S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports

http://semr.math.nsc.ru

Том 15, стр. 246–257 (2018) УДК 517.54

DOI 10.17377/semi.2018.15.023 MSC 30C65

THE COEFFICIENT OF QUASIMÖBIUSNESS IN PTOLEMAIC SPACES

V.V. ASEEV

Abstract. In ptolemaic spaces the class of η-quasimöbius mappings $f : X \to Y$ with control function $\eta(t) = C\max\{t^{\alpha}, t^{1/\alpha}\}$ may be completely characterized by the inequality $K^{-1} \le (1+\log P(fT))/(1+\log P(T)) \le K$ for all tetrads $T \subset X$ where $P(T)$ denotes the ptolemaic characteristic of a tetrad. The number K has properties quite similar to those of coefficients of quasiconformality, so the concept of K-quasimöbius mapping may be introduced. In particular, the stability theorem is proved for $(1+\varepsilon)$-quasimöbius mappings in $\bar{R}^n$.

Keywords: ptolemaic space, Möbius mapping, quasimöbius mapping, (power) quasimöbius mapping, quasisymmetric mapping, stability theorem.

1. Tetrads

The *tetrad* in the semimetric space (X, ρ) is a quadruple $T = (x_1, x_2, x_3, x_4)$ of mutually distinct points. *The absolute cross-ratio* of a tetrad T is

$$R(T) = R(x_1, x_2, x_3, x_4) := \frac{\rho(x_1, x_2)\rho(x_3, x_4)}{\rho(x_1, x_3)\rho(x_2, x_4)} . \tag{1.1}$$

The semimetric space (X, ρ) is called *ptolemaic* if the *Ptolemy inequality*

$$\rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_1, x_4)\rho(x_2, x_3) \ge \rho(x_1, x_3)\rho(x_2, x_4) \tag{1.2}$$

Aseev, V.V., The coefficient of quasimöbiusness in Ptolemaic spaces.

The work is supported by the program of fundamental scientific researches of the SB RAS No. 1.1.2., project No. 0314-2016-0007.

Received June, 28, 2017, published March, 16, 2018.

holds for any quadruple (x_1, x_2, x_3, x_4) of point in X. *Ptolemaic characteristic* of a tetrad in (X, ρ) is

$$P(T) = P(x_1, x_2, x_3, x_4) := \frac{\rho(x_1, x_2)\rho(x_3, x_4) + \rho(x_1, x_4)\rho(x_2, x_3)}{\rho(x_1, x_3)\rho(x_2, x_4)} \ . \tag{1.3}$$

The inequality $P(T) \geq 1$ is true for any tetrad T in ptolemaic space (X, ρ).

For some general properties of ptolemaic spaces see [1, §32, pp.78-80].

2. Quasimöbius mappings

Let $\eta : [0, +\infty) \to [0, +\infty)$ be a homeo-morphism of $[0, +\infty) \subset \mathsf{R}^1$ onto itself. An injective mapping $f : X \to Y$ in semimetric spaces (X, ρ), (Y, σ) is called *η-quasimöbius* if the estimate $R(fT) \leq \eta(R(T))$ holds for any tetrad T in X. In that case we write $f \in \eta$-QM and call η the *control function* for f. For the definition and basic properties of quasimöbius mappings, as well as for their connections with quasiconformal mappings, see [2]. In particular, $\mu : X \to Y$ is a *möbius* mapping if $R(fT) = R(T)$ for each tetrad T in X.

The case where the control function $\eta(t)$ is of the form

$$\eta(t) = C \cdot \max\{t^{1/\alpha}, t^\alpha\}, \text{ with } C \geq 1, \alpha \geq 1 \tag{2.1}$$

defines a special important subclass of quasimöbius mappings, so called *(power) quasimöbius* mappings. The function (2.1) will be called *(power) control function.*

The inverse function to (2.1) is

$$\eta^{-1}(t) = \min\{(t/C)^\alpha, (t/C)^{1/\alpha}\},$$

and

$$\tilde{\eta}(t) := \frac{1}{\eta(1/t)} = \frac{1}{C} \min\{t^\alpha, t^{1/\alpha}\} \ , \tag{2.2}$$

$$\eta^*(t) := \frac{1}{\eta^{-1}(1/t)} = \max\{(Ct)^\alpha \ , (Ct)^{1/\alpha}\} \leq C^\alpha \max\{t^\alpha, t^{1/\alpha}\} \ . \tag{2.3}$$

For tetrads $T = (x_1, x_2, x_3, x_4)$ and $T' = (x_1, x_3, x_2, x_4)$ as well as for their images $fT = (y_1, y_2, y_3, y_4)$ and $fT' = (y_1, y_3, y_2, y_4)$, where $y_j = f(x_j)$, we have the equalities

$$R(T') = \frac{1}{R(T)} \ , \ R(fT') = \frac{1}{R(fT)}.$$

If $f : X \to Y$ is η-quasimöbius then the estimate $R(fT') \leq \eta(R(T'))$ holds, which is equivalent to

$$\frac{1}{R(fT)} \leq \eta\left(\frac{1}{R(T)}\right).$$

That is the inequality

$$R(fT) \geq (\eta(1/R(T))^{-1} = \tilde{\eta}(R(T))$$

holds for any tetrad T in X.

It follows that $f : X \to Y$ being η-quasimöbius with (power) control function (2.1) the two-sided estimate

$$\tilde{\eta}(R(T)) \leq R(fT) \leq \eta(R(T)) \tag{2.4}$$

holds for any tetrad T in X, and the mapping $f^{-1} : f(X) \to X$ is also (power) quasimöbius with the (power) control function $C^\alpha \max\{t^\alpha, t^{1/\alpha}\} = C^{\alpha-1}\eta(t)$.

The notion of (power) control function was initially introduced in [3] for quasisymmetric mappings and then in [2] for qiasimöbius mappings. The complete characterization for the class of all metric spaces where every quasisymmetric embedding has a power control function (2.1) was obtained in [4, Th. 6.21] in terms of upper sets introduced by D.A. Trotsenko. In particular, this class contains all uniformly perfect metric spaces.

3. Distortion of ptolemaic characteristic

As a special case of more general properties presented in [5, Propositions 3.3, 3.4] we state the following

Lemma 1. *Let $f : X \to Y$ be a (power) quasimöbius mapping in ptolemaic spaces $(X,\rho), (Y,\sigma)$ with the (power) control function $\eta(t)$ from (2.1). Then the following inequality*

$$\frac{1}{2^\alpha C}[P(T)]^{1/\alpha} \leq P(fT) \leq 2C[P(T)]^\alpha \ . \tag{3.1.1}$$

holds for every tetrad T in X.

Proof. Given a tetad $T = (x_1, x_2, x_3, x_4)$ in X we denote $y_j = f(x_j)$ for $j = 1, 2, 3, 4$. Then η-quasimöbius property of f together with the ptolemaic condition $P(T) \geq 1$ implies the right-hand inequality in (3.1.1):

$$P(fT) = \frac{\sigma(y_1,y_2)\cdot\sigma(y_3,y_4)}{\sigma(y_1,y_3)\cdot\sigma(y_2,y_4)} + \frac{\sigma(y_1,y_4)\cdot\sigma(y_2,y_3)}{\sigma(y_1,y_3)\cdot\sigma(y_2,y_4)} \leq 2\eta(P(T)) = 2C[P(T)]^\alpha \ .$$

One of items in the sum

$$\frac{\rho(x_1,x_2)\cdot\rho(x_3,x_4)}{\rho(x_1,x_3)\cdot\rho(x_2,x_4)} + \frac{\rho(x_1,x_4)\cdot\rho(x_2,x_3)}{\rho(x_1,x_3)\cdot\rho(x_2,x_4)} = P(T) \geq 1$$

must be $\geq P(T)/2$. So the left-hand inequality in (2.4) implies the estimate

$$P(fT) \geq \tilde\eta\left(\frac{\rho(x_1,x_2)\cdot\rho(x_3,x_4)}{\rho(x_1,x_3)\cdot\rho(x_2,x_4)}\right) + \tilde\eta\left(\frac{\rho(x_1,x_4)\cdot\rho(x_2,x_3)}{\rho(x_1,x_3)\cdot\rho(x_2,x_4)}\right) \geq \tilde\eta\left(\frac{P(T)}{2}\right) \ .$$

Then the expression (2.2) for $\tilde\eta(t)$ together with the ptolemaic inequality $P(T) \geq 1$ leads to left-hand part in the desired estimate (3.1.1): $P(fT) \geq$

$$\frac{1}{C}\min\left\{\left(\frac{P(T)}{2}\right)^\alpha, \left(\frac{P(T)}{2}\right)^{1/\alpha}\right\} \geq \frac{\min\{P(T)^\alpha, P(T)^{1/\alpha}\}}{2^\alpha C} = \frac{[P(T)]^{1/\alpha}}{2^\alpha C} \ .$$

□

Theorem 1. *Let $f : X \to Y$ be an injective mapping in ptolemaic spaces (X,ρ), (Y,σ). Then*

(i) *f being a (power) η-quasimöbius with $\eta(t) = C\cdot\max\{t^\alpha, t^{1/\alpha}\}$ the estimate*

$$\frac{1}{K} \leq \frac{1+\log(P(fT))}{1+\log(P(T))} \leq K := \alpha(1+\log(2C^\alpha)) \tag{3.2.1}$$

holds for any tetrad T in X;

(ii) *if the estimate*

$$\frac{1}{K} \leq \frac{1+\log(P(fT))}{1+\log(P(T))} \leq K := 1+\varepsilon \ , \tag{3.2.3}$$

holds for any tetrad T in X then f is η-quasimöbius with (power) control function

$$\eta(t) = (9e^2)^\varepsilon \max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\}. \tag{3.2.4}$$

Proof. (i) Let T be a tetrad in X. Then by the right-hand of (3.1.1) we have

$$\log(P(fT)) \le \log(2C) + \alpha \cdot \log(P(T)) ,$$

and consequently

$$\frac{1+\log(P(fT))}{1+\log(P(T))} \le \frac{\log(2C)}{1+\log(P(T))} + \frac{1+\alpha\cdot\log(P(T))}{1+\log(P(T))}$$

$$\le \log(2C) + \alpha \le \alpha(1+\log(2C)) \le \alpha(1+\log(2C^\alpha)) = K .$$

Thus the right-hand inequality in (3.2.1) has been obtained.

The mapping $f^{-1} : f(X) \to X$ is (power) η^*-quasimöbius with the (power) control function $\eta^*(t) = C^\alpha \max\{t^\alpha, t^{1/\alpha}\}$ (see §2). Applying the right-hand inequality in (3.2.1) to the mapping f^{-1} with C^α instead of C and to a tetrad fT in $f(X)$ we obtain the inequality

$$\frac{1+\log(P(T))}{1+\log(P(fT))} \le \alpha(1+\log(2C^\alpha)) = K$$

which gives us the left-hand estimate in (3.2.1). The proof of (i) is complete.

(ii) It follows from (3.2.3) that

$$e^{\frac{1-K}{K}}[P(T)]^{\frac{1}{K}} \le P(fT) \le e^{K-1}[P(T)]^K$$

for any tetrad T in X. That is

$$e^{-\frac{\varepsilon}{1+\varepsilon}}[P(T)]^{\frac{1}{1+\varepsilon}} \le P(fT) \le e^\varepsilon[P(T)]^{1+\varepsilon} . \tag{3.2.5}$$

Given a tetrad $T = (x_1, x_2, x_3, x_4)$ in X and it's image $fT = (y_1, y_2, y_3, y_4)$ in Y where $y_j = f(x_j)$, $j = 1, 2, 3, 4$, we need in the estimate $R(fT) \le \eta(R(T))$.

It has been proved in [5, Corollary 2.9] that any ptolemaic metric four-point set A may be embedded into the extended complex plane by möbius transformations which do not change the characteristics R and P of all tetrads in A. Then considering the restriction $f|T$ we may assume without loss of generality that tetrads T and fT are in the extended complex plane, and $T = (0, z, 1, \infty)$, $fT = (0, w, 1, \infty)$. The required estimate $R(fT) \le \eta(R(T))$ is equivalent to the inequality

$$|w| \le \eta(|z|) . \tag{3.2.6}$$

For tetrads T and fT we have $p_1 := P(T) = |z|+|1-z|$, $q_1 := P(fT) = |w|+|1-w|$. The right-hand estimate in (3.2.5) means that

$$q_1 \le e^\varepsilon p_1^{1+\varepsilon} . \tag{3.2.6}$$

Considering the tetrad $T' = (0, 1, z, \infty)$ in T we have

$$p_2 := P(T') = \frac{1+|1-z|}{|z|} \ ; \ q_2 := P(fT') = \frac{1+|1-w|}{|w|} .$$

By the left-hand inequality in (3.2.5) we have

$$q_2 \ge e^{-\frac{\varepsilon}{1+\varepsilon}} p_2^{\frac{1}{1+\varepsilon}} . \tag{3.2.7}$$

Now the equality $(1+q_1)/(1+q_2) = |w|$ together with (3.2.6)-(3.2.7) leads to the estimate

$$|w| = \frac{1+q_1}{1+q_2} \le \frac{1+e^{\varepsilon}p_1^{1+\varepsilon}}{1+e^{-\frac{\varepsilon}{1+\varepsilon}}p_2^{\frac{1}{1+\varepsilon}}} \le \frac{e^{\varepsilon}p_1^{\varepsilon}+e^{\varepsilon}p_1^{1+\varepsilon}}{e^{\frac{\varepsilon}{1+\varepsilon}}p_2^{\frac{\varepsilon}{1+\varepsilon}}+p_2} e^{\frac{\varepsilon}{1+\varepsilon}}p_2^{\frac{\varepsilon}{1+\varepsilon}}$$

$$\le \frac{1+p_1}{1+p_2} e^{\frac{\varepsilon}{1+\varepsilon}}p_2^{\frac{\varepsilon}{1+\varepsilon}} e^{\varepsilon}p_1^{\varepsilon} = |z| e^{\varepsilon\frac{2+\varepsilon}{1+\varepsilon}}(p_1p_2)^{\varepsilon}p_2^{-\frac{\varepsilon^2}{1+\varepsilon}} \ ,$$

which means that

$$|w| \le |z| e^{2\varepsilon}(p_1p_2)^{\varepsilon}p_2^{-\frac{\varepsilon^2}{1+\varepsilon}} \ . \tag{3.2.8}$$

<u>Case 1</u>. Let $|z| \ge 1$. Then

$$p_1p_2 = \frac{(|z|+|1-z|)(1+|1-z|)}{|z|} \le \frac{|z|(2+|z|)}{|z|} \le 3|z|$$

and

$$p_2^{-\frac{\varepsilon^2}{1+\varepsilon}} \le 1 \ .$$

So in this case we obtain the desired estimate

$$|w| \le |z|e^{2\varepsilon}3^{\varepsilon}|z|^{\varepsilon} < (9e^2)^{\varepsilon}|z|^{1+\varepsilon} \ . \tag{3.2.9}$$

<u>Case 2</u>. Let $|z| \le 1$. Then

$$p_1p_2 = \frac{(|z|+|1-z|)(1+|1-z|)}{|z|} \le \frac{(1+|1-z|)^2}{|z|} \le \frac{(2+|z|)^2}{|z|} \le \frac{9}{|z|}$$

and

$$p_2^{-\frac{\varepsilon^2}{1+\varepsilon}} = \left(\frac{|z|}{1+|1-z|}\right)^{\frac{\varepsilon^2}{1+\varepsilon}} \le |z|^{\frac{\varepsilon^2}{1+\varepsilon}}.$$

So in this case we also obtain the desired estimate

$$|w| \le (9e^2)^{\varepsilon}|z|^{1-\varepsilon+\frac{\varepsilon^2}{1+\varepsilon}} = (9e^2)^{\varepsilon}|z|^{\frac{1}{1+\varepsilon}} \ .$$

Thus

$$|w| \le (9e^2)^{\varepsilon}\max\{|z|^{1+\varepsilon}, |z|^{\frac{1}{1+\varepsilon}}\} = \eta(|z|) \ ,$$

and (ii) has been proved. □

4. Coefficient of quasimöbiusness

Theorem 1 makes it possible to measure the (power) quasimöbius property with just a one number instead of the control function and justifies the following concept.

Definition 1. *Given $K \ge 1$ the injective mapping $f: X \to Y$ in ptolemaic spaces (X,ρ), (Y,σ) is said to be K-quasimöbius if the two-sided estimate*

$$\frac{1}{K} \le \frac{1+\log(P(fT))}{1+\log(P(T))} \le K \ , \tag{4.1.1}$$

holds for any tetrad T in X. The minimal number K ensuring the ineqiality (4.1.1) for every tetrads in X may be regarded to as the coefficient of quasi-möbiusness of the mapping f, quite analogous to the notion of coefficient of quasiconformality.

This analogy appears in the following elementary basic properties.

Proposition 1. *If $f: X \to Y$ is K-quasimöbius then $f^{-1}: f(X) \to X$ is also K-quasimöbius.*

Proposition 2. *If $f : X \to Y$ is K_1-quasimöbius and $g : f(X) \to Z$ is K_2-quasimöbius then $g \circ f : X \to Z$ will be K-quasimöbius with $K = K_1K_2$.*

Proposition 3. *Every 1-quasimöbius mapping $f : X \to Y$ is a möbius mapping.*

Proof. It follows from the Definition 1 that 1-quasimöbius mapping preserves the ptolemaic characteristic of tetrads. Then by [5, Theorem 2.3] it preserves the absolute cross-ratio for every tetrad in X. Thus f is a möbius mapping. □

The notion of the coefficient of quasimöbiusness allows to consider various extremal problems in the class of (power) quasimöbius mappings just similar to these presented in [6] for quasiconformal mappings. Moreover, if X and Y consist of the same finite number of points the mapping of X onto Y with minimal coefficient of quasimöbiusness may be found by computer calculation.

5. Connection with [s]-characteristic of quasimöbiusness

In order to consider the problems of approximation and stability the following number characterictic [s] was first introduced in [7] for quasisymmetric and then in [2] for quasimöbius mappings.

Definition 2. *Let X,Y be semimetric spaces, and $s \in [0, 1]$. An η-quasimöbius (or η-quasisymmetric) mapping $f : X \to Y$ is called [s]-quasi-möbius (respectively, [s]-quasisymmetric) if $\eta(t) - t \le s$ whenever $t \le 1/s$.*

Proposition 4. *Let $s \in (0, 1]$, $C \ge e$,*

$$\varepsilon \le \frac{\log(1 + s^2)}{\log(C/s)} \ , \tag{5.2.1}$$

and $\eta(t) = C^\varepsilon \max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\}$. Then $\eta(t) - t \le s$ whenever $t \le 1/s$.

Proof. In case $1 \le t \le 1/s$ we have

$$t' - t \le t[C^\varepsilon t^\varepsilon - 1] \le \frac{1}{s}[(C/s)^\varepsilon - 1] \le \frac{1}{s}[(1 + s^2) - 1] = s \ ,$$

as desired.

In case $t \in (0, 1]$ we have

$$t' - t \le \varphi(t) := C^\varepsilon t^{\frac{1}{1+\varepsilon}} - t \ .$$

Since the function $\varphi(t)$ has unique point of extremum

$$t_0 = \left(\frac{C^\varepsilon}{1+\varepsilon}\right)^{\frac{1+\varepsilon}{\varepsilon}} = \left[\frac{C}{(1+\varepsilon)^{1/\varepsilon}}\right]^{1+\varepsilon} \ge \left[\frac{C}{e}\right]^{1+\varepsilon} > 1$$

it is increasing in $[0, 1]$, so that

$$t' - t \le \varphi(1) = C^\varepsilon - 1 \le (1 + s^2) - 1 \le s,$$

as desired. □

Now we can specify the rusult in [8, Lemma 4.1] on [s]-characteristic of the inverse mapping.

Theorem 2. *Let $s \in (0,1]$. If the injective mapping $f : X \to Y$ in ptolemaic spaces is K-quasimöbius with $K = 1 + \varepsilon$ and*

$$\varepsilon \leq \frac{\log(1+s^2)}{\log(9e^2/s)} \ , s \in (0,1] \ ,$$

then both mappings f and f^{-1} are $[s]$-quasimöbius in the sense of Definition 2.

Proof. Since both f and f^{-1} are K-quasimöbius (see Proposition 1), it follows from Theorem 1(ii) that they are η-quasimöbius with $\eta(t) = C^{\varepsilon} \max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\}$ where $C = 9e^2 > e$. Then Proposition 4 means that both f and f^{-1} are $[s]$-quasimöbius. □

6. Quasisymmetry in chordal metric

It is well known that quasi-möbius mapping in bounded metric spaces is quasisymmetric one.

Lemma 2 ([2], Theorem 3.12)**.** *Let (X,ρ) and (Y,σ) be bounded metric spaces of diameters $d(X)$ and $d(Y)$ respectively. Let for a given ω-quasimöbius mapping $f : X \to Y$ there exist points $a_1, a_2, a_3 \in X$ such that*

$$\rho(a_i, a_j) \geq \delta \ ; \ \sigma(f(a_i), f(a_j)) \geq \delta$$

for every distinct $i, j \in \{1,2,3\}$. Then f is η-quasisymmetric with the control function

$$\eta(t) = \frac{2d(Y)}{\delta}\omega\left(\frac{2d(X)}{\delta}t\right) . \tag{6.1.1}$$

Proof. Let be given mutually distinct points $x, y, z \in X$. By the triangle inequality in X and Y we can find an index $j \in \{1,2,3\}$ such that

$$\rho(x, a_j) \geq \delta/2 \ ; \ \sigma(f(z), f(a_j)) \geq \delta/2 \ .$$

Then the ω-quasimöbius property for the tetrad $T = (y, x, z, a_j)$ gives the desired estimate:

$$\frac{\delta}{2d(Y)} \frac{\sigma(f(x), f(y))}{\sigma(f(z), f(y))} \leq R(T) \leq \omega(R(fT)) \leq \omega\left(\frac{\rho(x,y)}{\rho(z,y)} \frac{2d(X)}{\delta}\right) \ .$$

□

In the special case where $X, Y \subset \bar{R}^n$ are equipped with chordal metric and f is a (power) quasimöbius mapping we need in some more precise estimate for the control function $\eta(t)$ of quasisymmetry.

The proof of the following lemma is based on elementary routine estimations and has been placed in Appendix at the end of the article.

Lemma 3. *Let the set $A \subset \bar{R}^n$ to contain the points $0, e_0, \infty$ where $|e_0| = 1$ which are fixed points for an ω-quasimöbius mapping $f : A \to \bar{R}^n$ with the control function $\omega(t) = C^{\varepsilon} \max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\}$. Then for any $x \in A$ in every one of the following two situations*

(1) *$a = e_0$ and ($|x| \geq 4C^2/\varepsilon$ or $|x| \leq \varepsilon/(4C^2)$);*
(2) *($a = \infty$ and $0 \leq |x| \leq 4C^2/\varepsilon$) or ($a = 0$ and $|x| \geq \varepsilon/(4C^2)$)*

the inequalities

$$\frac{1}{C_1^{\sqrt{\varepsilon}}} \leq \frac{\sigma(f(a),a)}{\sigma(x,a)^{\frac{1}{1+\varepsilon}}} \leq \frac{\sigma(f(x),a)}{\sigma(x,a)^{1+\varepsilon}} \leq C_1^{\sqrt{\varepsilon}} \tag{6.2.1}$$

are true with the constant $C_1 = 16\sqrt{2}C^5e^2$.

Theorem 3. *Let the set* $A \subset \bar{R}^n$ *contain the points* $0, e_0, \infty$ *where* $|e_0| = 1$ *which are fixed points for an* ω*-quasimöbius mapping* $f : A \to \bar{R}^n$ *with the control function* $\omega(t) = C^\varepsilon \max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\}$. *Then* f *is* η*-quasisimmetric mapping in chordal metric in* $\bar{R}^n$ *with the control function*

$$\eta(t) = C_2^{\sqrt{\varepsilon}} \max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\}$$

where $C_2 = 2^9C^{11}e^4$.

Proof. Given distinct points $x, y, x \in A$ let us denote their images under f by $\tilde{x}, \tilde{y}, \tilde{z}$ respectively. It should be proved the inequality

$$\frac{\sigma(\tilde{x},\tilde{y})}{\sigma(\tilde{z},\tilde{y})} \leq \eta\left(\frac{\sigma(x,y)}{\sigma(z,y)}\right) = C_2^{\sqrt{\varepsilon}} \max\left\{\left(\frac{\sigma(x,y)}{\sigma(z,y)}\right)^{1+\varepsilon}, \left(\frac{\sigma(x,y)}{\sigma(z,y)}\right)^{\frac{1}{1+\varepsilon}}\right\}.$$

If $a \in \{0, e_0, \infty\}$ and $a \neq x$, $a \neq z$ then ω-quasimöbius property of f produces the inequality

$$\frac{\sigma(\tilde{x},\tilde{y})\sigma(\tilde{z},a)}{\sigma(\tilde{z},\tilde{y})\sigma(\tilde{x},a)} \leq C^\varepsilon \max\left\{\left(\frac{\sigma(x,y)\sigma(z,a)}{\sigma(z,y)\sigma(x,a)}\right)^{1+\varepsilon}, \left(\frac{\sigma(x,y)\sigma(z,a)}{\sigma(z,y)\sigma(x,a)}\right)^{\frac{1}{1+\varepsilon}}\right\};$$

$$\frac{\sigma(\tilde{x},\tilde{y})}{\sigma(\tilde{z},\tilde{y})} \leq C^\varepsilon \max\left\{\left(\frac{\sigma(x,y)}{\sigma(z,y)}\right)^{1+\varepsilon}, \left(\frac{\sigma(x,y)}{\sigma(z,y)}\right)^{\frac{1}{1+\varepsilon}}\right\} \cdot \max\{T_1, T_2\},$$

where

$$T_1 = \frac{\sigma(z,a)^{1+\varepsilon}}{\sigma(\tilde{z},a)} \cdot \frac{\sigma(\tilde{x},a)}{\sigma(z,a)}^{1+\varepsilon}; \quad T_2 = \frac{\sigma(z,a)^{\frac{1}{1+\varepsilon}}}{\sigma(\tilde{z},a)} \cdot \frac{\sigma(\tilde{x},a)}{\sigma(z,a)}^{\frac{1}{1+\varepsilon}}.$$

So we have to obtain estimate $\max\{T_1, T_2\} \leq (C_1^2)^{\sqrt{\varepsilon}}$ where C_1^2 is the constant from Lemma 3.

In case ($|x| \geq \varepsilon/(4C^2)$ and $|z| \geq \varepsilon/(4C^2)$) we put $a = 0$ and use (6.2.1) both for x and z in Lemma 3, the situation (2).

In case ($|x| \leq 4C^2/\varepsilon$ and $|z| \leq 4C^2/\varepsilon$) we put $a = \infty$ and use (6.2.1) both for x and z in Lemma 3, the situation (2).

In case (($|x| \geq 4C^2/\varepsilon$ and $|z| \leq \varepsilon](4C^2)$ or $|x| \leq \varepsilon/(4C^2)$ and $|z| \geq 4C^2/\varepsilon$)) we put $a = e_0$ and use both for x and z the inequality (6.2.1) from Lemma 3, the situation (1).

Thus we obtain the desied estimate for T_1 and T_2 in all possible cases. □

7. Stability

We shall use the following stability theorem for $[s]$- quasi-symmetric mappings in R^n which had been obtained by J. Partanen in his dissertation in 1991. It is more convenient for our purposes to formulate it with R^{n+1} instead of R^n.

Theorem 4 ([8], Theorem 1.6)**.** *Let* $A' \subset R^{n+1}$ *be compact and* $B \subset A'$ *have at least two distinct points. Then there ezists a function* $\lambda(s; B, A')$ *which* $\to 0$ *as* $s \to 0$, *such that for any* $[s]$*-quasisymmetric mapping* $f : A' \to R^{n+1}$ *which is* *identical on B there exists an euclidean isometry $h : R^{n+1} \to R^{n+1}$ identical on B such that*

$$\max_{x\in A'} |f(x) - h(x)| \leq \lambda(s; A', B). \tag{7.1.1}$$

The estimate function λ in this theorem essentially depends on metric properties of the set A and may be detailed in some special cases, see [9].

We shall prove the following version of stability theorem for K-quasi-möbius mappings in $\bar{R}^n$.

Theorem 5. *Let the set $A \subset \bar{R}^n$ contain points $0, e_0, \infty$ where $|e_0| = 1$. Then for a given $\delta > 0$ there exists $\varepsilon_0 > 0$ with the following property: For every K-quasimöbius mapping $f : A \to \bar{R}^n$ with fixed points $0, e_0, \infty$ and $K = 1 + \varepsilon$ where $\varepsilon \leq \varepsilon_0$ there exists an euclidean isometry h with fixed points $0, e_0$ such that*

$$\max_{x\in A} \sigma(f(x), h(x)) \leq \delta\ ,$$

where $\sigma(.,.)$ denotes the chordal distance in $\bar{R}^n$.

Proof. By Theorem 1(ii) the K-quasimöbius mapping $f : A \to \bar{R}^n$ with $K = 1 + \varepsilon$ is (power) η-qusimöbius with the control function $\eta(t) = (9e^2)^\varepsilon \max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\}$. Since $f(\infty) = \infty$ it is η-quasisymmetric in euclidean metric in R^n. Then Theorem 3 says that f is also η_1-quasisymmetric in chordal metric with the distortion function $\eta_1(t) = (C_3)^{\sqrt{\varepsilon}} \max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\}$ where $C_3 = 2^9 9^{11} e^{26}$. Since $\varepsilon \leq \sqrt{\varepsilon}$ and $\max\{t^{1+\varepsilon}, t^{\frac{1}{1+\varepsilon}}\} \leq \max\{t^{1+\sqrt{\varepsilon}}, t^{\frac{1}{1+\sqrt{\varepsilon}}}\}$ the mapping f is η_2-quasisymmetric in chordal metric with the control function $\eta_2(t) = C_3^{\sqrt{\varepsilon}} \max\{t^{1+\sqrt{\varepsilon}}, t^{\frac{1}{1+\sqrt{\varepsilon}}}\}$.

The stereographic projection $\pi : \bar{R}^n \to S \subset R^{n+1}$, $\pi(0) = 0$ is the isometry of the space $\bar{R}^n$ equipped with chordal metric to the sphere $S \subset R^{n+1}$ equipped with euclidean metric. So we may identify $\bar{R}^n$ as a sphere S in R^{n+1}. We denote $e_1 = \pi(e_0)$, $p = \pi(\infty)$.

Then the mapping $g = \pi \circ f \circ \pi^{-1} : A' = \pi(A) \to S$ is η_2-quasisymmetric in euclidean metric in R^{n+1} and is identical on the set $B = \{0, e_1, p\} \subset \pi(A) \subset S$. As the function $\lambda(s, A', B)$ in the Theorem 4 tends to 0 as $s \to 0$ we can find for a given $\delta > 0$ such $s_0 > 0$ that $\lambda(s_0, A', B) \leq \delta$. Then we can find ε_0 such that

$$\sqrt{\varepsilon} \leq \frac{\log(1 + s_0^2)}{\log(C_3/s_0)} \tag{7.2.2}$$

for all $\varepsilon < \varepsilon_0$. In this case we have by Proposition 4 the estimate $\eta_2(t) - t \leq s_0$ to be valid for all $0 \leq t \leq s_0$. That means that g is $[s_0]$-quasisymmetric in euclidean metric in R^{n+1}. By Theorem 4 there exists an euclidean isometry $h' : R^{n+1} \to R^{n+1}$ with fixed points $0, e_1, p$ such that

$$\max_{y\in\Sigma(A)} |g(y) - h'(y)| \leq \lambda(s_0, A', B) \leq \delta\ .$$

The isometry h' with fixed points $0, p$ is identical on the whole line through these points, so the center y_0 of sphere S is also a fixed point for h'. It means that $h'(S) = S$ and $h = \pi^{-1} \circ h' \circ \pi$ is an isometry in chordal metric in $\bar{R}^n$. Hence h being a möbius mapping with fixed points $0, e_0, \infty$ is mere an euclidean isometry in R^n with fixed line through 0 and e_0. Next we have the equality $|g(y) - h'(y)| = \sigma(f(\pi^{-1}(y)), h(\pi^{-1}(y))) = \sigma(f(x), h(x))$ for all $x \in A$ with $y = \pi(x)$. Thus

$$\max_{y\in A'} |g(y) - h'(y)| = \max_{x\in A} \sigma(f(x), h(x)) \leq \delta$$

provided $\varepsilon \le \varepsilon_0$. □

8. Appendix

Here we present the proof to Lemma 3.

Case (1). Since the mapping $j(x) = x/|x|^2$ ($j(0) = \infty$, $j(\infty) = 0$, $j(e_0) = e_0$) preserves chordal distances the mapping $g = j \circ f \circ j$ satisfies on $j(A)$ the same conditions as f. Then for each $\alpha \in \{1+\varepsilon, 1/(1+\varepsilon)\}$ the equality

$$\frac{\sigma(g(j(x)), e_0)}{\sigma(j(x)), e_0)^\alpha} = \frac{\sigma(j(f(x)), e_0)}{\sigma(j(x), e_0)^\alpha} = \frac{\sigma(f(x), e_0)}{\sigma(x, e_0)^\alpha}$$

holds for all $x \in A$. Thus in the sutuation (1) it suffices to proove (1) for the case $|x| \ge 4C^2/\varepsilon$. Applying to $\tilde{x} := f(x)$ the ω-quasimöbius property we have:

$$\frac{1}{|\tilde{x}|} \le \frac{C^\varepsilon}{|x|^{\frac{1}{1\varepsilon}}} \le C^\varepsilon \left(\frac{\varepsilon}{4C^2}\right)^{\frac{1}{1\varepsilon}} \le C^\varepsilon \frac{\sqrt{\varepsilon}}{2C} \le \frac{\sqrt{\varepsilon}}{2} \le \frac{1}{2}\ ;$$

$$\frac{1}{2}\ln\frac{(1+|\tilde{x}|)^2}{1+|\tilde{x}|^2} \le \frac{|\tilde{x}|}{1+|\tilde{x}|^2} \le \frac{1}{|\tilde{x}|} \le \frac{\sqrt{\varepsilon}}{2} \le \sqrt{\varepsilon}\ ;$$

$$\frac{|\tilde{x} - e_0|}{\sqrt{1+|\tilde{x}|^2}} \le \sqrt{\frac{(1+|\tilde{x}|)^2}{1+|\tilde{x}|^2}} \le e^{\sqrt{\varepsilon}}\ ; \tag{0.1}$$

$$\frac{1}{2}\ln\frac{1+|\tilde{x}|^2}{(|\tilde{x}|-1)^2} \le \frac{|\tilde{x}|}{(|\tilde{x}|-1)^2} = \frac{1}{|\tilde{x}|}\cdot\frac{1}{(1-1/|\tilde{x}|)^2} \le \frac{\sqrt{\varepsilon}}{2(1/2)^2} = 2\sqrt{\varepsilon}\ ;$$

$$\frac{|\tilde{x} - e_0|}{\sqrt{1+|\tilde{x}|^2}} \ge \sqrt{\frac{(|\tilde{x}|-1)^2}{1+|\tilde{x}|^2}} \ge \frac{1}{(e^2)^{\sqrt{\varepsilon}}}\ . \tag{0.2}$$

$$\frac{1+\varepsilon}{2}\ln\frac{1+|x|^2}{(|x|-1)^2} \le \frac{2|x|}{(|x|-1)^2} = \frac{2}{|x|(1-1/|x|)^2} \le \frac{2\varepsilon}{4C^2(1/2)^2} \le 2\sqrt{\varepsilon}\ ;$$

$$\left(\frac{\sqrt{1+|x|^2}}{|x-e_0|}\right)^{1+\varepsilon} \le \left(\frac{\sqrt{1+|x|^2}}{|x|-1}\right)^{1+\varepsilon} \le (e^2)^{\sqrt{\varepsilon}}\ ; \tag{0.3}$$

$$\frac{1}{2(1+\varepsilon)}\ln\frac{(1+|x|)^2}{1+|x|^2} \le \frac{|x|}{2(1+|x|^2)} \le \frac{\varepsilon}{8C^2} \le \sqrt{\varepsilon}\ ;$$

$$\left(\frac{\sqrt{1+|x|^2}}{|x-e_0|}\right)^{\frac{1}{1+\varepsilon}} \ge \left(\frac{\sqrt{1+|x|^2}}{|x|+1}\right)^{\frac{1}{1+\varepsilon}} \ge \frac{1}{e^{\sqrt{\varepsilon}}}\ . \tag{0.4}$$

Thus using the inequalities $(\sqrt{2})^\varepsilon \le (\sqrt{2})^{\sqrt{\varepsilon}}$, $(\sqrt{2})^{\frac{1}{1+\varepsilon}-1} = (\sqrt{2})^{-\frac{\varepsilon}{1+\varepsilon}} \ge 1/(\sqrt{\varepsilon})^{\sqrt{\varepsilon}}$ and (0.1)-(0.4) we obtain the desired estimate

$$\frac{1}{C_1^{\sqrt{\varepsilon}}} \le \frac{1}{(e^3\sqrt{2})^{\sqrt{\varepsilon}}} \le \frac{\sigma(\tilde{x}, e_0)}{\sigma(x, e_0)^{\frac{1}{1+\varepsilon}}} \le \frac{\sigma(\tilde{x}, e_0)}{\sigma(x, e_0)^{1+\varepsilon}} \le (e^3\sqrt{2})^{\sqrt{\varepsilon}} \le C_1^{\sqrt{\varepsilon}}\ .$$

Case (2). Since j is the chordal isometry the equality

$$\frac{\sigma(g(j(x)), 0)}{\sigma(j(x), 0)^\alpha} = \frac{\sigma(j(f(x)), 0)}{\sigma(j(x), 0)^\alpha} = \frac{\sigma(f(x), \infty)}{\sigma(x, \infty)^\alpha}$$

holds. So in the situation (2) it suffices to prove (6.2.1) for the case $a = \infty$.

If $1 \le |x| \le 4C^2/\varepsilon$, the ω-quasimöbius property together with the inequality $1/\varepsilon^\varepsilon \le 2^2\sqrt{\varepsilon}$ produces the following estimates

$$\frac{\sigma(\tilde{x},\infty)}{(\sigma(x,\infty))^{1+\varepsilon}} = \sqrt{\frac{(1+|x|^2)^{1+\varepsilon}}{1+|\tilde{x}|^2}} \le \sqrt{C^{2\varepsilon}\frac{(1+|x|^2)^{1+\varepsilon}}{1+|\tilde{x}|^{\frac{2}{1+\varepsilon}}}}$$

$$\le (C\sqrt{2})^\varepsilon |x|^\varepsilon \sqrt{\frac{1+|x|^2}{1+|x|^{\frac{2}{1+\varepsilon}}}} \le (C\sqrt{2})^{\sqrt{\varepsilon}}|x|^\varepsilon \sqrt{\frac{1+|x|^{\frac{2}{1+\varepsilon}}|x|^{\frac{2\varepsilon}{1+\varepsilon}}}{1+|x|^{\frac{2}{1+\varepsilon}}}}$$

$$\le (C\sqrt{2})^\varepsilon |x|^\varepsilon |x|^{\frac{\varepsilon}{1+\varepsilon}} \le (C\sqrt{2})^\varepsilon |x|^{2\varepsilon} \le (16\sqrt{2}C^5e^2)^{\sqrt{\varepsilon}}\ ;$$

$$\frac{\sigma(\tilde{x},\infty)}{\sigma(x,\infty)^{\frac{1}{1+\varepsilon}}} = \sqrt{\frac{(1+|x|^2)^{\frac{1}{1+\varepsilon}}}{1+|\tilde{x}|^2}} \ge \frac{1}{C^\varepsilon}\sqrt{\frac{(1+|x|^2)^{\frac{1}{1+\varepsilon}}}{1+|x|^{2(1+\varepsilon)}}}$$

$$\ge \frac{1}{C^\varepsilon |x|^\varepsilon (1+|x|^2)^{\frac{\varepsilon}{2(1+\varepsilon)}}} \ge \frac{1}{(C\sqrt{2})^\varepsilon |x|^{2\varepsilon}} \ge \frac{1}{(16\sqrt{2}C^5e^2)^{\sqrt{\varepsilon}}}\ .$$

Thus we obtain the estimates (6.2.1) in the case $a = \infty, 1 \le |x| \le 4C^2/\varepsilon$:

$$\frac{1}{C_1^{\sqrt{\varepsilon}}} \le \frac{1}{(16\sqrt{2}C^5e^2)^{\sqrt{\varepsilon}}} \le \frac{\sigma(\tilde{x},\infty)}{\sigma(x,\infty)^{\frac{1}{1+\varepsilon}}} \le \frac{\sigma(\tilde{x},\infty)}{\sigma(x,\infty)^{1+\varepsilon}} \le 16\sqrt{2}C^5e^2)^{\sqrt{\varepsilon}} = C_1^{\sqrt{\varepsilon}}\ .$$

If $a = \infty$, $\varepsilon \le |x| \le 1$ then $|x|^{1+\varepsilon} \le |\tilde{x}| \le C^\varepsilon |x|^{\frac{1}{1+\varepsilon}}$, and

$$\frac{\sigma(\tilde{x},\infty)}{\sigma(x,\infty)^{1+\varepsilon}} = \sqrt{\frac{(1+|x|^2)^{1+\varepsilon}}{1+|\tilde{x}|^2}} \le (\sqrt{2})^\varepsilon \sqrt{C^{2\varepsilon}\frac{1+|x|^2}{C^{2\varepsilon}+|x|^2|x|^{2\varepsilon}}}$$

$$\le (C\sqrt{2})^\varepsilon \sqrt{\frac{1+|x|^2}{|x|^{2\varepsilon}(1+|x|^2)}} \le \frac{(C\sqrt{2})^\varepsilon}{\varepsilon^\varepsilon} \le (C\sqrt{2}e^2)^{\sqrt{\varepsilon}} \le C_1^{\sqrt{\varepsilon}}\ ;$$

$$\frac{\sigma(x,\infty)^{\frac{1}{1+\varepsilon}}}{\sigma(\tilde{x},\infty)} = \sqrt{\frac{1+|\tilde{x}|^2}{(1+|x|^2)^{\frac{1}{1+\varepsilon}}}} \le \sqrt{\frac{1+C^{2\varepsilon}|x|^{\frac{2}{1+\varepsilon}}}{1+|x|^2}(1+|x|^2)^{\frac{\varepsilon}{1+\varepsilon}}}$$

$$\le (C\sqrt{2})^\varepsilon \sqrt{\frac{1+|x|^2|x|^{-\frac{2\varepsilon}{1+\varepsilon}}}{1+|x|^2}} \le \frac{(C\sqrt{2})^\varepsilon}{|x|^\varepsilon} \le \frac{(C\sqrt{2})^\varepsilon}{\varepsilon^\varepsilon} \le (C\sqrt{2}e^2)^{\sqrt{\varepsilon}} \le C_1^{\sqrt{\varepsilon}}.$$

Thus (6.2.1) is true in the case $a = \infty$, $\varepsilon \le |x| \le 1$.

At last, in the case $a = \infty$, $0 \le |x| \le \varepsilon$ we have

$$\frac{\sigma(\tilde{x},\infty)}{\sigma(x,\infty)^{1+\varepsilon}} = \sqrt{\frac{(1+|x|^2)^{1+\varepsilon}}{1+|\tilde{x}|^2}} \le (1+\varepsilon)^{\frac{1+\varepsilon}{2}} \le e^{(\varepsilon^2)} \le e^{\sqrt{\varepsilon}} \le C_1^{\sqrt{\varepsilon}}\ ;$$

$$\frac{\sigma(x,\infty)^{\frac{1}{1+\varepsilon}}}{\sigma(\tilde{x},\infty)} = \sqrt{\frac{1+|\tilde{x}|^2}{(1+|x|^2)^{\frac{1}{1+\varepsilon}}}} \le C^\varepsilon \sqrt{1+|x|^{\frac{2}{1+\varepsilon}}} \le C^\varepsilon(1+\varepsilon) \le (Ce)^{\sqrt{\varepsilon}} \le C_1^{\sqrt{\varepsilon}}.$$

So (6.2.1) is true in the case $a = \infty$, $0 \le |x| \le \varepsilon$ as well.

Now the lemma 3 has been comletely proved.

The author is thankful to referee for his(her) critical remarks on the content of this article.

References

[1] M.L. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford: Clarendod Press, 1953. MR0054981

[2] J. Väisälä, *Quasimöbius maps*, J. Anal. Math., **44** (1984/85), 218–234. MR0801295

[3] P. Tukia, J. Väisälä, *Quasisymmetric embeddings of metric spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math., **5**:1 (1980), 97–114. MR0595180

[4] D.A. Trotsenko, J. Väisälä, *Upper sets and quasisymmetric maps*, Ann. Acad. Sci. Fenn. Ser. A I Math., **24**:2 (1999), 465–488. MR1724387

[5] V.V. Aseev, A.V. Sychev, A.V. Tetenov, *Möbius-invariant metrics and generalized angles in ptolemeic spaces*, Sib. Math. J., **46**:2 (2005), 189–204. MR2141193

[6] F.W. Gehring, J. Väisälä, *The coefficients of quasiconformality of domains in space*, Acta Math., **114**:1–2 (1965), 1–70. MR0180674

[7] P. Tukia, J. Väisälä, *Extension of embeddings close to isometries or similarities*, Ann. Acad. Sci. Fenn. Ser. A I Math., **9** (1984), 153–175. MR0752401

[8] J. Partanen, *Invariance theorems for the bilipschitz and quasisymmetric extension properties*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissert., **80** (1991), 1–40. MR1098344

[9] P. Alestalo, D.A. Trotsenko, *On mappings that are close to a similarity*, Math. Rep., Buchar., **15(65)**:4 (2013), 313–318. MR3237474

Vladislav Vasilyevich Aseev
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
E-mail address: btp@math.nsc.ru