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ON PIECEWISE CONTINUOUS MAPPINGS OF PARACOMPACT SPACES

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Abstract. It is proved that every resolvably measurable mapping $f\colon X \to Y$ of a first-countable perfectly paracompact space X to a regular space Y is piecewise continuous. If X is additionally completely Baire, then f is resolvably measurable if and only if it is piecewise continuous.

Keywords: resolvably measurable mapping, piecewise continuous mapping, $\mathcal{F}_\sigma$-measurable mapping, completely Baire space.

The study of the relationship between different types of mappings is a classical topological problem. In particular, the close connection between resolvably measurable mappings and piecewise continuous ones for metrizable spaces was announced (without proof) by V. Vinokurov [13]. T. Banakh and B. Bokalo [1] continued this line of investigation for perfectly paracompact spaces. Recently, A. Ostrovsky [10] showed that every resolvably measurable function $f\colon X \to Y$ is countably continuous for subsets X and Y of real numbers $\mathbb{R}$. On the other hand, J. Jayne and C. Rogers [5] proved that a mapping $f\colon X \to Y$ is $\mathbf{\Delta}^0_2$-measurable if and only if it is piecewise continuous for an absolute Souslin-$\mathcal{F}$ set X and a metric space Y. M. Kačena, L. Motto Ros, and B. Semmes [7] showed that the last statement holds for a regular space Y. The useful technique for investigation of piecewise continuous mappings was developed by S. Solecki [11]. The relationship between resolvably measurable mappings and weakly discontinuous ones was obtained by T. Banakh and B. Bokalo [2].

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The main result of the paper (see Theorem 1) states that every resolvably measurable mapping $f\colon X \to Y$ of a first-countable perfectly paracompact space X to a regular space Y is piecewise continuous. For mappings on completely Baire spaces this statement can be reversed, see Theorem 2. Notice that under the Martin Axiom, T. Banakh and B. Bokalo [1, Example 9.3] joint with L. Zdomskyy constructed a $\mathbf{\Delta}^0_2$-measurable mapping between metrizable separable spaces which is not piecewise continuous.

Lemma 6 gives an internal characterization of completely Baire space via comparison of $\mathbf{\Delta}^0_2$-sets and resolvable ones. In fact, this is a generalization of the classical Hurewisz's theorem [6] about embedding of the space of rational numbers into metrizable spaces.

Theorem 3 shows that in the study of piecewise continuous mappings defined on perfectly paracompact completely Baire spaces it suffices to consider separable subspaces.

Notation. For all undefined terms, see [4].

A space X is called *perfectly paracompact* if X is paracompact and each closed subset of X is of type G_δ in X.

A subset E of a space X is *resolvable* if it can be represented as

$$E = (F_1 \setminus F_2) \cup (F_3 \setminus F_4) \cup \ldots \cup (F_\xi \setminus F_{\xi+1}) \cup \ldots,$$

where $\langle F_\xi \rangle$ forms a decreasing transfinite sequence of closed sets in X.

Denote by $\mathcal{F}_\sigma(X)$ the family of F_σ-sets in X. Let $\mathbf{\Delta}^0_2(X)$ be the family of $\mathbf{\Delta}^0_2$-sets in X, i.e., sets that are both F_σ and G_δ.

A mapping $f\colon X \to Y$ is said to be

(1) *resolvably measurable* if $f^{-1}(U)$ is a resolvable subset of X for every open set $U \subset Y$,
(2) *$\mathcal{F}_\sigma$-measurable* if $f^{-1}(U) \in \mathcal{F}_\sigma(X)$ for every open set $U \subset Y$,
(3) *$\mathbf{\Delta}^0_2$-measurable* if $f^{-1}(U) \in \mathbf{\Delta}^0_2(X)$ for every open set $U \subset Y$,
(4) *piecewise continuous* if X can be covered by a sequence $X_0, X_1, \ldots$ of closed sets such that the restriction $f|X_n$ is continuous for every $n \in \omega$.

A simple example shows that there exists a piecewise continuous mapping which is not resolvably measurable.

Example 1. *Let $f\colon \mathbb{Q} \to D$ be a one-to-one mapping of the space $\mathbb{Q}$ of rational numbers onto the countable discrete space D. Clearly, f is piecewise continuous. Take a set $A \subset \mathbb{Q}$ such that A and $\mathbb{Q} \setminus A$ are both dense in $\mathbb{Q}$. Then A is a nonresolvable subset of $\mathbb{Q}$. The set $f(A)$ is open in D. Since $A = f^{-1}(f(A))$, the mapping f is not resolvably measurable.*

To prove Theorem 1, we shall slightly modify the technique due to Kačena, Motto Ros, and Semmes [7]. So, the ideas from [7] are used heavily.

The closure of a set $A \subset X$ is denoted by $\overline{A}$. Given a mapping $f\colon X \to Y$, let us denote by $\mathcal{I}_f$ the family of all subsets $A \subset X$ for which there is a set $F \in \mathcal{F}_\sigma(X)$ such that $A \subset F$ and the restriction $f|F$ is piecewise continuous. In particular, f is piecewise continuous if and only if $X \in \mathcal{I}_f$.

The sets $A, B \subset Y$ are *strongly disjoint* if $\overline{A} \cap \overline{B} = \emptyset$.

Let $x \in X$, $X' \subset X$, and $A \subset Y$. Define $A^f = f^{-1}(Y \setminus \overline{A})$. The pair (x, X') is said to be *f-irreducible outside A* if for every neighborhood $V \subset X$ of x we have $A^f \cap X' \cap V \notin \mathcal{I}_f$. Otherwise we say that (x, X') is *f-reducible outside A*, i.e., there exist a neighborhood V of x and a set $F \in \mathcal{F}_\sigma(X)$ such that $A^f \cap X' \cap V \subset F$ and $f|F$ is piecewise continuous. Clearly, $x \in \overline{A^f \cap X'}$ if (x, X') is f-irreducible outside A. One can check that if a pair (x, X') is f-irreducible outside A and $A' \subset A$, then (x, X') is also f-irreducible outside A'.

The following lemma shows that the σ-ideal $\mathcal{I}_f$ is locally generated.

Lemma 1. *Let $f\colon X \to Y$ be a mapping of a perfectly paracompact space X to a space Y. Given $X' \subset X$, let $G = \bigcup\{U \text{ is open in } X\colon U \cap X' \in \mathcal{I}_f\}$. Then $X' \cap G \in \mathcal{I}_f$.*

Proof. Denote by $\mathcal{U}$ the family $\{U \text{ is open in } X\colon U \cap X' \in \mathcal{I}_f\}$. For every $U \in \mathcal{U}$ choose a sequence of closed sets $F_n(U)$ such that $U \bigcap X' \subset \bigcup\{F_n(U)\colon n \in \omega\}$ and all restrictions $f|F_n(U)$ are continuous. Since X is a perfectly paracompact space, we have $G = \bigcup\{G_k\colon k \in \omega\}$, where each G_k is closed in X. By the Michael theorem (see [4, Theorem 5.1.28]), G is paracompact itself. Let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{U}$. For every $V \in \mathcal{V}$ fix $U_V \in \mathcal{U}$ such that $V \subset U_V$. Define $F_n^k = \bigcup\{\overline{V} \cap F_n(U_V) \cap G_k\colon V \in \mathcal{V}\}$. The local finiteness of the family $\mathcal{V}$ implies that all F_n^k are closed in X and all restrictions $f|F_n^k$ are continuous. Since $X' \cap G \subset \bigcup\{F_n^k\colon k \in \omega, n \in \omega\}$, we obtain $X' \cap G \in \mathcal{I}_f$. □

The following lemma is a slight modification of [7, Lemma 3]; in [7] it was proved for a metrizable space X.

Lemma 2. *Suppose $f\colon X \to Y$ is an $\mathcal{F}_\sigma$-measurable mapping. Let X' be a subset of a perfectly paracompact space X and A be a subset of a regular space Y such that $X' \subset A^f$. Then the following assertions are equivalent:*

(1) *$X' \notin \mathcal{I}_f$,*
(2) *there exist a point $x \in \overline{X'} \cap A^f$ and an open set $U \subset Y$ strongly disjoint from A such that $f(x) \in U$ and the pair (x, X') is f-irreducible outside U.*

Proof. (2) ⇒ (1): If (x, X') is f-irreducible outside U, then $U^f \cap X' \cap X \notin \mathcal{I}_f$. Hence, $X' \notin \mathcal{I}_f$.

(1) ⇒ (2): Denote by G the open set $\bigcup\{O \text{ is open in } X\colon O \cap X' \in \mathcal{I}_f\}$. By Lemma 1 we have $X' \cap G \in \mathcal{I}_f$.

Assume toward a contradiction that (ii) does not hold, i.e., for every $x \in \overline{X'} \cap A^f$ and every open set $U \subset Y$ such that $f(x) \in U$ and $\overline{U} \cap \overline{A} = \emptyset$ we have that (x, X') is f-reducible outside U.

The intersection $Z = A^f \cap (\overline{X'} \setminus G)$ is an F_σ-set in X. We claim that the restriction $f|Z$ is continuous. Suppose otherwise, so that there are $x \in Z$ and an open set $U \subset Y$ such that $f(x) \in U$ and there is no neighborhood V of x with $f(V \cap Z) \subset \overline{U}$. Because $f(x) \notin \overline{A}$ by the assumption, we can assume that $\overline{U} \cap \overline{A} = \emptyset$. Fix a neighborhood V of x satisfying $U^f \cap X' \cap V \in \mathcal{I}_f$. By our hypothesis there is $x' \in V \cap Z$ such that $f(x') \notin \overline{U} \cup \overline{A}$. Using regularity of Y, we can find a neighborhood U' of $f(x')$ which is strongly disjoint from U and A. Let now V' be a neighborhood of x' given by the failure of (ii), i.e. $(U')^f \cap X' \cap V' \in \mathcal{I}_f$. Since $\overline{U} \cap \overline{U'} = \emptyset$, we have $X' \cap V \cap V' \subset U^f \cup (U')^f$. Therefore $X' \cap V \cap V' \in \mathcal{I}_f$. In other words, $X' \cap V \cap V' \subset X' \cap G$. But this implies $x' \notin Z$, a contradiction. Thus $f|Z$ is continuous. Hence $X' \cap Z \in \mathcal{I}_f$.

On the whole, we get $X' = (X' \cap Z) \cup (X' \cap G) \in \mathcal{I}_f$, a contradiction with (i). □

Lemma 3 ([7]). *Let $f\colon X \to Y$ be a mapping of a space X to a space Y, $x \in X$, $X' \subset X$, $A \subseteq Y$, and let $U_0, \dots, U_k$ be a sequence of pairwise strongly disjoint open subsets of Y. If (x, X') is f-irreducible outside A, then there is at most one $i \in \{0, \dots, k\}$ such that (x, X') is f-reducible outside $A \cup U_i$.*

For the sake of completeness, we reproduce the concise proof of Lemma 3 from [7].

Proof. Assume that there are two indices $i, j \in \{0, \dots, n\}$, $i \neq j$, such that (x, X') is f-reducible outside both $A \cup U_i$ and $A \cup U_j$. Then there are neighbourhoods V_i and V_j of x such that $(A \cup U_i)^f \cap X' \cap V_i \in \mathcal{I}_f$ and $(A \cup U_j)^f \cap X' \cap V_j \in \mathcal{I}_f$. Since U_i and U_j are strongly disjoint, this implies that

$$A^f \cap X' \cap V_i \cap V_j \in \mathcal{I}_f,$$

and thus $V_i \cap V_j$ contradicts the fact that (x, X') is f-irreducible outside A. □

Lemma 4. *Let X be a first-countable perfectly paracompact space and Y be a regular space. Suppose $f\colon X \to Y$ is an $\mathcal{F}_\sigma$-measurable mapping which is not piecewise continuous.*

Then there exist a countable subset $Q = \{q_n : n \in \omega\}$ of X and a sequence $\langle U_n : n \in \omega\rangle$ of disjoint open sets in Y such that

(1) *$f(q_n) \in U_n$ for every $n \in \omega$,*
(2) *Q is homeomorphic to the space of rational numbers,*
(3) *the restriction $f|Q$ is a bijection.*

Proof. Denote by $2^{<\omega}$ the set of all binary sequences of finite length. The construction will be carried out by induction with respect to the order $\preceq$ on $2^{<\omega}$ defined by

$$s \preceq t \iff \text{length}(s) < \text{length}(t) \vee (\text{length}(s) = \text{length}(t) \wedge s \leq_{\text{lex}} t),$$

where $\leq_{\text{lex}}$ is the usual lexicographical order on $2^{\text{length}(s)}$. We write $s \prec t$ if $s \preceq t$ and $s \neq t$.

The map υ assigns to each $t \in 2^{<\omega}$ the length of a string of zeros at the end of t; for example, $\upsilon(10100) = 2$.

We will construct a sequence $\langle x_s : s \in 2^{<\omega}\rangle$ of points of X, a base $\langle W_n^s : n \in \omega\rangle$ at the point x_s, a sequence $\langle V_s : s \in 2^{<\omega}\rangle$ of subsets of X, and a sequence $\langle U_s : s \in 2^{<\omega}\rangle$ of open subsets of Y such that for every $s \in 2^{<\omega}$:

(1) $\overline{W_{n+1}^s} \subset W_n^s$ and $\overline{W_0^{s\frown 1}} \subset W_{\upsilon(s)}^s \setminus \overline{W_{1+\upsilon(s)}^s}$ for each $n \in \omega$,
(2) $x_s \in \overline{V_s} \subset W_0^s$,
(3) if $t \subset s$ then $V_s \subset V_t$,
(4) if $s = t\frown 0$ then $x_s = x_t$ and $U_s = U_t$,
(5) $f(x_s) \in U_s$,
(6) if the last digit of s is 1 then $f(\overline{V_s}) \cap \overline{\bigcup_{r \prec s} U_r} = \emptyset$,
(7) (x_t, V_t) is f-irreducible outside A for every $t \preceq s$, where $A = \bigcup_{r \preceq s} U_r$,
(8) the family $\{V_t : t \in 2^n\}$ is pairwise strongly disjoint for every $n \in \omega$,
(9) the family $\{U_t : t \preceq s \wedge (\text{the last digit of } t \text{ is } 1) \text{ or } t = \emptyset\}$ is pairwise strongly disjoint,
(10) $\overline{V_{s\frown 0}} \subset W_{1+\upsilon(s)}^s$ and $\overline{V_{s\frown 1}} \subset W_{\upsilon(s)}^s \setminus \overline{W_{1+\upsilon(s)}^s}$.

For the base step of the induction, let x and U be given as in Lemma 2 applied to $X' = X$ and $A = \emptyset$. Then put $x_\emptyset = x$ and $U_\emptyset = U$. Using regularity of X, find a base $\langle W_n^\emptyset : n \in \omega\rangle$ at the point $x_\emptyset$ satisfying condition (1). Define $V_\emptyset = W_1^\emptyset$.

Assume that x_t, V_t, and U_t have been constructed for any $t \preceq s$.

Let $A = \bigcup\{U_t : t \prec s\hat{}0\}$ and $O = Y \setminus \overline{A}$. Notice that $f^{-1}(O) = A^f$ is an F_σ-set in X. By the inductive hypothesis, condition (7) says that (x_s, V_s) is f-irreducible outside A. Then

$$f^{-1}(O) \cap V_s \cap W^s_{\upsilon(s)} \notin \mathcal{I}_f.$$

Since X is perfectly paracompact, every set $W^s_n \setminus \overline{W^s_{n+1}}$ is F_σ in X. Now we can find some $k > 0$ with

$$f^{-1}(O) \cap V_s \cap \left(W^s_{\upsilon(s)} \setminus \overline{W^s_{k+\upsilon(s)}}\right) \notin \mathcal{I}_f.$$

To simplify notation, we can assume that $k = 1$. Since f is $\mathcal{F}_\sigma$-measurable, there exists a set C closed in X such that

$$C \subset f^{-1}(O) \cap \left(W^s_{\upsilon(s)} \setminus \overline{W^s_{1+\upsilon(s)}}\right)$$

and $X' = C \cap V_s \notin \mathcal{I}_f$.

Put $x_{s\hat{}0} = x_s$, $U_{s\hat{}0} = U_s$, $W^{s\hat{}0}_n = W^s_{n+1}$, and $V_{s\hat{}0} = V_s \cap W^{s\hat{}0}_1$.

Claim. There are $x_{s\hat{}1} \in \overline{X'}$ and $U_{s\hat{}1} \subset Y$ such that $f(x_{s\hat{}1}) \in U_{s\hat{}1}$, $U_{s\hat{}1}$ is open and strongly disjoint from A, (x_t, V_t) is f-irreducible outside $A \cup U_{s\hat{}1}$ for every $t \prec s\hat{}1$ and $(x_{s\hat{}1}, X')$ is f-irreducible outside $A \cup U_{s\hat{}1}$.

Proof of the claim. Let $k = |\{t \in 2^{<\omega} : t \prec s\hat{}1\}|$. Using Lemma 2, for $j = 0, \dots, k$ recursively construct x_j and U_j such that $f(x_j) \in U_j$, U_j is open and strongly disjoint from $A \cup U_{<j}$ (where $U_{<j} = \emptyset$ if $j = 0$ and $U_{<j} = \bigcup\{U_i : i < j\}$ otherwise), and $(x_j, X' \cap (U_{<j})^f)$ is f-irreducible outside U_j. According to Lemma 3, for each $t \prec s\hat{}1$ there is at most one $j_t \in \{0, \dots, k\}$ such that (x_t, V_t) is f-reducible outside $A \cup U_{j_t}$. The pigeonhole principle implies that the claim is satisfied with $x_{s\hat{}1} = x_{j^*}$ and $U_{s\hat{}1} = U_{j^*}$ for some $j^* \in \{0, \dots, k\}$. △

Choose a base $\langle W^{s\hat{}1}_n : n \in \omega\rangle$ at the point $x_{s\hat{}1}$ such that $\overline{W^{s\hat{}1}_0} \subset W^s_{\upsilon(s)} \setminus \overline{W^s_{1+\upsilon(s)}}$ and $\overline{W^{s\hat{}1}_{n+1}} \subset W^{s\hat{}1}_n$. Finally, set $V_{s\hat{}1} = X' \cap W^{s\hat{}1}_1$. Obviously, $V_{s\hat{}1} \subset V_s$. Since $V_{s\hat{}1} \subset C$, we obtain condition (6).

By construction, $\overline{V_{s\hat{}0}} \subset W^s_{1+\upsilon(s)}$ and $\overline{V_{s\hat{}1}} \subset W^s_{\upsilon(s)} \setminus \overline{W^s_{1+\upsilon(s)}}$. Then $\overline{V_{s\hat{}0}} \bigcap \overline{V_{s\hat{}1}} = \emptyset$. Together with (3), this implies condition (8). One can check that all the conditions (1)–(10) are satisfied.

The set $Q = \{x_s : s \in 2^{<\omega}\}$ is countable and has no isolated points by (1) and (2). Since X is first-countable, the Urysohn theorem [4, Theorem 4.2.9] implies that Q is metrizable. According to the Sierpiński theorem (see [4, Exercise 6.2.A]), Q is homeomorphic to the space of rational numbers.

Denote by J the index set $\{s \in 2^{<\omega} :$ the last digit of s is 1 or $s = \emptyset\}$. Condition (4) implies that $Q = \{x_s : s \in J\}$. Given two distinct points $x_s, x_t \in Q$ with $s, t \in J$, we have $t \neq s$. By (9), $U_t \bigcap U_s = \emptyset$. Since $f(x_t) \in U_t$ and $f(x_s) \in U_s$, we conclude that the restriction $f|Q$ is a bijection.

It remains to number the distinct elements of Q as $\{q_n : n \in \omega\}$ and elements of $\langle U_s : s \in J\rangle$ as $\{U_n : n \in \omega\}$ by the rule $f(q_n) \in U_n$. □

We are now ready to give the main result of the paper.

Theorem 1. *Every resolvably measurable mapping $f : X \to Y$ of a first-countable perfectly paracompact space X to a regular space Y is piecewise continuous.*

Proof. Suppose towards a contradiction that there is a resolvably measurable mapping $f\colon X \to Y$ which is not piecewise continuous. By Lemma 5 (see below), every resolvable subset of X is an F_σ-set in X. Hence, the mapping f is $\mathcal{F}_\sigma$-measurable.

Using Lemma 4, we can find a set $Q \subset X$ such that Q is homeomorphic to the space of rational numbers, the restriction $f|Q$ is a bijection, and $f(Q)$ is relatively discrete. Since f is a resolvably measurable mapping, $f|Q$ is the same. However, the restriction $f|Q$ fails to be resolvably measurable as shown by Example 1. □

Corollary 1. *Let $f\colon X \to Y$ be a bijection between first-countable perfectly paracompact spaces X and Y such that f and f^{-1} are both resolvably measurable mappings. Then:*

1. *there is a cover $\{X_n\colon n \in \omega\}$ of X by closed sets and a cover $\{Y_n\colon n \in \omega\}$ of Y by closed sets such that for every $n \in \omega$ the restriction $f|X_n$ is a homeomorphism of X_n onto Y_n,*
2. $\dim X = \dim Y$,
3. *X is σ-compact if and only if so is the space Y.*

Proof. (1) Theorem 1 implies that f and f^{-1} are both piecewise continuous. Then $X = \bigcup\{A_i\colon i \in \omega\}$, where each A_i is closed in X and each restriction $f|A_i$ is continuous. Each set $f(A_i)$ is F_σ in Y because f^{-1} is $\mathbf{\Delta}^0_2$-measurable. Using piecewise continuity of f^{-1}, for every $i \in \omega$ we can find a sequence $\langle B_{ij}\colon j \in \omega\rangle$ of closed sets in Y such that $f(A_i) = \bigcup\{B_{ij}\colon j \in \omega\}$ and each restriction $f^{-1}|B_{ij}$ is continuous. From continuity of $f|A_i$ it follows that each $f^{-1}(B_{ij})$ is closed in A_i; hence, in X. One can readily verify that $X = \bigcup\{f^{-1}(B_{ij})\colon i \in \omega, j \in \omega\}$. Clearly, each restriction $f|f^{-1}(B_{ij})$ is a homeomorphism. It remains to enumerate nonempty sets $\{f^{-1}(B_{ij})\colon i \in \omega, j \in \omega\}$ as $\{X_n\colon n \in \omega\}$ and set $Y_n = f(X_n)$.

(2) The equality $\dim X = \dim Y$ follows from part (1) and the countable sum theorem [4, Theorem 7.2.1].

(3) is an immediate consequence of (1). □

A topological space X is called a *Baire space* if the intersection of countably many dense open sets in X is dense; or equivalently every non-empty open set in X is not of the first category. A space X is *completely Baire* if every closed subspace of X is a Baire space.

Lemma 5 generalizes Theorem 5 of [8, p. 362] from metrizable spaces to perfectly paracompact ones. We prove Lemma 5 repeating almost word by word the proof of the last theorem. In completely metrizable spaces, resolvable sets coincide with $\mathbf{\Delta}^0_2$-sets, see [8, p. 418]. Lemma 6 expands the class of spaces having such a property.

Lemma 5. *Let E be a resolvable set in a perfectly paracompact space X. Then $E \in \mathbf{\Delta}^0_2(X)$.*

Proof. Let $E = \bigcup\{F_\xi \setminus H_\xi\colon \xi < \alpha\}$, where $F_\xi \supset H_\xi \supset F_\zeta$ for every $\xi < \zeta$ and all sets F_ξ, H_ξ are closed in X.

Firstly, we shall show that E is locally $\mathbf{\Delta}^0_2$ at each point, that is, every point $x \in E$ has a neighborhood U such that $U \cap E \in \mathbf{\Delta}^0_2(X)$.

Clearly, this is true for $\alpha = 1$. Assume that the statement is true for each $\beta < \alpha$. Given a point $x \in E$, fix $\beta < \alpha$ with $x \in F_\beta \setminus H_\beta$. The set $U = X \setminus H_\beta$ is open in X and $x \in U$. For each $\zeta > \beta$ we have $F_\zeta \subset H_\beta$, whence $F_\zeta \cap U = \emptyset$ and $U \cap \bigcup\{F_\zeta \setminus H_\zeta\colon \zeta > \beta\} = \emptyset$.

Therefore

$$U \cap E = [U \cap (F_\beta \setminus H_\beta)] \cup [U \cap \bigcup\nolimits_{\xi<\beta} (F_\xi \setminus H_\xi)].$$

By hypothesis, $\bigcup_{\xi<\beta}(F_\xi \setminus H_\xi)$ is a $\mathbf{\Delta}^0_2$-set; hence so is $U \cap E$.

According to [12, Lemma 4], every locally $\mathbf{\Delta}^0_2$-set is a $\mathbf{\Delta}^0_2$-set. □

Lemma 6. *For a first-countable perfectly paracompact space X the following conditions are equivalent:*

(1) *no closed subspace of X is homeomorphic to the space $\mathbb{Q}$ of rational numbers,*
(2) *X is a completely Baire space,*
(3) *the family $\mathbf{\Delta}^0_2(X)$ coincides with the family of resolvable sets in X.*

Proof. (1)⇒(2): Suppose towards a contradiction that X is not a completely Baire space. Then there is a closed set $F \subset X$ which is not Baire. Hence we can find a nonempty open (in F) set $U \subset F$ of the first category in F. The closure $\overline{U}$ is of the first category on itself. According to [9, Theorem 2] (see also [3]) $\overline{U}$ contains a closed copy of $\mathbb{Q}$, a contradiction.

(2)⇒(3): By Lemma 5, every resolvable set in X is a $\mathbf{\Delta}^0_2$-set.

Conversely, to obtain a contradiction, we suppose that there is a non-resolvable set $E \in \mathbf{\Delta}^0_2(X)$. By [8, p. 99], we have $F = \overline{F \cap E} = \overline{F \setminus E}$ for some nonempty closed set F. Then $F \cap E$ and $F \setminus E$ are both F_σ-sets of the first category on F. Hence, $F = (F \cap E) \cup (F \setminus E)$ is a set of the first category. On the other hand, F is a Baire space as a closed subset of X, a contradiction.

(3)⇒(1): Striving for a contradiction, suppose that X contains a closed set F which is homeomorphic to $\mathbb{Q}$. Take a set $A \subset F$ such that A and $F \setminus A$ are both dense in F. Then A is a non-resolvable subset of F (and of X). Clearly, $A \in \mathbf{\Delta}^0_2(F)$; hence, $A \in \mathbf{\Delta}^0_2(X)$ because F is closed in X. A contradiction. □

Theorem 2. *Let $f\colon X \to Y$ be a mapping of a first-countable perfectly paracompact, completely Baire space X to a regular space Y. Then the following conditions are equivalent:*

(1) *f is resolvably measurable,*
(2) *f is piecewise continuous,*
(3) *f is $\mathbf{\Delta}^0_2$-measurable.*

Proof. The implication (1)⇒(2) follows from Theorem 1.

(2)⇒(3): By the definition, there are closed sets $X_n \subset X$, $n \in \omega$, such that $\bigcup_{n\in\omega} X_n = X$ and each restriction $f|X_n$ is continuous. Then

$$f^{-1}(A) = \bigcup\{X_n \cap f^{-1}(A)\colon n \in \omega\}$$

is an F_σ-set in a perfectly paracompact space X for every open (or closed) set $A \subset Y$. Hence $f^{-1}(A) \in \mathbf{\Delta}^0_2(X)$. Thus, f is resolvably measurable.

Lemma 6 implies (3)⇒(1). □

Remark. The equivalence of conditions (2) and (3) in Theorem 2 can be obtained by combining Theorem 8.1 and Corollary 8.2 from the Banakh-Bokalo paper [1].

Theorem 3. *Let $f\colon X \to Y$ be an $\mathcal{F}_\sigma$-measurable mapping of a first-countable perfectly paracompact, completely Baire space X to a regular space Y. Then f is* *piecewise continuous if and only if the restriction $f|K$ is piecewise continuous for any separable closed subset K of X.*

Proof. Clearly, if f is piecewise continuous, then the restriction $f|K$ is piecewise continuous for every closed subset K of X.

Let the restriction $f|K$ be piecewise continuous for any separable closed subset $K \subset X$. Suppose towards a contradiction that f is not piecewise continuous. By Lemma 4, there exist a set $Q = \{q_n : n \in \omega\}$ and a sequence $\langle U_n : n \in \omega \rangle$ of disjoint open sets in Y such that $f(q_n) \in U_n$ for every $n \in \omega$ and Q is homeomorphic to the space of rational numbers. Consider the closure K of Q. Because f is by assumption piecewise continuous, there are closed sets $K_n \subset X$, $n \in \omega$, such that $\bigcup_{n\in\omega} K_n = K$ and every $f|K_n$ is continuous. Since K is a Baire space, there exists a K_j with the nonempty interior V_j (in K). Then $V_j \cap Q$ is homeomorphic to the space of rational numbers. Clearly, the restriction $f|\overline{V_j \cap Q}$ is continuous. Take a point $q \in V_j \cap Q$. Fix a neighbourhood $U_q \subset Y$ of $f(q)$ such that $U_q \cap f(Q) = f(q)$. From continuity of $f|\overline{V_j \cap Q}$ it follows that there is a neighbourhood $V \subset V_j$ (in K) of q such that $f(V) \subset U_q$. Then $V \cap Q = \{q\}$, i.e., q is an isolated point of Q. This contradicts the fact that the set $V_j \cap Q$ has no isolated points. □

Corollary 2. *Let $f\colon X \to Y$ be an $\mathcal{F}_\sigma$-measurable mapping of a first-countable perfectly paracompact, completely Baire space X to a regular space Y. Then precisely one of the following holds:*

(1) *f is piecewise continuous,*
(2) *there exists a separable closed set $K \subset X$ such that the restriction $f|K$ is not $\mathbf{\Delta}^0_2$-measurable.*

Proof. Suppose f is not piecewise continuous. By Theorem 3, there exists a separable closed set $K \subset X$ such that the restriction $f|K$ is not piecewise continuous. Theorem 2 implies that the restriction $f|K$ is not $\mathbf{\Delta}^0_2$-measurable. □

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