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GREEDY CYCLES IN THE STAR GRAPHS

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Abstract. We apply the greedy approach to construct greedy cycles in Star graphs $S_n, n \geqslant 3$, defined as Cayley graphs on the symmetric group Sym_n with generating set $t = \{(1\,i), 2 \leqslant i \leqslant n\}$ of transpositions. We define greedy sequences presented by distinct elements from t, and prove that any greedy sequence of length k, $2 \leqslant k \leqslant n-1$, forms a greedy cycle of length $2 \cdot 3^{k-1}$. Based on these greedy sequences we give a construction of a maximal set of independent greedy cycles in the Star graphs S_n for any $n \geqslant 3$.

Keywords: Cayley graph; Star graph; greedy sequence; greedy cycle

1. Introduction

The *Star graph* $S_n = Cay(\mathrm{Sym}_n, t), n \geqslant 2$, is defined as the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1 \pi_2 \ldots \pi_i \ldots \pi_n]$ with the generating set $t = \{t_i \in \mathrm{Sym}_n, 2 \leqslant i \leqslant n\}$ of all transpositions $t_i = (1\ i)$ swapping the 1-st and i-th elements of a permutation π. It is a connected bipartite $(n-1)$-regular graph of order $n!$ containing all even ℓ-cycles, $6 \leqslant \ell \leqslant n!$, with the sole exception when $\ell = 4$ [2]. The hamiltonicity of this graph also follows from [7].

Greedy hamiltonian cycles were investigated in [6, 11] for constructing prefix–reversal Gray codes in the Pancake graphs $P_n = (\mathrm{Sym}_n, PR)$ defined as Cayley graphs on Sym_n with the generating set $PR = \{r_i \in \mathrm{Sym}_n, 2 \leqslant i \leqslant n\}$ of all prefix–reversals r_i inversing the order of any substring $[1, i]$ of a permutation when multiplied on the right. A *greedy approach* for constructing greedy cycles in P_n

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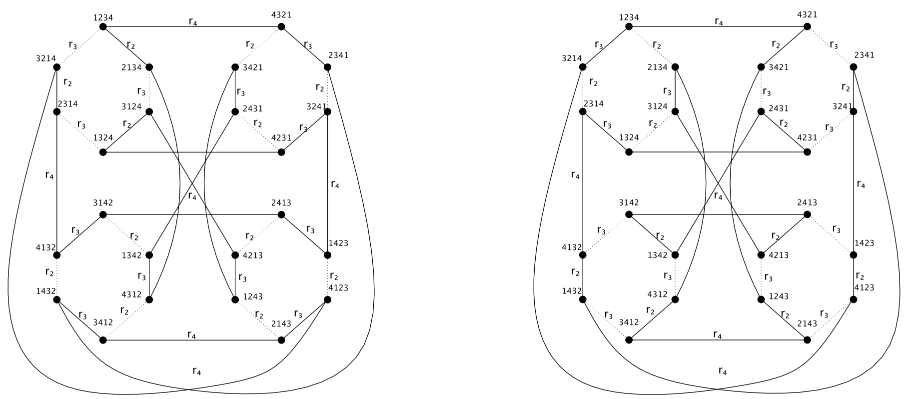
was initiated in [10]. Consider a sequence $GP = (r_{m_1}, r_{m_2}, \ldots, r_{m_k})$ of distinct $k \leqslant n-1$ prefix–reversals $r_{m_j}, 2 \leqslant m_j \leqslant n$, from the set PR. A *greedy cycle* is formed by consecutive application of the leftmost suitable prefix–reversal from GP which is called a *greedy sequence* of length k in this setting. There are at least four different greedy sequences forming distinct greedy hamiltonian cycles in the Pancake graphs [6, 11, 12]. For example, greedy hamiltonian cycles formed by greedy sequences (r_4, r_3, r_2) and (r_4, r_2, r_3) in the Pancake graph P_4 are presented in Figure 1 (with dotted edges not belonging to the cycles).



(r_4, r_3, r_2)-greedy hamiltonian cycle (r_4, r_2, r_3)-greedy hamiltonian cycle

FIG 1. Greedy hamiltonian cycles in the Pancake graph P_4

The main goal of this paper is to investigate greedy sequences in the Star graphs with respect to the corresponding greedy cycles. Let us define a sequence

$$GS = (t_{m_1}, t_{m_2}, \ldots, t_{m_k}) \tag{1}$$

of distinct $k \leqslant n-1$ transpositions $t_{m_j}, 2 \leqslant m_j \leqslant n$, from the set t in the Star graph S_n for $n \geqslant 3$. A cycle C is called a *GS-greedy cycle* if it is formed greedily by a sequence GS which is called a *greedy sequence of length k*. A GS-greedy cycle without one edge is called a *GS-greedy path.* If a GS-greedy path has terminal vertices π and τ, then we write

$$\pi \xrightarrow{GS} \tau \tag{2}$$

taking into account all permutations within this path.

In this paper we prove the following main result.

Theorem 1. *In the Star graph $S_n, n \geqslant 3$, any greedy sequence GS of length k, where $2 \leqslant k \leqslant n-1$, forms a GS-greedy cycle of length $2 \cdot 3^{k-1}$.*

From this theorem we immediately have that there are no greedy hamiltonian cycles in the Star graphs. The following question naturally arises here: is it possible to get a cyclic covering of the Star graphs using their GS-greedy cycles from Theorem 1? Let us remind, that a two vertex–disjoint cycles are called independent. An independent set of cycles is a set of pairwise independent cycles in a graph. A set of independent cycles covering all vertices of a graph is called maximal. A maximal set of independent GS-greedy cycles in the Star graph is called *GS-greedy cyclic* *covering*. Cycle coverings are interesting not only in their own right [1, 8], they are an important tool for the design of the traveling salesman problems [9] and for vehicle routing problems [3] aiming to get hamiltonian cycles in graphs.

Short cycles of the Star graphs were investigated in [6], where the characterization of 6-cycles and 8-cycles was given. In particular, 6-cycles are described by greedy sequences (t_{m_1}, t_{m_2}), where $2 \leqslant m_1 < m_2 \leqslant n$. Moreover, a greedy cyclic covering can be found by a sequence (t_{m_1}, t_{m_2}) for given m_1 and m_2, $2 \leqslant m_1 < m_2 \leqslant n$. Such a trivial greedy covering has $\frac{n!}{6}$ cycles of length 6. This is the only known greedy covering with cycles of fixed length. In the case of 8-cycles, there are only non-greedy cyclic coverings each of which has $\frac{n!}{8}$ cycles of length 8.

Theorem 1 gives a tool to get a GS-greedy cyclic covering with cycles of length $2 \cdot 3^{k-1}$ for all $2 \leqslant k \leqslant n-1$. In this paper we construct such a maximal set of independent GS-greedy cycles in the Star graph S_n, $n \geqslant 3$. This is the first example of a greedy cyclic covering of the Star graph consisting cycles of non-fixed length.

The paper is organized as follows. In Section 2 we prove Theorem 1 and its corollaries. In Section 3 we construct a maximal set of independent GS-greedy cycles in the Star graph S_n, $n \geqslant 3$, that consists of greedy cycles of length $2 \cdot 3^{k-1}$ for all $2 \leqslant k \leqslant n-1$. In proofs below we use hierarchical structure of S_n having n copies of induced subgraphs $S_{n-1}(i), 1 \leqslant i \leqslant n$, on $(n-1)!$ permutations of form $[\pi_1 \dots \pi_{n-1}\ i]$ with the fixed last element [4].

2. Proof of Theorem 1

Proof. Since for any $n \geqslant 3$ the graph S_n is vertex–transitive as any Cayley graph and edge–transitive [4], without loss of generality we prove the theorem by induction for a given sequence

$$GS_m = (t_2, t_3, \dots, t_m),\ 3 \leqslant m \leqslant n, \tag{3}$$

of the transpositions t_i, $2 \leqslant i \leqslant m$, from the set t in the Star graph S_n with the starting vertex corresponding to the identity permutation $I_n = [1\,2\ \dots\ n]$. If $n = 3$, then $S_3 \cong C_6$ and $m = 3$, hence $GS_3 = (t_2, t_3)$ is a greedy sequence of length two generating six permutations as follows:

$$GS_3 : \qquad [123] \xrightarrow{t_2} [213] \xrightarrow{t_3} [312] \xrightarrow{t_2} [132] \xrightarrow{t_3} [231] \xrightarrow{t_2} [321],$$

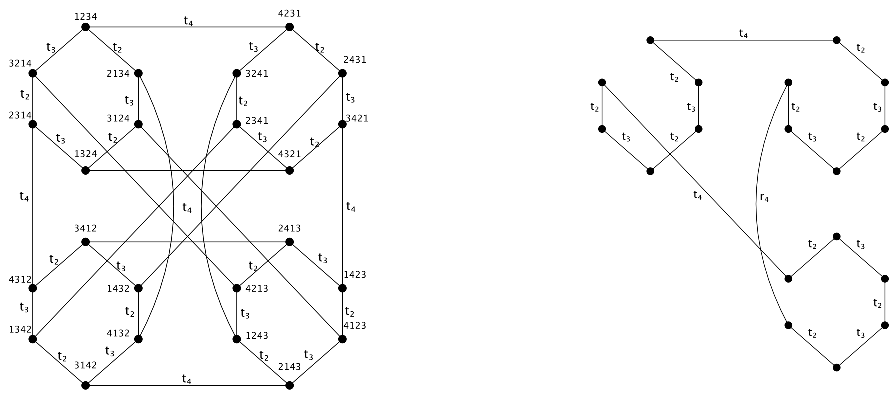
which obviously form a cycle of length $2 \cdot 3^{2-1} = 6$. If $n = 4$, then either $m = 3$ or $m = 4$. By hierarchical structure, S_4 has four copies of S_3, hence GS_3 gives the same result as above. If $m = 4$, then applying $GS_4 = (t_2, t_3, t_4)$ greedily to the vertex $I_n = [1234]$ we obtain 18 distinct permutations that form the (t_2, t_3, t_4)-greedy 18-cycle in S_4:

$$[1234] \xrightarrow{t_2} [2134] \xrightarrow{t_3} [3124] \xrightarrow{t_2} [1324] \xrightarrow{t_3} [2314] \xrightarrow{t_2} [3214] \xrightarrow{t_4}$$

$$[4213] \xrightarrow{t_2} [2413] \xrightarrow{t_3} [1423] \xrightarrow{t_2} [4123] \xrightarrow{t_3} [2143] \xrightarrow{t_2} [1243] \xrightarrow{t_4}$$

$$[3241] \xrightarrow{t_2} [2341] \xrightarrow{t_3} [4321] \xrightarrow{t_2} [3421] \xrightarrow{t_3} [2431] \xrightarrow{t_2} [4231],$$

where $[4231] \circ t_4 = [1234]$. The Star graph S_4 and its (t_2, t_3, t_4)-greedy cycle of length 18 are presented in Figure 2.



Star graph S_4 GS_4-greedy cycle

FIG 2. Star graph S_4 and its (t_2, t_3, t_4)-greedy 18-cycle

By notation (2) the list of permutations above can be represented as follows:

$$\begin{aligned} GS_4:\ &[1234] \xrightarrow{GS_3} [3214] \xrightarrow{t_4} \\ &[4213] \xrightarrow{GS_3} [1243] \xrightarrow{t_4} \\ &[3241] \xrightarrow{GS_3} [4231], \end{aligned}$$

where GS_3 acts on six vertices of three copies of S_3 in S_4. From this, we conclude that GS_4 is a greedy sequence that forms a greedy cycle of length $6 \cdot 3 = 2 \cdot 3^{3-1}$.

Suppose that the statement of the theorem holds for greedy sequences of length $m \leqslant k-1$, where $k \geqslant 5$, a sequence $GS_k = (t_2, t_3, \ldots, t_k)$ forms a greedy cycle of length $2 \cdot 3^{k-2}$ in S_k and has a GS_k-greedy path as follows:

$$[1\,2\,3\,\ldots\,k-1\,k] \xrightarrow{GS_k} [k\,2\,3\,\ldots\,k-1\,1]. \tag{4}$$

Since S_{k+1} has $k+1$ copies of S_k, hence GS_k forms the same cycle in S_{k+1}.

Let us prove the theorem for a greedy sequence $GS_{k+1} = (t_2, t_3, \ldots, t_k, t_{k+1})$ of length $k \leqslant n-1$ started at $I_{k+1} = [1\,2\,\ldots\,k\,k+1]$. According to the greedy approach, we have to apply consecutively the leftmost suitable transposition from GS_{k+1} which is the same as to apply the leftmost suitable transposition from GS_k, or t_{k+1} otherwise. This process is repeated until the permutation $[k+1\,2\,3\,\ldots\,k-1\,k\,1]$ is reached, which is the one adjacent to I_{k+1} in the Star graph S_{k+1}. Since both starting and ending permutations were generated greedily by the GS_k, the final form GS_{k+1} can be identified using the induction hypothesis (4):

$$\begin{aligned} GS_{k+1}:\ &[1\,2\,3\,\ldots\,k-1\,k\,k+1] \xrightarrow{GS_k} [k\,2\,3\,\ldots\,k-1\,1\,k+1] \xrightarrow{t_{k+1}} \\ &[k+1\,2\,3\,\ldots\,k-1\,1\,k] \xrightarrow{GS_k} [1\,2\,3\,\ldots\,k-1\,k+1\,k] \xrightarrow{t_{k+1}} \\ &[k\,2\,3\,\ldots\,k-1\,k+1\,1] \xrightarrow{GS_k} [k+1\,2\,3\,\ldots\,k-1\,k\,1], \end{aligned}$$

where $[k+1\,2\,3\,\ldots\,k-1\,k\,1] \circ t_{k+1} = I_{k+1}$ and each of three GS_k-greedy sequences generates $2 \cdot 3^{k-2}$ distinct permutations by the induction hypothesis, hence GS_{k+1} generates $2 \cdot 3^{k-2} \cdot 3 = 2 \cdot 3^{k-1}$ distinct permutations and this completes the proof. $\square$

In a GS-greedy cycle formed by $GS = (t_{m_1}, t_{m_2}, \dots, t_{m_k})$, where $k \leqslant n-1$, an edge generated by a transposition $t_{m_j}, 2 \leqslant m_j \leqslant n$, is called an *edge of type* t_{m_j}.

Corollary 1. *Let C be a GS-greedy cycle formed by $GS = (t_{m_1}, t_{m_2}, \dots, t_{m_k})$ in the Star graph $S_n, n \geqslant 3$, for $2 \leqslant k \leqslant n-1$. Then C contains 3^{k-1} edges of type t_{m_1}, $2 \cdot 3^{k-j}$ edges of type t_{m_j}, for each $2 \leqslant j \leqslant k-1$, and 3 edges of type t_{m_k}.*

Proof. As previously discussed, let us prove the statement for $GS_m = (t_2, t_3, \dots, t_m)$, $3 \leqslant m \leqslant n$. If $m = 3$, then there are two greedy sequences (t_2, t_3) and (t_3, t_2), each of which forms a 6-cycle with 3 edges of each of types t_2 and t_3. If $m = 4$, then from the proof of Theorem 1 it follows that the number of edges of types t_2, t_3, and t_4 in any GS_4-greedy cycle of length 18 in S_n is distributed as follows:

GS_4-greedy sequence	t_2	t_3	t_4
# edges in GS_4-greedy cycle	9	6	3

In general, for any $m = k+1$, where $2 \leqslant k \leqslant n-1$, by the induction hypothesis of Theorem 1 and by hierarchical structure of Star graphs we have the following distribution of edges in the GS_{k+1}-greedy cycle of length $2 \cdot 3^{k-1}$:

GS_{k+1}-greedy sequence	t_2	t_3	$\dots$	t_{k-1}	t_k	t_{k+1}
# edges in GS_{k+1}-greedy cycle	3^{k-1}	$2 \cdot 3^{k-2}$	$\dots$	$2 \cdot 3^2$	$2 \cdot 3$	3

,

which completes the proof. □

By Corollary 1, any two different greedy sequences of length at least three should form cycles with specific edge distributions. A combination of this fact with Theorem 1 gives us the following statement.

Corollary 2. *Any two different greedy sequences of length k, $3 \leqslant k \leqslant n$, form distinct cycles of length $2 \cdot 3^{k-1}$ in the Star graph $S_n, n \geqslant 4$.*

Corollary 2 does not hold for $k = 2$ since greedy sequences (t_2, t_3) and (t_3, t_2) produce the same 6-cycles in the graph S_n for any $n \geqslant 3$.

In the next corollary we use the following simple property of greedy sequences.

Mirror property. *Any greedy sequence $GS = (t_{m_1}, t_{m_2}, \dots, t_{m_k})$ forms the same cycle at starting vertices π and $\tau = \pi \circ t_{m_k}$ so that vertices of a greedy path $\pi \xrightarrow{GS} \tau$ are written in the reverse order in a greedy path $\tau \xrightarrow{GS} \pi$.*

Corollary 3. *Any greedy sequence of length k, $2 \leqslant k \leqslant n-1$, gives $\frac{n!}{6}$ distinct cycles of length $2 \cdot 3^{k-1}$ in the Star graph $S_n, n \geqslant 3$.*

Proof. We prove this statement with $GS_{k+1} = (t_2, t_3, \dots, t_k, t_{k+1})$ as a greedy sequence of length k, where $2 \leqslant k \leqslant n-1$. By Theorem 1, the greedy sequence GS_{k+1} forms a GS_{k+1}-greedy cycle of length $2 \cdot 3^{k-1}$ in S_n for any $n \geqslant 3$. We consider S_{k+1} and present this cycle by its GS_{k+1}-greedy path as follows:

$$(5) \qquad GS_{k+1}: \qquad I_{k+1} \xrightarrow{GS_k} \tau_1 \xrightarrow{t_{k+1}} \tau_2 \xrightarrow{GS_k} \tau_3 \xrightarrow{t_{k+1}} \tau_4 \xrightarrow{GS_k} \tau_5,$$

where

$$\tau_1 = [k\,2\,3\,\dots\,k-1\,1\,k+1],$$
$$\tau_2 = [k+1\,2\,3\,\dots\,k-1\,1\,k],$$

$$\tau_3 = [1\,2\,3\,\ldots\,k-1\,k+1\,k],$$
$$\tau_4 = [k\,2\,3\,\ldots\,k-1\,k+1\,1],$$
$$\tau_5 = [k+1\,2\,3\,\ldots\,k-1\,k\,1],$$

and $\tau_5 \circ t_{k+1} = I_{k+1}$. By Mirror property, the GS_{k+1}-greedy cycle (5) can be also presented by one of the following five GS_{k+1}-greedy paths:

$$\tau_1 \xrightarrow{GS_k} I_n \xrightarrow{t_{k+1}} \tau_5 \xrightarrow{GS_k} \tau_4 \xrightarrow{t_{k+1}} \tau_3 \xrightarrow{GS_k} \tau_2\,,$$

$$\tau_2 \xrightarrow{GS_k} \tau_3 \xrightarrow{t_{k+1}} \tau_4 \xrightarrow{GS_k} \tau_5 \xrightarrow{t_{k+1}} I_n \xrightarrow{GS_k} \tau_1\,,$$

$$\tau_3 \xrightarrow{GS_k} \tau_2 \xrightarrow{t_{k+1}} \tau_1 \xrightarrow{GS_k} I_n \xrightarrow{t_{k+1}} \tau_5 \xrightarrow{GS_k} \tau_4\,,$$

$$\tau_4 \xrightarrow{GS_k} \tau_5 \xrightarrow{t_{k+1}} I_n \xrightarrow{GS_k} \tau_1 \xrightarrow{t_{k+1}} \tau_2 \xrightarrow{GS_k} \tau_3\,,$$

$$\tau_5 \xrightarrow{GS_k} \tau_4 \xrightarrow{t_{k+1}} \tau_3 \xrightarrow{GS_k} \tau_2 \xrightarrow{t_{k+1}} \tau_1 \xrightarrow{GS_k} I_n\,,$$

which totally with (5) give us six different representations of the GS_{k+1}-greedy cycle. However, whenever we apply the greedy sequence GS_{k+1} to a permutation which is not in the set $\{I_n, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$ we never get the same GS_{k+1}-greedy cycle again. Thus, GS_{k+1} forms $\frac{n!}{6}$ distinct cycles of length $2 \cdot 3^{k-1}$ in the Star graph $S_n, n \geqslant 3$. By edge–transitivity of S_n, this holds for any greedy sequence and completes the proof. $\square$

As illustration of Corollary 3, a schematic representation of a GS_5-greedy cycle of length $54 = 2 \cdot 3^3$ with fixed permutations I_n, τ_1, τ_2, τ_3, τ_4, τ_5 is given in Figure 3.

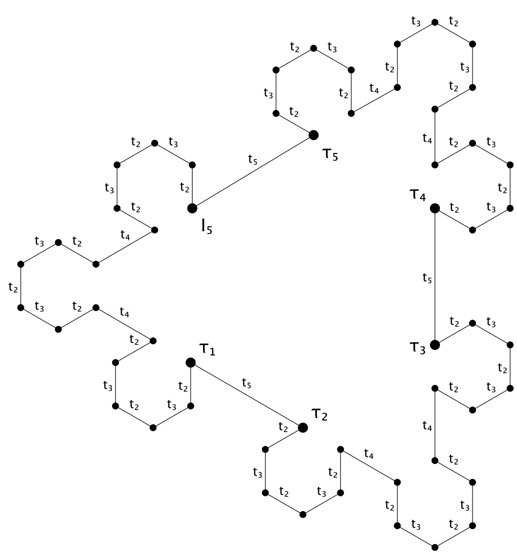


FIG 3. (t_2, t_3, t_4, t_5)-greedy cycle

We finished to analyzing greedy sequences and the corresponding greedy cycles in Star graphs. We use the obtained results in the next section to show that there is a cyclic covering by greedy cycles in Star graphs.

3. GS-GREEDY CYCLIC COVERING

Let $\mathfrak{F} = \{GS_k = (t_2, t_3, \ldots, t_k),\ 3 \leqslant k \leqslant n\}$ be a family of greedy sequences.

Theorem 2. *In the Star graph $S_n,\ n \geqslant 3$, there exists a maximal set of independent cycles formed by greedy sequences from the family $\mathfrak{F}$ consisting of the following cycles:*

(1) *one cycle of length $2 \cdot 3^{n-2}$, and*
(2) *$n-3$ cycles of length $2 \cdot 3^{n-3}$ when $n \geqslant 4$, and*
(3) *N_m cycles of length $2 \cdot 3^{n-m-2}$ for all $2 \leqslant m \leqslant n-3$ when $n \geqslant 5$, where*

$$N_m = \left(\prod_{l=2}^{m} (n-l+2) \right) \cdot (n-m-2).$$

Proof. If $n = 3$, then $S_3 \cong C_6$ which is formed by the greedy sequence $GS_3 = (t_2, t_3)$ and theorem holds. If $n = 4$, then by Theorem 1 greedy cycles of length 18 and 6 are formed by $GS_4 = (t_2,\, t_3,\, t_4)$ and $GS_3 = (t_2, t_3)$ correspondingly. Two such independent greedy cycles covering all vertices of S_4 are presented in Figure 4 (with dotted edges not belonging to the cycles). Thus, Theorem 2 holds for $n = 4$.

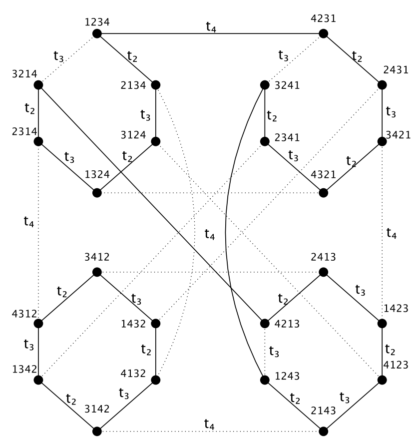


FIG 4. GS-greedy cyclic covering of S_4

Now we construct independent greedy cycles in the Star graphs S_n for $n \geqslant 5$.
First we obtain one cycle of length $2 \cdot 3^{n-2}$ using $GS_n = (t_2,\, t_3, \ldots, t_n)$ as follows:

$$\begin{aligned} GS_n: \quad & [1\,2\,3\,\ldots\,n-1\,n] \xrightarrow{GS_{n-1}} [n-1\,2\,3\,\ldots\,1\,n] \xrightarrow{t_n} \\ & [n\,2\,3\,\ldots\,1\,n-1] \xrightarrow{GS_{n-1}} [1\,2\,3\,\ldots\,n\,n-1] \xrightarrow{t_n} \\ & [n-1\,2\,3\,\ldots\,n\,1] \xrightarrow{GS_{n-1}} [n\,2\,3\,\ldots\,n-1\,1], \end{aligned}$$

where GS_{n-1} acts on vertices of copies $S_{n-1}(n)$, $S_{n-1}(n-1)$ and $S_{n-1}(1)$ in S_n, where a copy $S_{n-1}(i)$, $1 \leqslant i \leqslant n$, contains $(n-1)!$ permutations $[\pi_1 \ldots \pi_{n-1}\ i]$.

Then we take $n-3$ copies of $S_{n-1}(i)$ such that $i \notin \{1, n-1, n\}$. By Theorem 1, we can always form greedy cycles of length $2 \cdot 3^{n-3}$ using the greedy sequence $GS_{n-1} = (t_2, t_3, \ldots, t_{n-1})$ at arbitrary chosen starting vertices in these $n-3$ copies. Since we take different copies of $S_n - 1$, we obtain independent cycles. Thus, we construct $n-3$ independent greedy cycles of length $2 \cdot 3^{n-3}$.

By the hierarchical structure of S_n, each of n copies $S_{n-1}(i)$, $1 \leqslant i \leqslant n$, contains $n-1$ copies of $S_{n-2}(ji), 1 \leqslant j \neq i \leqslant n$, each of which is an induced subgraph on $(n-2)!$ permutations of form $[\pi_1 \ldots \pi_{n-2} ji]$ with two last elements fixed. By the arguments above, for any i there are three copies of $S_{n-2}(ji)$ each of which already has a GS_{n-2}-greedy path as a part of one of the constructed above $n-3$ independent GS_{n-1}-greedy cycles of length $2 \cdot 3^{n-3}$ or of the cycle of length $2 \cdot 3^{n-2}$. So, we take $n-4$ copies of $S_{n-2}(ji)$ whose vertices were not covered before and construct GS_{n-2}-greedy cycles of length $2 \cdot 3^{n-4}$ starting from arbitrary chosen vertices in each of $n-4$ copies. We can always do this by Theorem 1, and moreover all these cycles are independent by construction. Thus, totally we construct $n \cdot (n-4)$ independent cycles of length $2 \cdot 3^{n-4}$ using the greedy sequence $GS_{n-2} = (t_2, t_3, \ldots, t_{n-2})$.

In general, we consider $n \cdot (n-1) \cdot \ldots \cdot (n-m+2)$ copies of S_{n-m+1} each of which contains $(n-m+1)!$ permutations with $(m-1)$ last elements fixed. For each copy of S_{n-m+1}, we take $(n-m-2)$ copies of S_{n-m} whose vertices were not covered by previously constructed greedy cycles and construct GS_{n-m}-greedy cycles of length $2 \cdot 3^{n-m-2}$ starting from arbitrary chosen vertices in each of $(n-m-2)$ copies. We are always able to do this by Theorem 1. By construction, all these cycles are independent. We repeat the procedure above until we construct $n \cdot (n-1) \cdot \ldots \cdot 5$ independent cycles of length 6 using the greedy sequence $GS_3 = (t_2, t_3)$ in each of $\frac{n!}{24}$ copies of S_4 for $n \geqslant 5$. In total we construct

$$N_m = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-m+2) \cdot (n-m-2)$$

cycles of length $2 \cdot 3^{n-m-2}$ using the greedy sequence $GS_{n-m} = (t_2, t_3, \ldots, t_{n-m})$ for any $2 \leqslant m \leqslant n-3$ and $n \geqslant 5$, each of which is independent.

Thus, we have

$$2 \cdot 3^{n-2} + (n-3) \cdot 2 \cdot 3^{n-3} + n \cdot (n-4) \cdot 2 \cdot 3^{n-4} + n \cdot (n-1) \cdot (n-5) \cdot 2 \cdot 3^{n-5} +$$
$$+ n \cdot (n-1) \cdot (n-2) \cdot (n-6) \cdot 2 \cdot 3^{n-6} + n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-7) \cdot 2 \cdot 3^{n-7} +$$
$$\ldots$$
$$+ n \cdot (n-1) \cdot \ldots \cdot 5 \cdot 1 \cdot 2 \cdot 3 = n!$$

vertices covered by greedy cycles which completes the proof. □

Concluding remarks

A maximal set of independent cycles described in Theorem 2 by greedy sequences from the family $\mathfrak{F}$ contains one cycle of length $2 \cdot 3^{n-2}$, and also shorter cycles. One can obtain cycle coverings based on other families of greedy cycles, also it is possible to find construction consisting of a few longest $2 \cdot 3^{n-2}$-cycles. However, there is a way to get a hamiltonian cycle from our construction. It would be also interesting to construct a greedy cyclic covering with minimal number of cycles. This is a topic for future research.

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