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# EIGENFUNCTIONS SUPPORTS OF MINIMUM CARDINALITY IN CUBICAL DISTANCE-REGULAR GRAPHS

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ABSTRACT. In this paper we focus on cubical distance-regular graphs and for 10 of them we find eigenfunctions with the minimum number of nonzero positions and provide the classification of their possible structures.

**Keywords:** eigenfunctions, minimum supports, distance-regular graphs, cubical graphs.

# 1. INTRODUCTION

Let G = (V, E) be an undirected graph without loops and multiple edges with a vertex set  $V = V(G) = \{1, 2, ..., n\}$  and an edge set E = E(G). For  $u, v \in V$ , if there is an edge between vertices u and v, we will call them adjacent (or neighbors) and denote  $u \sim v$ . The adjacency matrix A of order n is defined as follows:

$$A_{uv} = \begin{cases} 1, \text{ when } u \sim v \\ 0, \text{ when } u \nsim v \end{cases}$$

G is regular if each vertex has the same number k of neighbors. This parameter k is called the *degree* of a graph. For any vertices  $v, u \in V$  the *distance* d(v, u) is the number of edges in a shortest path that connects them. The greatest distance between any pairs of vertices is called the *diameter* D of a graph. By  $G_i(v)$  we denote the set of vertices that are at distance i from v. A connected graph G is called *distance-regular* if it is regular of degree k and for any two vertices  $v, u \in V$  at distance i = d(v, u) there are precisely  $c_i$  neighbors of u in  $G_{i-1}(v)$  and  $b_i$  neighbors of u in  $G_{i+1}(v)$ ; where  $c_i$  and  $b_i$  do not depend on the choice of vertices u, v but depend only on d(u, v). Numbers  $b_i, c_i, a_i = k - b_i - c_i$  are called the *intersection* 

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numbers of G. A set  $\{b_0, \ldots, b_{D-1}; c_1, \ldots, c_D\}$  is called an *intersection array* of a distance-regular graph G.

For an adjacency matrix A, let  $\Lambda = \{\lambda_1^{(m_1)}, \ldots, \lambda_t^{(m_t)}\}$  be a set of its eigenvalues, where  $m_i$  is a multiplicity of an eigenvalue  $\lambda_i$ . A function  $f: V \to \mathbb{R}$  that is not constantly zero and satisfies the equation

(1) 
$$\lambda f(u) = \sum_{v \in G_1(u)} f(v) \quad \forall u \in V$$

is called an *eigenfunction* of a graph G corresponding to the eigenvalue  $\lambda$ . A support supp(f) of a function f is defined as follows  $supp(f) = \{v \in V \mid f(v) \neq 0\}$ . Further we will refer to a support of an eigenfunction as an eigensupport. An eigenfunction can be considered as a column-vector  $\bar{f} = (f(1), f(2), \ldots, f(n))^{\mathrm{T}}$ . Under this notation (1) can be rewritten in the following form:

(2) 
$$A\bar{f} = \lambda \bar{f}$$

in other words  $\bar{f}$  is an eigenfunction of the adjacency matrix of a graph.

We are interested in finding eigenfunctions with supports of minimum cardinality for different families of distance-regular graphs. The motivation for this problem comes from the deep connection between eigenfunctions and important combinatorial configurations, since many combinatorial objects can be represented as eigenfunctions on graphs. Therefore, one of the crucial problem of finding the minimum possible difference between two combinatorial objects is strongly related to the study of minimum eigensupports. More details regarding these relations can be found in [8], where also the minimum support for the smallest eigenvalue of the Grassman graph is described. For the Hamming graphs H(n,q) the problem of finding minimum eigensupports is completely solved for q = 2 and partially solved in case of arbitrary q, see [6], [9]. Results for the Johnson graphs can be found in [10]. In this work we focus on cubical distance-regular graphs. The goal of the paper is to find the minimum cardinalities of the eigensupports and also classify its possible structures.

The paper is organized as follows. In section 2 we provide the notation and several simple statements. Sections 3, 4 are devoted to bipartite graphs and bipartite double covering construction. In Section 5 we recall what a *weight distribution* is and how it can be used to get the lower bound on the eigensupport cardinality. Sections 6, 7 describe the techniques that we use in our work. Main results are presented in Section 8. Appendix contains the details of computer calculations performed.

# 2. Preliminaries

- $S(\lambda)$  an induced subgraph on the vertices from eigensupport supp(f) corresponding to an eigenvalue  $\lambda$ . In other words it is a subgraph of G with a vertex set supp(f) together with any edges from E whose endpoints are in this set.
- $|\mathcal{S}(\lambda)|$  the number of vertices in  $\mathcal{S}(\lambda)$ , i.e. the cardinality of the vertex set  $V(\mathcal{S})$ .
- $\mathcal{O}(\mathcal{S}) = V(G) \setminus V(\mathcal{S}).$

Since the aim of this paper is not only to calculate the cardinalities of the minimum eigensupports, but also to characterize their structures as the subgraphs of the original graph, we introduce the following notation:

•  $\ll t_1 + \ldots + t_s \gg -$  any graph that consists of s connected components that are of cardinalities  $t_1, \ldots, t_s$  correspondingly. For example,  $\ll 1 \gg$ an isolated vertex;  $\ll 2 \gg -$  an edge;  $\ll 1 + 1 \gg -$  two isolated vertices;  $\ll 1 + 2 \gg -$  an isolated vertex and an edge;  $\ll 3 \gg -$  it can be either a cycle  $C_3$  on three vertices, or two edges that share one vertex.

Now we provide several simple lemmas that will be useful further.

**Lemma 1.**  $S(\lambda)$  has an isolated vertex if and only if  $\lambda = 0$ . In other words,  $\ll 1 \gg \subseteq S(\lambda) \Leftrightarrow \lambda = 0.$ 

*Proof.* Let  $u \in \mathcal{S}(\lambda)$  be an isolated vertex. Since all neighbors of u in G do not belong to the support  $\mathcal{S}(\lambda)$ , we have  $\lambda f(u) = 0$ . The left side of the equation can be zero if and only if  $\lambda = 0$ .

**Lemma 2.** There does not exist a vertex from  $\mathcal{O}(\mathcal{S})$  that has only one neighbor in  $\mathcal{S}(\lambda).$ 

*Proof.* Suppose the opposite. Let there exist  $z \in \mathcal{O}(\mathcal{S})$  such that  $G_1(z) \cap \mathcal{S}(\lambda) = \{v\}$ . Then  $0 = \lambda f(z) = \sum_{u \in G_1(z)} f(u) = f(v) \neq 0$ . Contradiction.

**Lemma 3.**  $\mathcal{S}(0)$  does not have a vertex with a degree equal to 1.

Proof. Obvious.

The following proposition contains the results for some simple families of graphs:

**Proposition 1.** (1) The complete graph  $K_n$ , whose eigenvalues are

- $\{(n-1)^{(1)}, -1^{(n-1)}\},\ has\ the\ following\ minimum\ eigensupports:$ •  $\mathcal{S}(n-1) = K_n$ 

  - $\mathcal{S}(-1) = \ll 2 \gg$  and any two vertices with opposite non-zero values yield the minimum eigensupport  $\ll 2 \gg$ .
- (2) The complete bipartite graph  $K_{m,h}$ , with eigenvalues  $\{\pm \sqrt{mh}^{(1)}, 0^{(m+h-2)}\}$ , has the following minimum eigensupports
  - $\mathcal{S}(\pm\sqrt{mh}) = K_{m,h}$
  - $S(0) = \ll 1 + 1 \gg$ , where any two non-adjacent vertices with opposite non-zero values yield the minimum eigensupport  $\ll 1 + 1 \gg$

Proof. Obvious.

#### 3. BIPARTITE GRAPHS

Well known is the following lemma about eigenvalues of a bipartite graph (see [4], for example)

**Lemma 4.** If G is a bipartite graph, then its spectrum is symmetric with respect to zero, in other words if  $\lambda$  is an eigenvalue of G then  $-\lambda$  is also its eigenvalue.

Vertices V of a bipartite graph G can be particulated into two disjoint sets X and Y (called parts) in such a way that any edge connects a vertex from X to some vertex in Y. Let  $v \in X$ . Consider the distance partition  $\{G_i(v) \mid 0 \le i \le D\}$  with respect to the vertex v. It is clear that  $G_i(v)$  consists only of vertices from the same part, which is X if i is even and Y if i is odd.

The following lemma is true:

**Lemma 5.** Let G be a connected bipartite graph and let  $f^{\lambda}$  be an eigenfunction corresponding to the eigenvalue  $\lambda$ . Suppose  $g^{-\lambda}$  is constructed in the following way:

- (1) g(x) = f(x) for all vertices  $x \in X$
- (2) g(y) = -f(y) for all vertices  $y \in Y$

Then  $g^{-\lambda}$  is an eigenfunction for the eigenvalue  $-\lambda$ .

*Proof.* For any  $x \in X$  we have  $\lambda f(x) = \sum_{y \in G_1(x)} f(y)$ , where  $y \in Y$ . For any  $y \in Y$  we have  $\lambda f(y) = \sum_{x \in G_1(y)} f(x)$ , where  $x \in X$ . Now consider the eigenvalue  $-\lambda$  and function g constructed as described above. For all  $x \in X$  we have  $-\lambda g(x) = -\lambda f(x) = -\sum_{y \in G_1(x)} f(y) = \sum_{y \in G_1(x)} g(y)$  and for all  $y \in Y$  we have  $-\lambda g(y) = -\lambda(-f(y)) = \sum_{x \in G_1(y)} f(x) = \sum_{x \in G_1(y)} g(x)$ . Therefore, g is an eigenfunction for the eigenvalue  $-\lambda$ 

eigenvalue  $-\lambda$ .

**Corollary 1.** For a bipartite graph G, minimum eigensupports for eigenvalues  $\lambda$  and  $-\lambda$  have the same cardinalities.

**Proposition 2.** Let G be a bipartite graph with parts X and Y. The following statements are true:

- (1) Eigensupport for  $\lambda \neq 0$  has vertices in both parts.
- (2) Under the condition that both parts are of the same cardinality, the minimum eigensupport for  $\lambda = 0$  is fully contained in one of the parts, therefore it consists of vertices no two of which are adjacent.
- *Proof.* (1) Suppose the opposite, let all vertices from S lie in X. Then for a vertex  $x \in X$  we have  $\lambda f(x) = \sum_{y \in G_1(x)} f(y) = 0$  since  $G_1(x) \subseteq Y$ .

Contradiction.

(2) Without loss of generality we can assume that vertices are labeled in such an order that:  $X = \{1, ..., n/2\}$  and  $Y = \{n/2 + 1, ..., n\}$ . Hence, the adjacency matrix A will have the following form

$$A = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix},$$

where B is a square matrix of order n/2. Consider any eigenfunction f and denote

$$\bar{f}_x = (f(1), f(2), \dots, f(n/2))^{\mathrm{T}}$$
  
 $\bar{f}_y = (f(n/2+1), \dots, f(n))^{\mathrm{T}}$ 

Thus, we obtain:

$$A\bar{f} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \bar{f}_x \\ \bar{f}_y \end{pmatrix} = \begin{pmatrix} B\bar{f}_y \\ B^T\bar{f}_x \end{pmatrix} = \bar{0}^n,$$

where  $\bar{0}^n$  is an all-zero column-vector of length *n*. Consequently,  $\bar{f}_y$  and  $\bar{f}_x$  are eigenfunctions of *B* and  $B^T$  correspondingly for eigenvalue  $\lambda = 0$ . It is clear that if  $(\bar{f}_x, \bar{f}_y)$  is the eigenfunction for *A* then  $(0, \bar{f}_y)$  and  $(\bar{f}_x, 0)$  are also eigenfunctions for the same eigenvalue  $\lambda = 0$ . So the minimum support is totally contained in one of the parts and, hence, consists of vertices non-adjacent to each other.

#### 4. BIPARTITE DOUBLE COVER

Before we proceed to bipartite double covers of graphs we need to introduce the concept of graphs tensor product, which is also called the Kronecker product of graphs. Suppose we have graphs G and G' with vertex sets V and V', correspondingly. Their adjacency matrices are A and A'. Then the tensor product  $G \otimes G'$  of these graphs is defined as follows:

- (1) The vertex set of  $G \otimes G'$  is the Cartesian product  $V \times V'$ , i.e.  $V(G \otimes V) = \{(v, v') \mid \forall v \in V, \forall v' \in V'\}$
- (2) Any two vertices (u, u') and (v, v') are adjacent in  $G \otimes G'$  if and only if  $u \sim v$  in graph G and  $u' \sim v'$  in graph G'.

The adjacency matrix B of the obtained graph will be the Kronecker product of the adjacency matrices of original graphs, i.e.  $B = A \otimes A'$  (see [2]). Well known is the following fact: if matrices A and A' have eigenvalues  $\{\lambda_1, \ldots, \lambda_n\}$  and  $\{\mu_1, \ldots, \mu_m\}$  then their Kronecker product will have eigenvalues  $\{\lambda_i \mu_j \mid i = 1 \ldots n; j = 1, \ldots m\}$  (for example, [3]).

The bipartite double cover BDC(G) of some graph G is the tensor product  $G \otimes K_2 = BDC(G)$ , where  $K_2$  is a complete graph on 2 vertices (edge), and can be presented as follows: for each vertex *i* of G we build two vertices  $i^a$  and  $i^b$  of the new graph BDC(G). If  $i \sim j$  in the original graph G then  $i^a \sim j^b$  and  $i^b \sim j^a$  in the new graph. If a graph G has eigenvalues  $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$  then BDC(G) has eigenvalues  $\{\pm \lambda_1, \ldots, \pm \lambda_n\}$ .

eigenvalues  $\{\pm \lambda_1, \ldots, \pm \lambda_n\}$ . Let  $\bar{f} = (f(1), \ldots, f(n))^{\mathrm{T}}$  be an eigenfunction of G. Now we define  $\bar{f}_{\mathrm{BDC}} = (f_{\mathrm{BDC}}(1^a), f_{\mathrm{BDC}}(1^b), \ldots, f_{\mathrm{BDC}}(n^a), f_{\mathrm{BDC}}(n^b))^{\mathrm{T}}$  as follows:  $f_{\mathrm{BDC}}(i^a) = f_{\mathrm{BDC}}(i^b) = f(i)$ .

**Lemma 6.** (Bipartite Double Cover) For a graph G let  $\lambda$  be such an eigenvalue that G does not have  $-\lambda$  as its eigenvalue. Under the notation above f is an eigenfunction of G corresponding to the eigenvalue  $\lambda \iff f_{BDC}$  is an eigenfunction of BDC(G).

*Proof.*  $\Rightarrow$  Obvious.

 $\Leftarrow$  For BDC(G) let  $\bar{g}_+$  be an eigenfunction corresponding to  $\lambda$ . According to Lemma 5 we build an eigenfunction  $\bar{g}_-$  corresponding to  $-\lambda$ . Thus we can write the following equalities:

$$g_{+}(i^{a}) = g_{-}(i^{a}) = g(i^{a}) \quad \forall i^{a}$$
$$g_{+}(i^{b}) = -g_{-}(i^{b}) = g(i^{b}) \quad \forall i^{b}$$

Now we build functions  $\bar{f}_+$  and  $\bar{f}_-$  in the following way:

$$f_{+}(i) = g_{+}(i^{a}) + g_{-}(i^{b}) = g(i^{a}) + g(i^{b})$$
$$f_{-}(i) = g_{-}(i^{a}) + g_{-}(i^{b}) = g(i^{a}) - g(i^{b})$$

Since  $\bar{g}_+$  is an eigenfunction, we have

$$\lambda g(i^a) = \sum_{j \in N(i)} g(j^b) \quad \forall i^a$$
$$\lambda g(i^b) = \sum_{j \in N(i)} g(j^a) \quad \forall i^b$$

Which gives us

$$\lambda f_{+}(i) = \lambda g(i^{a}) + \lambda g(i^{b}) = \sum_{j \in N(i)} g(j^{b}) + \sum_{j \in N(i)} g(j^{a})$$
$$= \sum_{j \in N(i)} (g(j^{a}) + g(j^{b})) = \sum_{j \in N(i)} f_{+}(j)$$

According to 1 this proofs that  $\bar{f}^+$  is an eigenfunction of G. Similarly we get

$$-\lambda f_{-}(i) = \sum_{j \in N(i)} f_{-}(j)$$

But  $-\lambda$  is not an eigenvalue of G. Hence,  $f_{-}(i) = 0$  for all i, in other words we have  $g(i^{a}) = g(i^{b})$ . Therefore,  $f_{+}(i) \neq 0 \iff g(i^{a}) \neq 0$ . As a result we obtain the following: if  $g^{+}$  is an eigenfunction of BDC(G) then  $f_{+}$ , constructed as described above, is an eigenfunction of G.

Directly obtained from the proof is the following

**Corollary 2.** Suppose we have a graph G with an eigenfunction  $\lambda$ , provided that  $-\lambda$  is not an eigenvalue of G. Let  $S(\lambda)$  and  $S_{BDC}(\lambda)$  be the minimum eigensupports of G and BDC(G) correspondingly. Then  $|S(\lambda)| = \frac{1}{2}|S_{BDC}(\lambda)|$ 

# 5. Weight Distribution Bound

Let  $f^{\lambda}$  be an eigenfunction of a distance-regular graph G corresponding to an eigenvalue  $\lambda$ . By definition of an eigenfunction we can choose a vertex v such that  $f^{\lambda}(v) \neq 0$ . Without loss of generality, suppose that  $f^{\lambda}(v) = 1$ . Let

$$W_i^v(f^\lambda) = \sum_{u \in G_i(v)} f(u)$$

For distance-regular graphs the value  $W_i^v(f^{\lambda})$  does not depend on the choice of a vertex v or an eigenfunction f, therefore

$$W_i^v(f^\lambda) = W_i(\lambda) \quad \forall v \in V, \, \forall f^\lambda$$

The recurrence takes place:

$$W_0(\lambda) = 1$$
  
 $W_1(\lambda) = \lambda$   
 $W_i(\lambda) = \frac{\lambda W_{i-1}(\lambda) - b_{i-2}W_{i-2}(\lambda) - a_{i-1}W_{i-1}(\lambda)}{c_i}$ , где  $i = 2, \dots, ..D$ 

The set  $\{W_0(\lambda), W_1(\lambda), \ldots, W_D(\lambda)\}$  is called the *weight distribution*, corresponding to the eigenvalue  $\lambda$ . Well known is the following fact (see [7], for example):

**Lemma 7.** For the cardinality of a support  $S(\lambda)$  the following estimation is true:  $|S(\lambda)| \ge \sum_{i=0}^{D} |W_i(\lambda)|.$ 

If some of  $W_i(\lambda)$  values are not integral the bound can be improved:

**Lemma 8.** For the cardinality of a support  $S(\lambda)$  the following estimation is true:  $|S(\lambda)| \ge \sum_{i=0}^{D} \lceil |W_i(\lambda)| \rceil.$ 

We will refer to this bound as WDB. To avoid lengthy expressions further we will write  $W_i$  instead of  $W_i(\lambda)$  if the corresponding eigenvalue is clear from the context.

## 6. Method

This paper is based on two techniques.

For graphs with a small number of vertices we are using a *manual* one that consists of the following steps:

- (1) Fix an eigenvalue  $\lambda$  for a considered graph G
- (2) Calculate  $\lambda$ -conditions:
  - (a) Consider an eigensupport  $S(\lambda)$  as a subgraph of G with its own eigenvalues  $\{\nu_1, \nu_2, \ldots, \nu_s\}$  (multiplicities are omitted) where  $\lambda = \nu_i$  for some *i*.
  - (b) If the minimum eigensupport S(λ) is of cardinality k, then all possible support structures can be represented with different compositions of k: ≪1+1+...+1≫, ≪1+...+2≫,...,≪k≫
  - (c) Let  $S(\lambda) = \ll m_1 + m_2 + \ldots + m_t \gg$ . This implies that each  $\ll m_i \gg$  has  $\lambda$  as its eigenvalue. Thus we can state the following:

**Necessary**  $\lambda$ -conditions. Each  $\ll m_i \gg \subseteq S$  must have  $\lambda$  among its eigenvalues and there must exist a corresponding eigenfunction that is non-zero on all vertices of  $\ll m_i \gg$ .

- (3) Count Weight Distribution Bound.
- (4) Considering  $\lambda$ -conditions and WDB, list all structures that possibly can be a support  $S(\lambda)$  of some eigenfunction.
- (5) Choose one of the possible structures of  $S(\lambda)$ . For all  $v \in V$  check the eigenfunction equality (1). Here we have 2 cases:
  - Case  $v \in S(\lambda)$ : the equality holds since Step 2.(c) gives us necessary  $\lambda$ -conditions. All we need to do is to calculate the eigenfunction values.
  - Case  $v \in \mathcal{O}(\mathcal{S})$ : the equality needs to be verified. If it holds for all  $v \in \mathcal{O}(\mathcal{S})$ , the minimum eigensupport is found.
- (6) Repeat Step 5 for the remaining structures.
- (7) If no eigensupports are found, increase k and go to Step 2.(b)

For graphs with a big number of vertices the manual technique produces too many cases to be considered, so the problem of finding minimum eigensupports for them was solved with the help of computer calculations. For the details of the algorithm the reader is referred to the Appendix at the end of the paper.

# 7. $\lambda$ -conditions: Subgraphs

From all graphs with the number of vertices  $n \leq 6$  (the full list of small graphs can be found in [11]) we choose connected graphs that are feasible as induced subgraphs of cubical distance-regular graphs and calculate their eigenvalues. Because of the properties of cubical distance-regular graphs we do not consider subgraphs with cycles of length less or equal to 4 and with the maximal degree greater than 3. Listed below are the possible candidates.

Remark: in the table of subgraphs by X we denote an eigenvalue of some complicated form that is out of our interest since the graphs considered in the current work do not have it as an eigenvalue.





 $\begin{array}{ccc} {\rm S5.1} & {\rm S5.2} \\ \Lambda = \{2, \frac{-1\pm\sqrt{5}}{2}\} & \Lambda = \{\pm\sqrt{3}, \pm 1, 0\} & \Lambda = \{\pm\sqrt{2}\pm\sqrt{2}, 0\} \end{array}$ 





## 8. Graphs

8.1. Cubical distance-regular graphs. It is known [1] that up to isomorphism there are only 13 cubical distance-regular graphs:  $K_4$ ,  $K_{3,3}$ , the Petersen graph,

the cube, the Heawood graph, the Pappus graph, the Coxeter graph, the Tutte-Coxeter graph, the dodecahedron, the Desargues graph, the Foster graph, the Tutte 12-cage, the Biggs-Smith graph. In this section we find the cardinalities of minimum eigensupports and describe their possible structures for first 10 graphs. For any regular graph of degree k the maximum eigenvalue  $\lambda_1$  is equal to k and its corresponding eigenfunction is an all-one vector  $1^n$  of weight n. For bipartite regular graphs an eigenfunction for  $\lambda = -k$  is also of weight n. So here and further we study only eigenvalues  $|\lambda| < 3$ .

Graph	Eigenvalues	Minimum support
$K_4$	$\{-1^{(3)},3^{(1)}\}$	$\{2,4\}$
K <sub>3,3</sub>	$\{0^{(4)},\pm 3^{(1)}\}$	$\{2, 6\}$
Cube	$\{\pm 1^{(3)}, \pm 3^{(1)}\}$	$\{4, 8\}$
Petersen	$\{-2^{(4)}, 1^{(5)}, 3^{(1)}\}$	$\{6, 4, 10\}$
Heawood	$\{\pm\sqrt{2}^{(6)},\pm3^{(1)}\}$	$\{6, 14\}$
Pappus	$\{0^{(4)}, \pm\sqrt{3}^{(6)}, \pm3^{(1)}\}$	$\{6, 8, 18\}$
Dodecahedral	$\{-2^{(4)}, 0^{(4)}, 1^{(5)}, \pm\sqrt{5}^{(3)}, 3^{(1)}\}$	$\{12, 8, 8, 16, 20\}$
Desargues	$\{\pm 1^{(5)}, \pm 2^{(4)}, \pm 3^{(1)}\}$	$\{8, 12, 20\}$
Coxeter	$\{(-1\pm\sqrt{2})^{(6)}, -1^{(7)}, 2^{(8)}, 3^{(1)}\}$	$\{16, 12, 14, 28\}$
Tutte-Coxeter	$\{0^{(10)}, \pm 2^{(9)}, \pm 3^{(1)}\}$	$\{6, 14, 30\}$

The summary table of the results.

8.2. Complete graph on 4 verices.  $K_4$  graph has eigenvalues  $\Lambda = \{3^{(1)}, -1^{(3)}\}$ . According to Proposition 1 we have  $S(-1) = \ll 2 \gg$  as the minimum eigensupport and any two vertices with opposite non-zero values yield the minimum support.

8.3. Complete bipartite graph on 6 vertices.  $K_{3,3}$  graph has eigenvalues  $\Lambda = \{\pm 3^{(1)}, 0^{(4)}\}$ . According to Proposition 1 we have that  $\mathcal{S}(0) = \ll 1 + 1 \gg$  is the minimum eigensupport. Any two non-adjacent vertices with opposite non-zero values yield the minimum support.

8.4. Cube graph. The eigenvalues of Cube are  $\Lambda = \{\pm 3^{(1)}, \pm 1^{(3)}\}$ . It is known that Cube graph is the bipartite double cover of  $K_4$  graph. So according to Lemma 6 we have  $S(1) = \ll 2 + 2 \gg = \{u \sim v, x \sim y\}$ , where u, v are not adjacent to x, y with f(u) = f(v) = -f(x) = -f(y).

Using Lemma 5 we obtain the minimum eigensupport  $S(-1) = \ll 2 + 2 \gg = \{u \sim v, x \sim y\}.$ 

8.5. The Petersen graph. The Petersen graph G has D = 2; eigenvalues  $\Lambda = \{3^{(1)}, 1^{(5)}, -2^{(4)}\}$ ; and its intersection array is (3, 2; 1, 1). Calculating WD, we get the following:  $\mathcal{W}(\lambda) = \{W_0 = 1, W_1 = \lambda, W_2 = \lambda^2 - 3\}$ . Consider the following cases:

λ = 1. We have WDB(1) = 4 and λ-conditions give us ≪ 1 ≫, ≪ 3 ≫, ≪ 4 ≫⊈ S. So the only option to examine is ≪ 2 + 2 ≫. Let u ~ v and x ~ y with the condition u, v ~ x, y. We have f(u) = f(v) and f(x) = f(y). Since G is a strongly regular graph with parameters (10, 3, 0, 1), each pair of nonadjacent vertices (u, x), (u, y), (v, x), (v, y) has only one common neighbor and all these neighbors are different because adjacent vertices do not have a shared neighbor. From this we obtain f(x) = f(y) = -f(u) = -f(v). Note that degrees of u, v, x, y are already equal to 3, so the remaining 2 vertices are adjacent to some vertices from  $\mathcal{O}$ . **Result:** the minimum eigensupport is  $\mathcal{S}(1) = \ll 2 + 2 \gg$  with the structure described above. An example is presented below, see Fig. 1:



Figure 1: Petersen graph,  $\mathcal{S}(1) = \ll 2 + 2 \gg$ 

•  $\lambda = -2$ . For this eigenvelue we have WDB(-2) = 4. According to  $\lambda$ conditions  $\ll 1 \gg, \ll 2 \gg, \ll 3 \gg, \ll 4 \gg, \ll 5 \gg \not\subseteq S$ , therefore we have  $|S(-2)| \ge 6$ . Suppose the equality holds. From the above it follows that
the only possible case is  $S(-2) = \ll 6 \gg$ , where S can have a form of  $C_6$  or
H-graph (see S6.1 and S6.4 in the table). We will now prove that both of
them are feasible as minimum eigensupports.

Case  $C_6$ : The cycle  $C_6$  is a regular bipartite graph with a degree 2. So for an eigenvalue  $\lambda = -2$  there exists a corresponding eigenfunction with 6 non-zero values. Let  $C_6 = \{u_0, \ldots, u_5\}$ , where  $u_i \sim u_{i+1} \quad \forall i \in \{0, \ldots, 5\}$ . Note, that all calculations of indices here are considered as mod 6. It is easy to show that  $(f(u_0), \ldots, f(u_5)) = (1, -1, 1, -1, 1, -1)$  up to multiplication by a constant. Because of the Petersen graph being strongly regular with parameters (10, 3, 0, 1), we have the following:

- $-u_i$  and  $u_{i+1}$  do not have a common neighbor
- $-u_i$  and  $u_{i+2}$  have only one common neighbor that is  $u_{i+1}$
- $-u_i$  and  $u_{i+3}$  have one common neighbour  $v_i$ , besides two different pairs have different shared neighbors, otherwise there would be a vertex with a degree more than 3. Note that  $v_i = v_{i+3}$ .

So we have six vertices  $u_i$  that are adjacent to  $u_{i-1}, u_{i+1}, v_i$ ; and three vertices  $v_i$  that are adjacent to  $u_i$  and  $u_{i+3}$ . We have the last remaining vertex w that is obviously adjacent to all  $v_i$  vertices. It is easy to see that an eigenfunction equality (1) holds, so  $C_6$  can be a minimum eigensupport for  $\lambda = -2$ .

Case of H-graph: Let S(-2) has a vertex set  $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ , where  $u_1, u_2 \sim u_3$ ;  $u_3 \sim u_4$ ;  $u_4 \sim u_5, u_6$ . Denote  $f_i = f(u_i)$ . We have the / ) (')

following system:

$$\begin{cases} \lambda f_1 = f_3 \\ \lambda f_2 = f_3 \\ \lambda f_3 = f_1 + f_2 + f_4 \\ \lambda f_4 = f_3 + f_5 + f_6 \\ \lambda f_5 = f_4 \\ \lambda f_6 = f_4 \end{cases}$$

From the above we easily get the solution

$$(f_1, f_2, f_3, f_4, f_5, f_6) = (f_1, f_1, -2f_1, 2f_1, -f_1, -f_1).$$

Now we will prove that this support is feasible in the graph. The vertices  $u_3$  and  $u_4$  are of full degree 3. Consider other four vertices from the support:  $u_1, u_2, u_5, u_6$ . Each pair  $(u_i, u_j)$  that consists of the vertices with the values of different sign has one common neighbor and all of this neighbors are different. As there are four such neighbors, they fully construct the set  $\mathcal{O}$ . If we consider a pair of different vertices  $(u_i, u_j)$  with the values of the same sign, they obviously have one common neighbor which is  $u_3$  or  $u_4$ . So for any vertex  $v \in \mathcal{O}$  it has two neighbors  $u_i$  and  $u_j$  from  $\mathcal{S}$  with the condition that  $f_i + f_j = 0$ . So S6.4 is feasible as the minimum support.

**Result:** The minimum support  $S(-2) = \ll 6 \gg$  with two possible structures described above. Examples are presented below, see Fig. 2, 3:



Figure 2: Petersen graph,  $\mathcal{S}(-2)$ : cycle  $C_6$ 



Figure 3: Petersen graph,  $\mathcal{S}(-2)$ : H-graph

8.6. The Heawood graph. The Heawood graph G has D = 3;  $\Lambda = \{\pm 3^{(1)}, \pm \sqrt{2}^{(6)}\}$ ; and its intersection array is (3, 2, 2; 1, 1, 3). Consider the following cases:

•  $\lambda = \sqrt{2}$ . Calculating WD we get:  $\mathcal{W}(\lambda) = \{W_0 = 1, W_1 = \lambda, W_2 = \lambda^2 - 3, W_3 = \frac{\lambda^3 - 5\lambda}{3}\}$ . For  $\lambda = \pm\sqrt{2}$  we have  $\mathcal{W}(\pm\sqrt{2}) = \{1, \pm\sqrt{2}, -1, \pm\sqrt{2}\}$ ; that gives WDB = 6.

According to  $\lambda$ -conditions  $\ll 1 \gg, \ll 2 \gg, \ll 4 \gg, \ll 5 \gg, \ll 6 \gg \not\subseteq S(\sqrt{2})$ . Therefore, if WDB is achievable the minimum support can only be of the following form  $S(\sqrt{2}) = \ll 3 + 3 \gg$ .

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Suppose  $\ll 3 + 3 \gg = \{u_1 \sim u_2 \sim u_3; v_1 \sim v_2 \sim v_3\}$ ; where  $u_i$  and  $v_j$ being not adjacent. As the Heawood graph is bipartite, we may color all vertices in red and blue colors with the condition that vertices of the same color are not adjacent. Let  $u_1, u_3$  be of red color and  $u_2$  of blue color. As the graph is of degree 3 we have  $u_1 \sim x_1, x_2$  (blue), and  $u_2 \sim x_3$ , (red) and  $u_3 \sim x_4, x_5$  (blue). So by now we have 3 red vertices and 5 blue vertices. Since the Heawood graph has 14 vertices in total, there remain only 2 blue vertices  $y_1, y_2$ ; it is easy to see that  $x_3 \sim y_1, y_2$ . Because of non-existence of  $C_4$  in the Heawood graph,  $y_1$  and  $y_2$  cannot belong to S simultaneously and also they are not adjacent to  $u_1, u_2$ . Granting this,  $y_1 \sim w_1, w_2$  and  $y_2 \sim w_3, w_4$ , where all vertices  $w_1, w_2, w_3, w_4$  are different. Without loss of generality suppose  $x_1 \sim w_1$ . Therefore,  $x_1 \nsim w_2$  and  $x_2 \nsim w_1$  (otherwise there would be  $C_4$  in the graph). Consequently,  $x_1$  is adjacent to  $w_3$  or  $w_4$ and w.l.o.g. we can assume that  $x_1 \sim w_3$ . In the same manner we obtain  $x_1 \nsim w_4$  and  $x_2 \nsim w_3$ . Hence,  $x_2 \sim w_2, w_4$ . By a parallel argument  $x_4$  and  $x_5$  are adjacent to only one neighbor of  $y_1$  and  $y_2$  and all these neighbors are different. Thus we can denote that  $x_4 \sim w_1, w_3$  and  $x_5 \sim w_2, w_4$ .

We will now prove that either  $\{y_1, w_1, w_2\}$  or  $\{y_2, w_3, w_4\}$  can be taken as vertices  $\{v_1, v_2, v_3\}$  from the support  $S(\sqrt{2})$ . To show that  $S_6 = \ll 3+3 \gg$ is feasible, we need to verify the eigenfunction equality for all vertices. First we consider vertices from the support. The following equalities hold for  $u_i$ :

$$\begin{cases} \lambda f(u_1) = f(u_2) \\ \lambda f(u_2) = f(u_1) + f(u_3) \\ \lambda f(u_3) = f(u_2) \end{cases}$$

For  $\lambda = \sqrt{2}$  we get  $f(u_1) = f(u_3) = f(u_2)/\sqrt{2}$ . Replacing  $u_i$  by  $v_i$  and carrying out the similar arguments, we show that  $f(v_1) = f(v_3) = f(v_2)/\sqrt{2}$ .

Now consider vertices from  $\mathcal{O}(\mathcal{S})$ . For  $x_3$  we get  $\lambda f(x_3) = f(u_2) + f(y_1) + f(y_2)$ . Equations for remaining vertices have the following form:

$$\begin{cases} \lambda f(x_1) = f(u_1) + f(w_1) + f(w_3) \\ \lambda f(x_2) = f(u_1) + f(w_2) + f(w_4) \\ \lambda f(x_4) = f(u_1) + f(w_1) + f(w_3) \\ \lambda f(x_5) = f(u_1) + f(w_2) + f(w_4) \end{cases}$$

It is easy to see that  $\ll 3 \gg = \{y_1, w_1, w_2\}$  satisfies the system. The same is also true for  $\ll 3 \gg = \{y_2, w_3, w_4\}$ . Therefore, the eigenfunction equality holds. By construction, the minimum support can only be of the form described above.

•  $\lambda = -\sqrt{2}$ . The minimum support for this case can be built according to Lemma 5 and Corollary 1, it has the same cardinality, the same form and differs only in signs.

**Result:** The minimum support  $S(\pm\sqrt{2}) = \ll 3 + 3 \gg$  and has a structure described above. Fig. 4 presents an example.

# 8.7. The Pappus graph. The Pappus graph G has

$$D = 4, \quad \Lambda = \{\pm 3^{(1)}, \pm \sqrt{3}^{(0)}, 0^{(4)}\};$$

(6)



Figure 4: Heawood graph,  $\mathcal{S}(\sqrt{2}) = \ll 3 + 3 \gg$ 

and its intersection array is (3, 2, 2, 1; 1, 1, 2, 3). Calculations of WD give us:  $\mathcal{W}(\lambda) = \{1, \lambda, \lambda^2 - 3, \frac{\lambda^3 - 5\lambda}{2}, \frac{\lambda^4 - 9\lambda^2 + 12}{6}\}$ . We have the following cases:

•  $\lambda = 0$ . The Pappus graph is bipartite. Let us denote its parts as X and Y. According to Proposition 2 the minimum support has the form  $\ll 1 + 1$  $\ldots + 1 \gg$  and consists of vertices from the same part. We will prove that  $\mathcal{S}(0) = \ll 1 + 1 + 1 + 1 + 1 + 1 \gg$  is feasible as a support of an eigenfunction. Since WDB = 6, it would be the minimum support we are looking for. Let S consist of vertices  $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ , where  $x_i \in X$ . Without loss of generality suppose that  $f(x_1) = 1$ . Consider the distance partition with respect to  $x_1$ . It is clear that  $G_1(x_1) = \{v_1, v_2, v_3\}$ , where all vertices  $v_i$  are different and  $v_i \in Y$ . Each  $v_i$  has two other neighbors  $u_{i1}, u_{i2} \in X$ . The girth of the Pappus graph is 6, therefore  $u_{i1} \neq u_{i2}$ . As the result, we get  $G_2(x_1) = \{u_{11}, u_{12}, \dots, u_{32}\}$ . It is clear that  $G_3(x_1)$  consists of 6 different vertices  $y_i$  from Y and  $G_4(x_1)$  consists of 2 remaining vertices  $w_1$  and  $w_2$ from X, where  $w_1$  and  $w_2$  do not have common neighbors (otherwise there would more than 6 edges coming from  $G_3(x_1)$  to  $G_4(x_1)$ ). So w.l.o.g we can consider  $G_1(w_1) = \{y_1, y_2, y_3\}$  as neighbors of  $w_1$ . Note that for any j it is true that  $u_{j1}$  is adjacent to some vertex from  $G(w_1)$  and some vertex from  $G_1(w_2)$ , otherwise there would be a too short cycle. The same is true for  $u_{j2}$ .

It is easy to show that up to enumeration we can take  $\{x_2, x_3, x_4\} = \{u_{1i}, u_{2i}, u_{3i}\}$  and  $\{x_5, x_6\} = \{w_1, w_2\}$  to get the feasible structure; only these structures are possible as minimum supports for  $\lambda = 0$ . The corresponding eigenfunction has the following non-zero elements:

 ${f(x_1), f(x_2), \dots, f(x_6)} = {1, -1, -1, 1, 1}.$ 

**Result:** The minimum support  $\mathcal{S}(0) = \ll 1 + 1 + 1 + 1 + 1 + 1 \gg$ 

•  $\lambda = \sqrt{3}$ . Here we have  $\mathcal{W}(\lambda) = \{1, \sqrt{3}, 0, -\sqrt{3}, -1\}$  and WDB = 6. Our  $\lambda$ -conditions show that  $\ll 1 \gg, \ll 2 \gg, \ll 3 \gg, \ll 6 \gg \not\subseteq \mathcal{S}(\sqrt{3})$ . With the help of computer calculations we found that  $\ll 7 \gg$  and  $\ll 8 \gg$  cannot be the minimum supports, while  $\mathcal{S} = \ll 4 + 4 \gg$ , where each  $\ll 4 \gg$  has the

form S4.2, is feasible. Let  $S = \ll 4 + 4 \gg = \{x_1, x_2, x_3, x_4\} \cup \{z_1, z_2, z_3, z_4\}$ , where  $x_1 \sim x_2, x_3, x_4$ ; and  $x_2, x_3, x_4$  are not adjacent to each other; and the similar conditions hold for  $z_i$ . Hence, up to multiplication we have  $(f(x_1), f(x_2), f(x_3), f(x_4)) = (\sqrt{3}, 1, 1, 1)$  and  $(f(z_1), f(z_2), f(z_3), f(z_4)) = (-\sqrt{3}, -1, -1, -1)$ .

•  $\lambda = -\sqrt{3}$ . The minimum support is obtained with the help of Lemma 5 and Corollary 1.

**Result:** The minimum support  $S(\pm\sqrt{3}) = \ll 4 + 4 \gg$ , where each  $\ll 4 \gg$  is of type S4.2. Examples are presented below, see Fig. 5, 6:



Figure 5: Pappus graph,  $\mathcal{S}(0) = \ll 1 + ... + 1 \gg$ 



8.8. The Desargues graph. The Desargues graph G has D = 5; the set of eigenvalues  $\Lambda = \{\pm 3^{(1)}, \pm 2^{(4)}, \pm 1^{(5)}\}$ ; its intersection array is (3, 2, 2, 1, 1; 1, 1, 2, 2, 3). Since the Desargues graph can be considered as BDC of Petersen graph (see [5]), we get the following results:

- $\lambda = 1$ . The minimum support for this case has the structure  $\ll 2+2+2+2\gg$ .
- $\lambda = -2$ . In this case there are two possible structures for the minimum support. So  $S(-2) = \ll 6 + 6 \gg$  is either a pair of cycles  $C_6$ , or a pair of H-graphs.
- The minimum supports for λ = −1 and λ = 2 is obtained with the help of Lemma 5 and Corollary 1.

**Result:** The minimum support  $S(\pm 1) = \ll 2 + 2 + 2 + 2 \gg$ . An example is shown on Fig. 7:

**Result:** The minimum support  $S(\pm 2) = \ll 6 + 6 \gg$ . Examples are shown on Fig. 8, 9:

8.9. The dodecahedral graph. The Dodecahedral graph G has D = 5;  $\Lambda = \{3^{(1)}, \pm \sqrt{5}^{(3)}, 1^{(5)}, 0^{(4)}, -2^{(4)}\}$ ; and its intersection array is (3, 2, 1, 1, 1; 1, 1, 1, 2, 3). We obtain:

 $\mathcal{W}(\lambda) = \{1, \lambda, \lambda^2 - 3, \lambda^3 - \lambda^2 - 5\lambda + 3, \frac{\lambda^4 - 2\lambda^3 - 5\lambda^2 + 8\lambda}{2}, \frac{\lambda^5 - 2\lambda^4 - 7\lambda^3 + 10\lambda^2 + 10\lambda - 6}{6}\}.$  The cases of interest are:



Figure 7: Desargues graph,  $\mathcal{S}(1) = \ll 2 + 2 + 2 + 2 \gg$ 





Figure 8: Desargues graph,  $\mathcal{S}(-2) = \ll 6 + 6 \gg$ : cycles

Figure 9: Desargues graph,  $\mathcal{S}(-2) = \ll 6 + 6 \gg$ : H-graphs

•  $\lambda = 0$ . Here we have  $\mathcal{W}(\lambda) = \{1, 0, -3, 3, 0, -1\}$  and WDB = 8. Suppose that  $s_1 \in S$  and  $f(s_1) = 1$ . Consider the distance partition of the Dodecahedral graph with respect to  $s_1$ . Then we have  $G_1(s_1) = \{v_1, v_2, v_3\}$  and  $G_2(s_1) = \{u_{11}, u_{12}, u_{22}, u_{31}, u_{32}\}$ . Without loss of generality suppose that  $u_{11} \sim u_{21}, u_{22} \sim u_{31}$  and  $u_{32} \sim u_{12}$  (note that  $a_2 = 1$ ); then we have  $G_3(s_1) = \{w_{11}, w_{12}, w_{21}, w_{22}, w_{31}, w_{32}\}$ , where  $u_{ij} \sim w_{ij}$  and  $w_{i1} \sim w_{i2}$  for any suitable *i* and *j*.  $G_4(s_1) = \{x_1, x_2, x_3\}$  and  $w_{11}, w_{21} \sim x_1$ ;  $w_{22}, w_{31} \sim x_2$ ;  $w_{12}, w_{32} \sim x_3$ . And finally,  $G_5(s_1) = \{y_1\}$ , where  $x_i \sim y_1$  for each suitable *i*.

We will show that  $\ll 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 \gg$  is the minimum support. For WDB to be achieved, there should be three -1 values in  $G_3(s_1)$ , three 1 values in  $G_4(s_1)$  and  $y_1 \in \mathcal{S}$ . We also need each vertex  $v_i$ to have a non-zero neighbor in  $G_2(s_1)$ . This implies that without loss of generality we can suppose that  $u_{11}, u_{22}, u_{32} \in \mathcal{S}$ . This automatically leads

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to  $w_{12}, w_{21}, w_{31} \in S$ . Thus, we constructed  $\ll 1 + ... + 1 \gg$  with  $f(s_1) = f(w_{ij}) = 1$  and  $f(u_{mk}) = f(y_1)$  for suitable indices. It is easy to check that the eigenfunction equality holds for zero vertices. With additional calculations it can be shown that this is the only possible structure for the minimum support.

**Result:** The minimum support  $S(0) = \ll 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 1 \gg$ . See Fig. 10.

•  $\lambda = 1$ . For this eigenvalue  $\mathcal{W}(\lambda) = \{1, 1, -2, -2, 1, 1\}$  and WDB = 8. The minimum support in this case is  $\ll 2 + 2 + 2 + 2 \gg$ ; it is constructed as follows. Using the notation above, we can choose any of  $s_1 \sim v_i$  as the first pair of adjacent vertices. Without loss of generality suppose  $s_1, v_1 \in \mathcal{S}$ . Since each vertex z of the Dodecahedral graph has exactly one vertex at distance D from z and taking WD into the account, we can conclude that  $x_1, y_2 \in \mathcal{S}$  (as they are antipodes to  $s_1$  and  $v_1$  correspondingly). It is easy to show that either  $u_{21}, u_{32}, w_{21}, w_{32} \in \mathcal{S}$  or  $u_{22}, u_{31}, w_{11}, w_{12} \in \mathcal{S}$ . Then up to multiplication the eigenfunction with this support has values:  $f(s_1) = f(v_1) = f(x_1) = f(y_2) = 1$  and  $f(u_{21}) = f(u_{32}) = f(w_{21}) = f(w_{32}) = -1$  (or  $f(u_{22}) = f(u_{31}) = f(w_{11}) = f(w_{12}) = -1$ ).

**Result:** The minimum support  $S(1) = \ll 2 + 2 + 2 + 2 \gg$ . See Fig. 11



Figure 10: Dodecahedral graph,  $\mathcal{S}(0) = \ll 1 + \ldots + 1 \gg$ 



Figure 11: Dodecahedral graph,  $\mathcal{S}(1) = \ll 2 + 2 + 2 + 2 \gg$ 

- $\lambda = -2$ . In this case we get  $\mathcal{W}(\lambda) = \{1, -2, 1, 1, -2, 1\}$  and WDB = 8. The minimum support  $\mathcal{S}(-2)$  has cardinality 12 and can have only one of the following structures:
  - $-\mathcal{S}(-2) = \ll 6+6 \gg$ , where both  $\ll 6 \gg$  stand for H-graph, see Fig. 12.  $-\mathcal{S}(-2) = \ll 12 \gg$ , where  $\ll 12 \gg$  stands for a cycle  $C_{12}$ , see Fig. 13.
- $\lambda = \pm \sqrt{5}$ . For this eigenvalue we have  $\mathcal{W}(\lambda) = \{1, \pm \sqrt{5}, 2, -2, \pm \sqrt{5}, -1\}$ and WDB  $\geq 12$ . The minimum support has cardinality 16 and can have only one of the following structures:
  - $\mathcal{S}(\pm\sqrt{5}) = \ll 16 \gg, \text{ where } \ll 16 \gg \text{ stands for the graph from Fig. 14.}$ Let  $\mathcal{S}$  consist of the vertices  $\{s_1, v_1, u_{ij}, w_{ij}, y_2, x_1\}$ . Then the corresponding eigenfunction has values:  $f(s_1) = -f(x_1) = 1, f(v_1) = -f(y_2) = \lambda, f(u_{11}) = f(u_{12}) = -f(w_{22}) = -f(w_{31}) = 2, f(u_{21}) = f(u_{32}) = -f(w_{32}) = -f(w_{31}) = 2, f(u_{21}) = f(u_{32}) = -f(w_{32}) = -f(w_{31}) = 2, f(w_{31}) = -f(w_{32}) = -f(w_{33}) = -f(w_{33})$

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Figure 12: Dodecahedral graph,  $\mathcal{S}(-2) = \ll 6 + 6 \gg$ , H-graphs



Figure 13: Dodecahedral graph,  $\mathcal{S}(-2) = \ll 12 \gg$ , cycle

$$-f(w_{21}) = -f(w_{32}) = \frac{\lambda - 1}{2}, f(u_{22}) = f(u_{31}) = -f(w_{11}) = -f(w_{12}) = \frac{-\lambda - 1}{2}.$$



Figure 14: Dodecahedral's subgraph,  $\mathcal{S}(\sqrt{5})$ 

 $\begin{array}{l} - \ \mathcal{S}(\sqrt{5}) = \ll 16 \gg, \text{where } \ll 16 \gg \text{ stands for the graph from Fig. 15. Let} \\ \mathcal{S} \text{ has vertices } \{s_1, v_1, v_2, u_{11}, u_{12}, u_{21}, u_{22}, u_{31}, w_{11}, w_{12}, w_{22}, w_{31}, w_{32}, y_2, y_3, x_1\}. \text{ The corresponding eigenfunction will have the following} \\ \text{values } \{1, \frac{\sqrt{5}+1}{2}, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}+1}{2}, 1, 1, \frac{1-\sqrt{5}}{2}, -1, 1, \frac{\sqrt{5}-1}{2}, -1, \frac{-1-\sqrt{5}}{2}, -1, \frac{-1-\sqrt{5}}{2}, -1, \frac{-1-\sqrt{5}}{2}, -1\}. \end{array}$ 

8.10. The Coxeter graph. The Coxeter graph has D = 4;  $\Lambda = \{3^{(1)}, 2^{(8)}, -1^{(7)}, (-1 \pm \sqrt{2})^{(6)}\}$ ; and its intersection array is (3, 2, 2, 1; 1, 1, 1, 2). Calculations of WD give us the following:  $\mathcal{W}(\lambda) = \{1, \lambda, \lambda^2 - 3, \lambda^3 - 5, \frac{\lambda^4 - \lambda^3 - 7\lambda^2 + 5\lambda + 6}{2}\}$ .

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Figure 15: Dodecahedral's subgraph,  $\mathcal{S}(\sqrt{5})$ 

- $\lambda = -1$ . For this eigenvalue we have  $\mathcal{W}(\lambda) = \{1, -1, -2, 4, -2\}$  and WDB = 10. Using computer calculations, we found that the minimum support has cardinality 12 and can be one of the following structures:  $-S(-1) = \ll 2 + 2 + 2 + 2 + 2 + 2 = 2 \gg$ 
  - $S(-1) = \ll 2 + 2 + 2 + 2 + 2 + 2 \gg.$

 $-S(-1) = \ll 6 + 6 \gg$ , where both  $\ll 6 \gg$  stand for H-graph. Examples of supports are presented below on Fig. 16, 17:



Figure 16: Coxeter graph,  $\mathcal{S}(-1) = \ll 2 + \ldots + 2 \gg$ 



Figure 17: Coxeter graph,  $\mathcal{S}(-1) = \ll 6 + 6 \gg$ 

- $\lambda = 2$ . In this case  $\mathcal{W}(\lambda) = \{1, 2, 1, -2, -2\}$  and WDB = 8. Computer calculations showed that the minimum support has cardinality 12 and can have only the following structure:
  - $-\mathcal{S}(12) = \ll 6 + 6 \gg$ , where both  $\ll 6 \gg$  stand for H-graph.
- $\lambda = -1 + \sqrt{2}$ . Here  $\mathcal{W}(\lambda) = \{1, \sqrt{2} 1, -2\sqrt{2}, -2, \sqrt{2} + 2\}$  and WDB =  $4\sqrt{2} + 4$ . The minimum support has cardinality 16 and can have only the following structure:

-  $S(-1 + \sqrt{2}) = \ll 16 \gg$ , where  $\ll 16 \gg$  stands for a spider-graph, see Fig. 18. Let S consist of vertices  $\{s_1, \ldots, s_8, v_1, \ldots, v_8\}$  where  $s_i \sim v_i$  for each  $i, v_1 \sim v_2 \sim \ldots \sim v_8 \sim v_1$  and there are no other edges. In other words, vertices  $v_i$  form the inner cycle  $C_8$ . Then the corresponding eigenfunction will have the values:  $f(s_1) = f(s_3) =$  $f(s_5) = 1, f(s_2) = f(s_4) = f(s_6) = -1, f(v_1) = f(v_3) = f(v_6) =$  $f(v_7) = \lambda, f(v_2) = f(v_4) = f(v_5) = f(v_8) = -\lambda.$ 



Figure 18: Spider-graph,  $\mathcal{S}(-1+\sqrt{2})$ 

 λ = −1 − √2. The minimum support has the same structure as in case of λ = −1 + √2, but with different values.

8.11. The Tutte-Coxeter graph. The Tutte-Coxeter graph has D = 4;  $\Lambda = \{\pm 3^{(1)}, \pm 2^{(9)}, 0^{(10)}\}$ ; its intersection array is (3, 2, 2, 2; 1, 1, 1, 3); and its weight distribution is  $\mathcal{W}(\lambda) = \{1, \lambda, \lambda^2 - 3, \lambda^3 - 5\lambda, \frac{\lambda^4 - 7\lambda^2 + 6}{3}\}$ .

•  $\lambda = 0$ . Here we have  $\mathcal{W}(\lambda) = \{1, 0, -3, 0, 2\}$  and WDB = 6. This border is achievable with the support  $\ll 1 + 1 + 1 + 1 + 1 + 1 \gg$ .

**Result:** The minimum support for  $S(0) = \ll 1 + 1 + 1 + 1 + 1 + 1 \gg$ , see Fig. 19:

- $\lambda = 2$ . WDB = 8. Computer calculations showed that the cardinality of the minimum support is 14 and there are 3 possible structures:
  - S(2) =≪7 + 7≫, where both ≪7≫ stand for Y-graph (see Fig. 20). Let ≪7+7≫ consist of { $s, v_1, v_2, v_3, u_1, u_2, u_3$ }∪{ $t, x_1, x_2, x_3, y_1, y_2, y_3$ }; where  $s \sim v_1, v_2, v_3; v_i \sim u_i$  and  $t \sim y_1, y_2, y_3; y_i \sim x_i$  and there are no other edges in this S. Then up to multiplication the eigenfunction with this support has the following values:  $f(s) = 1, f(v_1) = f(v_2) =$  $f(v_3) = \frac{1}{3}, f(u_1) = f(u_2) = f(u_3) = \frac{2}{3}, f(x_1) = f(x_2) = f(x_3) = -\frac{2}{3},$  $f(y_1) = f(y_2) = f(y_3) = -\frac{1}{3}, f(t) = -1.$
  - $S(14) = \ll 7 + 7 \gg$ , where both  $\ll 7 \gg$  stand for I-graph (see Fig. 21). Let  $\ll 7+7 \gg$  consist of  $\{s_1, s_2, s_3, v_1, v_2, u_1, u_2\} \cup \{t_1, t_2, t_3, x_1, x_2, y_1, y_2\}$ ; where  $s_1 \sim v_1, v_2, s_2$ ;  $s_2 \sim s_3$ ;  $s_3 \sim y_1, y_2$  and  $t_1 \sim x_1, x_2, t_2$ ;  $t_2 \sim t_3$ ;  $t_3 \sim y_1, y_2$  and there are no other edges in this S. Then up to multiplication the eigenfunction with this support has the following

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Figure 19: Tutte-Coxeter graph,  $\mathcal{S}(0)$ 

values:  $f(s_1) = f(s_2) = f(s_3) = 1$ ,  $f(v_1) = f(v_2) = f(u_1) = f(u_2) = \frac{1}{2}$ ,  $f(t_1) = f(t_2) = f(t_3) = -1$ ,  $f(x_1) = f(x_2) = f(y_1) = f(y_2) = -\frac{1}{2}$ -  $S(2) = \ll 6 + 8 \gg$ , where  $\ll 8 \gg$  stands for a cycle  $C_8$  and  $\ll 6 \gg$  stands for H-graph. Let  $\ll 6 + 8 \gg$  consist of vertices  $\{s_1, s_2, u_1, u_2, u_3, u_4\} \cup \{v_1, v_2, \dots, v_8\}$ ; where  $s_1 \sim u_1, u_2, s_2; s_2 \sim u_3, u_4$  and  $v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_5 \sim v_6 \sim v_1$ ; and there are no other edges in S. Then up to multiplication the eigenfunction with this support has the following values:  $f(s_1) = f(s_2) = 1$ ,  $f(u_1) = f(u_2) = f(u_3) = f(u_4) = \frac{1}{2}$ ,  $f(v_i) = -\frac{1}{2}$ .

• The minimum support for  $\lambda = -2$  is obtained from the above case with the help of Lemma 5 and Corollary 1.



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#### APPENDIX. ALGORITHM

Suppose  $S(\lambda)$  with a vertex set  $\{s_1, \ldots, s_t\}$  to be a support for some eigenfunction f. Without loss of generality we can take  $f(s_1) = 1$  and consider the distance partition with respect to  $s_1$ . Denote  $L_i = |G_i(s_1)|$  and

$$t_i = \sum_{s_j \in G_i(s_1)} 1, \quad \text{for all } i \in \{1, \dots D\}$$

Then we have  $\sum_{i=1}^{D} t_i = t - 1$ , where  $t_i \leq L_i$  for all *i*. In other words, at each layer *i* there are  $t_i$  non-zero elements. The idea of the algorithm is pretty simple: we consider all compositions of an integer t - 1 into *D* parts and check if we can build an eigenfunction with non-zeros corresponding to the considered composition. Recall, that composition of an integer is its presentation as a sum of a sequence of positive integers (note, that in some cases we also consider the sums of non-zero integers). We are interested only in those that consist of *D* parts. Since the weight disribution gives us a lower bound on the number of non-zeros in each layer and the cardinality of the layer gives us an upper bound, we do not consider the compositions that do not satisfy these bounds. Let *Z* be a total number of the compositions that fulfil the conditions above.

So the algorithm can be described as follows:

- (1) Initializing  $t = \text{WDB}(\lambda)$ .
- (2) Build all compositions  $T^j = \{t_1^j, \ldots, t_D^j\}$ , where

$$\sum_{i=1}^{D} t_i^j = t - 1$$
$$t_i^j \le L_i$$
$$t_i^j \ge |W_i(\lambda)|$$

for all  $j \in \{1, ..., Z\}$  and  $i \in \{1, ..., D\}$ .

- (3) For each composition  $T^{j}$  we consider all possible combinations of vertices that construct our eigenfunction support. In other words, at each layer iwe choose the appropriate number  $t_{i}$  of vertices. As a result we get  $S(\lambda)$ consisting of vertices  $\{s_{1}, \ldots, s_{t}\}$ .
- (4) Solving the system: consider a submatrix Sub of A that consists of the columns of matrix A, that correspond to the vertices  $\{s_1, \ldots, s_t\}$ . Similarly, let  $\bar{f}'$  be a column-vector that consists of non-zero elements of  $\bar{f}'$ . In other words,  $\bar{f}' = (f(s_1), f(s_2), \ldots, f(s_t))^{\mathrm{T}}$ . Therefore, we obtain the system  $Sub * \bar{f}' = 0$ . If there is a non-zero solution the minimum support is found.
- (5) If the system does not have a non-zero solution then increase t and go to Step 2.

For the purposes of this paper the described algorithm was implemented in Matlab.

#### References

 N.L. Biggs, A.G. Boshier, J. Shawe-Taylor. *Cubic distance-regular graphs*, J. London Math.Soc., 33:2 (1986), 385–394.

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- [2] P.M. Weichsel, The Kronecker product of graphs, Proceedings of the American Mathematical Society, 13:1 (1962), 47–52.
- [3] R.A. Horn, C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
- [4] D. Cvetković, P. Rowlinson, S. Simić, *Eigenspaces of graphs*, Cambridge University Press, Cambridge, 1997.
- [5] W. Imrich, T. Pisanski. *Multiple Kronecker covering graphs*, European Journal of Combinatorics, 29:5 (2008), 1116-–1122.
- [6] V.N. Potapov, On perfect 2-colorings of the q-ary n-cube, Discrete Math., 312:6 (2012), 1269–1272.
- K.V. Vorob'ev, D.S. Krotov, Bounds for the size of a minimal 1-perfect bitrade in a Hamming graph, J. Appl. Ind. Math., 9:1 (2015), 141–146. Translated from Diskretn. Anal. Issled. Oper. 21(6) (2014) 3–10.
- [8] D.S. Krotov, I.Yu. Mogilnykh, V.N. Potapov, To the theory of q-ary Steiner and other-type trades, Discrete Math., 339:3 (2016), 1150–1157.
- [9] A.A. Valyuzhenich, Minimum supports of eigenfunctions of Hamming graphs, Discrete Math., 340:5 (2017), 1064–1068.
- [10] K. Vorob'ev, I. Mogilnykh, A. Valyuzhenich, Minimum supports of eigenfunctions of Johnson graphs, accepted to Discrete Mathematics.
- [11] http://www.graphclasses.org/smallgraphs.html

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