \leq m \leq n,
\]

of the transpositions \(t_i\), \(2 \leq i \leq m\), from the set \(t\) in the Star graph \(S_n\) with the starting vertex corresponding to the identity permutation \(I_n = [12 \ldots n]\). If \(n = 3\), then \(S_3 \cong C_6\) and \(m = 3\), hence \(GS_3 = (t_2, t_3)\) is a greedy sequence of length two generating six permutations as follows:

\[
GS_3 : \quad [123] \xrightarrow{t_2} [213] \xrightarrow{t_3} [312] \xrightarrow{t_2} [132] \xrightarrow{t_3} [231] \xrightarrow{t_2} [321],
\]

which obviously form a cycle of length \(2 \cdot 3^{2-1} = 6\). If \(n = 4\), then either \(m = 3\) or \(m = 4\). By hierarchical structure, \(S_4\) has four copies of \(S_3\), hence \(GS_4\) gives the same result as above. If \(m = 4\), then applying \(GS_4 = (t_2, t_3, t_4)\) greedily to the vertex \(I_n = [1234]\) we obtain 18 distinct permutations that form the \((t_2, t_3, t_4)\)-greedy 18-cycle in \(S_4\):

\[
[1234] \xrightarrow{t_2} [2134] \xrightarrow{t_3} [3124] \xrightarrow{t_2} [1324] \xrightarrow{t_3} [2314] \xrightarrow{t_2} [3214] \xrightarrow{t_4} [4231], \quad [1234] \xrightarrow{t_2} [2134] \xrightarrow{t_3} [3124] \xrightarrow{t_2} [1324] \xrightarrow{t_3} [2314] \xrightarrow{t_2} [3214] \xrightarrow{t_4} [4231],
\]

where \([4231] \circ t_4 = [1234]\). The Star graph \(S_4\) and its \((t_2, t_3, t_4)\)-greedy cycle of length 18 are presented in Figure 2.
By notation (2) the list of permutations above can be represented as follows:

\[
G_{S_4} : [1234] \xrightarrow{G_{S_3}} [3214] \xrightarrow{t_k} [4213] \xrightarrow{G_{S_3}} [1243] \xrightarrow{t_k} [3241] \xrightarrow{G_{S_3}} [4231],
\]

where \(G_{S_3}\) acts on six vertices of three copies of \(S_3\) in \(S_4\). From this, we conclude that \(G_{S_4}\) is a greedy sequence that forms a greedy cycle of length \(6 \cdot 3 = 2 \cdot 3^{k-1}\).

Suppose that the statement of the theorem holds for greedy sequences of length \(m \leq k - 1\), where \(k \geq 5\), a sequence \(G_{S_k} = (t_2, t_3, \ldots, t_k)\) forms a greedy cycle of length \(2 \cdot 3^{k-2}\) in \(S_k\) and has a \(G_{S_k}\)-greedy path as follows:

\[ [1 \ 2 \ 3 \ldots k \ 1 \ 1 \ 1] \xrightarrow{G_{S_k}} [1 \ 2 \ 3 \ldots k \ 2 \ 3 \ldots k-1]. \]

Since \(S_{k+1}\) has \(k + 1\) copies of \(S_k\), hence \(G_{S_k}\) forms the same cycle in \(S_{k+1}\).

Let us prove the theorem for a greedy sequence \(G_{S_{k+1}} = (t_2, t_3, \ldots, t_k, t_{k+1})\) of length \(k \leq n - 1\) started at \(I_{k+1} = [1 \ 2 \ldots k \ 1 \ 1 \ 1]\). According to the greedy approach, we have to apply consecutively the leftmost suitable transposition from \(G_{S_{k+1}}\) which is the same as to apply the leftmost suitable transposition from \(G_{S_k}\), or \(t_{k+1}\) otherwise. This process is repeated until the permutation \([k + 1 \ 2 \ 3 \ldots k - 1 \ 1 \ 1]\) is reached, which is the one adjacent to \(I_{k+1}\) in the Star graph \(S_{k+1}\). Since both starting and ending permutations were generated greedily by the \(G_{S_k}\), the final form \(G_{S_{k+1}}\) can be identified using the induction hypothesis (4):

\[
G_{S_{k+1}} : [1 \ 2 \ 3 \ldots k - 1 \ 1 \ 1 \ k + 1] \xrightarrow{G_{S_k}} [k \ 2 \ 3 \ldots k - 1 \ 1 \ 1 \ k + 1] \xrightarrow{t_{k+1}} [k \ 2 \ 3 \ldots k - 1 \ k + 1] \xrightarrow{G_{S_k}} [k \ 2 \ 3 \ldots k - 1 \ k + 1 \ 1] \xrightarrow{G_{S_k}} [k \ 2 \ 3 \ldots k - 1 \ k + 1].
\]

where \([k + 1 \ 2 \ 3 \ldots k - 1 \ 1] \circ t_{k+1} = I_{k+1}\) and each of three \(G_{S_k}\)-greedy sequences generates \(2 \cdot 3^{k-2}\) distinct permutations by the induction hypothesis, hence \(G_{S_{k+1}}\) generates \(2 \cdot 3^{k-2} \cdot 3 = 2 \cdot 3^{k-1}\) distinct permutations and this completes the proof. \(\square\)
In a GS-greedy cycle formed by \( GS = (t_{m_1}, t_{m_2}, \ldots, t_{m_k}) \), where \( k \leq n - 1 \), an edge generated by a transposition \( t_{m_j}, 2 \leq m_j \leq n \), is called an edge of type \( t_{m_j} \).

**Corollary 1.** Let \( C \) be a GS-greedy cycle formed by \( GS = (t_{m_1}, t_{m_2}, \ldots, t_{m_k}) \) in the Star graph \( S_n, n \geq 3 \), for \( 2 \leq k \leq n - 1 \). Then \( C \) contains \( 3^{k-1} \) edges of type \( t_{m_1}, 2 \cdot 3^{k-j} \) edges of type \( t_{m_j} \), for each \( 2 \leq j \leq k - 1 \), and \( 3 \) edges of type \( t_{m_k} \).

**Proof.** As previously discussed, let us prove the statement for \( GS_m = (t_2, t_3, \ldots, t_m) \), \( 3 \leq m \leq n \). If \( m = 3 \), then there are two greedy sequences \((t_2, t_3)\) and \((t_3, t_2)\), each of which forms a 6-cycle with 3 edges of each of types \( t_2 \) and \( t_3 \). If \( m = 4 \), then from the proof of Theorem 1 it follows that the number of edges of types \( t_2, t_3, t_4 \) and \( t_4 \) in any GS-greedy cycle of length 18 in \( S_n \) is distributed as follows:

<table>
<thead>
<tr>
<th>GS-greedy cycle</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
<th>( t_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td># edges in GS-greedy cycle</td>
<td>9</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

In general, for any \( m = k + 1 \), where \( 2 \leq k \leq n - 1 \), by the induction hypothesis of Theorem 1 and by hierarchical structure of Star graphs we have the following distribution of edges in the GS\(_{k+1}\)-greedy cycle of length \( 2 \cdot 3^{k-1} \):

<table>
<thead>
<tr>
<th>GS(_{k+1})-greedy cycle</th>
<th>( t_2 )</th>
<th>( t_3 )</th>
<th>( \ldots )</th>
<th>( t_{k-1} )</th>
<th>( t_k )</th>
<th>( t_{k+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td># edges in GS(_{k+1})-greedy cycle</td>
<td>( 3^{k-1} )</td>
<td>( 2 \cdot 3^{k-2} )</td>
<td>( \ldots )</td>
<td>( 2 \cdot 3^2 )</td>
<td>( 2 \cdot 3 )</td>
<td>( 3 )</td>
</tr>
</tbody>
</table>

which completes the proof. \( \square \)

By Corollary 1, any two different greedy sequences of length at least three should form cycles with specific edge distributions. A combination of this fact with Theorem 1 gives us the following statement.

**Corollary 2.** Any two different greedy sequences of length \( k \), \( 3 \leq k \leq n \), form distinct cycles of length \( 2 \cdot 3^{k-1} \) in the Star graph \( S_n, n \geq 4 \).

Corollary 2 does not hold for \( k = 2 \) since greedy sequences \((t_2, t_3)\) and \((t_3, t_2)\) produce the same 6-cycles in the graph \( S_n \) for any \( n \geq 3 \).

In the next corollary we use the following simple property of greedy sequences.

**Mirror property.** Any greedy sequence \( GS = (t_{m_1}, t_{m_2}, \ldots, t_{m_k}) \) forms the same cycle at starting vertices \( \pi \) and \( \tau = \pi \circ t_{m_k} \) so that vertices of a greedy path \( \pi \xrightarrow{GS} \tau \) are written in the reverse order in a greedy path \( \tau \xrightarrow{GS} \pi \).

**Corollary 3.** Any greedy sequence of length \( k \), \( 2 \leq k \leq n - 1 \), gives \( \frac{6}{k} \) distinct cycles of length \( 2 \cdot 3^{k-1} \) in the Star graph \( S_n, n \geq 3 \).

**Proof.** We prove this statement with \( GS_{k+1} = (t_2, t_3, \ldots, t_k, t_{k+1}) \) as a greedy sequence of length \( k \), where \( 2 \leq k \leq n - 1 \). By Theorem 1, the greedy sequence \( GS_{k+1} \) forms a GS\(_{k+1}\)-greedy cycle of length \( 2 \cdot 3^{k-1} \) in \( S_n \) for any \( n \geq 3 \). We consider \( S_{k+1} \) and present this cycle by its GS\(_{k+1}\)-greedy path as follows:

\begin{equation}
GS_{k+1} : I_{k+1} \xrightarrow{GS_k} \tau_1 \xrightarrow{t_{k+1}} \tau_2 \xrightarrow{GS_k} \tau_3 \xrightarrow{t_{k+1}} \tau_4 \xrightarrow{GS_k} \tau_5,
\end{equation}

where

\( \tau_1 = [k \ 23 \ldots k-1 \ 11 \ k + 1] \),

\( \tau_2 = [k + 1 \ 23 \ldots k - 11 \ k] \),
\[
\tau_3 = [1 \ 2 \ 3 \ldots k \ -1 \ k + k],
\]
\[
\tau_4 = [k \ 2 \ 3 \ldots k \ -1 \ k + 11],
\]
\[
\tau_5 = [k + 1 \ 2 \ 3 \ldots k \ -1 \ k 1],
\]
and \(\tau_5 \circ t_{k+1} = I_{k+1}\). By Mirror property, the \(GS_{k+1}\)-greedy cycle (5) can be also presented by one of the following five \(GS_{k+1}\)-greedy paths:

\[
\tau_1 \xrightarrow{GS_k} I_n \xrightarrow{t_{k+1}} \tau_5 \xrightarrow{GS_k} \tau_4 \xrightarrow{t_{k+1}} \tau_3 \xrightarrow{GS_k} \tau_2,
\]
\[
\tau_2 \xrightarrow{GS_k} \tau_3 \xrightarrow{t_{k+1}} \tau_4 \xrightarrow{GS_k} \tau_5 \xrightarrow{t_{k+1}} I_n \xrightarrow{GS_k} \tau_1,
\]
\[
\tau_3 \xrightarrow{GS_k} \tau_2 \xrightarrow{t_{k+1}} \tau_1 \xrightarrow{GS_k} I_n \xrightarrow{t_{k+1}} \tau_5 \xrightarrow{GS_k} \tau_4,
\]
\[
\tau_4 \xrightarrow{GS_k} \tau_5 \xrightarrow{t_{k+1}} I_n \xrightarrow{GS_k} \tau_1 \xrightarrow{t_{k+1}} \tau_2 \xrightarrow{GS_k} \tau_3,
\]
\[
\tau_5 \xrightarrow{GS_k} \tau_4 \xrightarrow{t_{k+1}} \tau_3 \xrightarrow{GS_k} \tau_2 \xrightarrow{t_{k+1}} \tau_1 \xrightarrow{GS_k} I_n,
\]

which totally with (5) give us six different representations of the \(GS_{k+1}\)-greedy cycle. However, whenever we apply the greedy sequence \(GS_{k+1}\) to a permutation which is not in the set \(\{I_n, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}\) we never get the same \(GS_{k+1}\)-greedy cycle again. Thus, \(GS_{k+1}\) forms \(n! / 6\) distinct cycles of length \(2 \cdot 3^{k-1}\) in the Star graph \(S_n, n \geq 3\). By edge–transitivity of \(S_n\), this holds for any greedy sequence and completes the proof. ~\(\square\)

As illustration of Corollary 3, a schematic representation of a \(GS_5\)-greedy cycle of length \(54 = 2 \cdot 3^3\) with fixed permutations \(I_n, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\) is given in Figure 3.
We finished analyzing greedy sequences and the corresponding greedy cycles in Star graphs. We use the obtained results in the next section to show that there is a cyclic covering by greedy cycles in Star graphs.

3. GS-greedy cyclic covering

Let \( \mathcal{F} = \{ GS_k = (t_2, t_3, \ldots, t_k), \ 3 \leq k \leq n \} \) be a family of greedy sequences.

**Theorem 2.** In the Star graph \( S_n, n \geq 3 \), there exists a maximal set of independent cycles formed by greedy sequences from the family \( \mathcal{F} \) consisting of the following cycles:

1. one cycle of length \( 2 \cdot 3^{n-2} \), and
2. \( n - 3 \) cycles of length \( 2 \cdot 3^{n-3} \) when \( n \geq 4 \), and
3. \( N_m \) cycles of length \( 2 \cdot 3^{n-m-2} \) for all \( 2 \leq m \leq n - 3 \) when \( n \geq 5 \), where

\[
N_m = \left( \prod_{l=2}^{m} (n-l+2) \right) (n-m-2).
\]

**Proof.** If \( n = 3 \), then \( S_3 \cong C_6 \) which is formed by the greedy sequence \( GS_3 = (t_2, t_3) \) and theorem holds. If \( n = 4 \), then by Theorem 1 greedy cycles of length 18 and 6 are formed by \( GS_4 = (t_2, t_3, t_4) \) and \( GS_3 = (t_2, t_3) \) correspondingly. Two such independent greedy cycles covering all vertices of \( S_4 \) are presented in Figure 4 (with dotted edges not belonging to the cycles). Thus, Theorem 2 holds for \( n = 4 \).

![Figure 4: GS-greedy cyclic covering of \( S_4 \)](image)

Now we construct independent greedy cycles in the Star graphs \( S_n \) for \( n \geq 5 \). First we obtain one cycle of length \( 2 \cdot 3^{n-2} \) using \( GS_n = (t_2, t_3, \ldots, t_n) \) as follows:

\[
\begin{align*}
GS_n: \ [1 \ 2 \ 3 \ \ldots \ n-1 \ n] & \xrightarrow{GS_{n-1}} [n-1 \ 2 \ 3 \ \ldots \ 1 \ n] \xrightarrow{t_n} \\
[n \ 2 \ 3 \ \ldots \ 1 \ n-1] & \xrightarrow{GS_{n-1}} [1 \ 2 \ 3 \ \ldots \ n \ n-1] \xrightarrow{t_n} \\
[n-1 \ 2 \ 3 \ \ldots \ n \ 1] & \xrightarrow{GS_{n-1}} [n \ 2 \ 3 \ \ldots \ n-1] 
\end{align*}
\]
where $GS_{n-1}$ acts on vertices of copies $S_{n-1}(n)$, $S_{n-1}(n-1)$ and $S_{n-1}(1)$ in $S_n$, where a copy $S_{n-1}(i)$, $1 \leq i \leq n$, contains $(n-1)!$ permutations $[\pi_1 \ldots \pi_{n-1}]$.

Then we take $n-3$ copies of $S_{n-1}(i)$ such that $i \not\in \{1, n-1, n\}$. By Theorem 1, we can always form greedy cycles of length $2 \cdot 3^{n-3}$ using the greedy sequence $GS_{n-1} = (t_2, t_3, \ldots, t_{n-1})$ at arbitrary chosen starting vertices in these $n-3$ copies. Since we take different copies of $S_n$, we obtain independent cycles. Thus, we construct $n-3$ independent greedy cycles of length $2 \cdot 3^{n-3}$.

By the hierarchical structure of $S_n$, each of $n$ copies $S_{n-1}(i)$, $1 \leq i \leq n$, contains $n-1$ copies of $S_{n-2}(ji)$, $1 \leq j \neq i \leq n$, each of which is an induced subgraph on $(n-2)!$ permutations of form $[\pi_1 \ldots \pi_{n-2}ji]$ with two last elements fixed. By the arguments above, for any $i$ there are three copies of $S_{n-2}(ji)$ each of which already has a $GS_{n-2}$-greedy path as a part of one of the constructed above $n-3$ independent $GS_{n-1}$-greedy cycles of length $2 \cdot 3^{n-3}$ or of the cycle of length $2 \cdot 3^{n-2}$. So, we take $n-4$ copies of $S_{n-2}(ji)$ whose vertices were not covered before and construct $GS_{n-2}$-greedy cycles of length $2 \cdot 3^{n-4}$ starting from arbitrary chosen vertices in each of $n-4$ copies. We can always do this by Theorem 1, and moreover all these cycles are independent by construction. Thus, totally we construct $n \cdot (n-4)$ independent cycles of length $2 \cdot 3^{n-4}$ using the greedy sequence $GS_{n-2} = (t_2, t_3, \ldots, t_{n-2})$.

In general, we consider $n \cdot (n-1) \ldots (n-m+2)$ copies of $S_{n-m+1}$ each of which contains $(n-m+1)!$ permutations with $(m-1)$ last elements fixed. For each copy of $S_{n-m+1}$, we take $(n-m-2)$ copies of $S_{n-m}$ whose vertices were not covered by previously constructed greedy cycles and construct $GS_{n-m}$-greedy cycles of length $2 \cdot 3^{n-m-2}$ starting from arbitrary chosen vertices in each of $(n-m-2)$ copies. We are always able to do this by Theorem 1. By construction, all these cycles are independent. We repeat the procedure above until we construct $n \cdot (n-1) \ldots 5$ independent cycles of length 6 using the greedy sequence $GS_3 = (t_2, t_3)$ in each of $\frac{n!}{2!}$ copies of $S_4$ for $n \geq 5$. In total we construct

$$N_m = n \cdot (n-1) \cdot (n-2) \ldots \cdot (n-m+2) \cdot (n-m-2)$$

cycles of length $2 \cdot 3^{n-m-2}$ using the greedy sequence $GS_{n-m} = (t_2, t_3, \ldots, t_{n-m})$ for any $2 \leq m \leq n-3$ and $n \geq 5$, each of which is independent.

Thus, we have

$$2 \cdot 3^{n-2} + (n-3) \cdot 2 \cdot 3^{n-3} + n \cdot (n-4) \cdot 2 \cdot 3^{n-4} + n \cdot (n-1) \cdot (n-5) \cdot 2 \cdot 3^{n-5} + \ldots + n \cdot (n-1) \ldots 5 \cdot 1 \cdot 2 \cdot 3 = n!$$

vertices covered by greedy cycles which completes the proof.

**Concluding remarks**

A maximal set of independent cycles described in Theorem 2 by greedy sequences from the family $\mathcal{F}$ contains one cycle of length $2 \cdot 3^{n-2}$, and also shorter cycles. One can obtain cycle coverings based on other families of greedy cycles, also it is possible to find construction consisting of a few longest $2 \cdot 3^{n-2}$-cycles. However, there is a way to get a hamiltonian cycle from our construction. It would be also interesting to construct a greedy cyclic covering with minimal number of cycles. This is a topic for future research.
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