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ON THE MINIMUM SUPPORTS OF SOME EIGENFUNCTIONS IN THE DOOB GRAPHS

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Abstract. We prove that the minimum size of the support of an eigenfunction in the Doob graph $D(m,n)$ corresponding to the second largest eigenvalue is $6 \cdot 4^{2m+n-2}$, and obtain characterisation of all eigenfunctions with minimum support. Similar results, with the minimum support size 2^{2m+n}, are obtained for the minimum eigenvalue of $D(m,n)$.

Keywords: eigenfunction, minimum support, Doob graph.

1. Introduction

Many combinatorial configurations are connected with eigenfunctions. The study of such configurations often leads to considering the symmetric difference of two configurations from same class. The finding of the minimum possible difference of two combinatorial objects can be helpful in the tasks of constructing new objects with the same parameters or finding the bounds on the number of such objects. For some configurations, this difference belongs to the class of objects known as trades. Since the spectre of characteristic $\{0, \pm 1\}$-function of a trade consists of the small number of eigenvalues of the graph, for many combinatorial objects (perfect codes, MDS codes, equitable partitions, combinatorial designs, etc.) trades are connected with the eigenfunctions of graphs. In this light, the problem of finding the minimum size of the support of eigenfunction is intriguing. For more information about trades see [1]. Currently, some bounds and exact values for the minimum cardinalities of the supports of eigenfunctions in the Hamming graphs ([1], [2], [3]), Johnson graphs [4], Paley graphs [12] and cubical distance-regular graphs [13] are known.

The Doob graph $D(m,n)$ is a distance-regular graph with the same parameters as the Hamming graph $H(2m+n,4)$. Some combinatorial objects related with

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eigenfunctions have already been studied in the Doob graphs, for example perfect codes [5], [6] and MDS codes [7], [8]. In the current paper we study the eigenfunctions of the Doob graphs. The adjacency matrix of the Doob graph $D(m, n)$ has the same eigenvalues as the adjacency matrix of the Hamming graph $H(2m+n, 4)$. For the eigenfunctions in the Hamming graphs, there are bounds on the cardinality of support obtained in [2]. Since the bound in [2] is based on the parameters of the distance-regular graph, these bounds also are valid for eigenfunctions in the Doob graphs. In the current paper we find the minimum cardinality of the support of eigenfunctions, corresponding to the second largest eigenvalue $\lambda_1 = 6m + 3n - 4$ and the minimum eigenvalue $\lambda_{2m+n} = -2m - n$, and describe all eigenfunctions with the minimum cardinality of the support. The main results are formulated in Theorem 1 and Theorem 2. In the proof we use the results of [3], where the same problem in the Hamming graphs is studied . In [3] it was proved, that the minimum cardinality of the support of eigenfunction in $H(n, q)$ corresponding to the second largest eigenvalue $\lambda_1 = n(q - 1) - q$ is equal to $2(q - 1)q^{n-2}$, and all functions with the minimum support are described. Note that for $q = 4$ the minimum possible cardinalities of the support of the eigenfunctions corresponding to the second largest eigenvalue coincide for the Doob graphs $D(m, n)$ and the Hamming graphs $H(2m + n, 4)$. However the techniques of the proof are different.

2. Definitions and auxiliary statements

The *Shrikhande graph* Sh is the Cayley graph of the group $\mathbb{Z}_4^2$ with the connecting set $\{01, 03, 10, 30, 11, 33\}$ (the vertices of the graph are the elements of the group $\mathbb{Z}_4^2$, denoted as 00, 01, 02, ..., 33; two vertices are adjacent if and only if their difference belongs to connecting set). The complete graph $K = K_4$ is the Cayley graph of the group $\mathbb{Z}_4$ with the connecting set $\{1, 2, 3\}$.

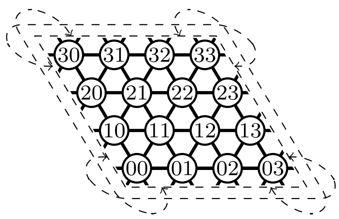

Figure 1. The Shrikhande graph.

Let m and n be non-negative integer numbers. We denote by $D(m, n) = \mathrm{Sh}^m \square K^n$ the Cartesian product of m copies of Sh and n copies of K_4. This graph is called a *Doob graph* if $m > 0$, while $D(0, n)$ is a Hamming graph $H(n, 4)$.

By vG we denote the set of vertices of a graph G. A function $f : \mathrm{v}D(m, n) \to \mathbb{R}$ is called an *eigenfunction* of the graph $D(m, n)$ with the eigenvalue λ, if $f \not\equiv 0$ and $Af = \lambda f$, where A is the adjacency matrix of the graph $D(m, n)$. The adjacency matrix of $D(m, n)$ has the following eigenvalues:

$$\lambda_i = 6m + 3n - 4i, \qquad i = 0, 1, \dots, 2m + n.$$

Denote the corresponding eigensubspaces by

$$V_i^{m,n} = \{f : \mathrm{v}D(m,n) \to \mathbb{R} \mid \sum_{\substack{d(x,y)=1\\ y\in \mathrm{v}D(m,n)}} f(y) = \lambda_i f(x),\ \forall x \in \mathrm{v}D(m,n)\}.$$

The *support* of a function $f : \mathrm{v}D(m,n) \to \mathbb{R}$ is the set

$$S(f) = \{x \in \mathrm{v}D(m,n) : f(x) \neq 0\}.$$

Lemma 1. *Let* $f \in V_1^{1,0}$. *Then*

(1) $\sum\limits_{a\in \mathrm{vSh}} f(a) = 0$*;*
(2) $f(a) + f(a+02) + f(a+20) + f(a+22) = 0$ *for any vertex* $a \in \mathrm{vSh}$*;*
(3) $f(a) + f(a+2s) = f(a+s) + f(a+3s)$ *for any vertex* $a \in \mathrm{vSh}$ *and any* $s \in \{01, 10, 11\}$.

Proof. For all items we will use that for the function f and for any vertex $x \in \mathrm{vSh}$ we have the following equality

$$2f(x) = \sum_{\substack{d(x,y)=1\\ y\in \mathrm{vSh}}} f(y).$$

1. Since the Shrikhande graph is a regular graph of degree 6, then

$$2\sum_{a\in \mathrm{vSh}} f(a) = 6\sum_{a\in \mathrm{vSh}} f(a),$$

whence statement 1 follows.

2. We may assume without loss of generality, that $a = 00$. Then

$$2(f(00) + f(02) + f(20) + f(22)) \\ = 2\sum_{\substack{b\in \mathrm{vSh}\\ b\neq 00,02,20,22}} f(b) = -2(f(00) + f(02) + f(20) + f(22))$$

whence statement 2 follows. The last equality comes from item 1.

3. We may assume without loss of generality, that $a = 00$, $s = 01$. Denote $\Sigma_1 = f(10) + f(12) + f(30) + f(32)$, $\Sigma_2 = f(11) + f(13) + f(31) + f(33)$. Then

$$2(f(00) + f(02)) = 2f(01) + 2f(03) + \Sigma_1 + \Sigma_2$$

and statement 3 follows from item 2. □

For $c \in \mathbb{R}$ and $a \in \mathrm{vSh}$ we define the following function on $\mathrm{v}D(1,0)$:

$$f_{a,c}(x) = \begin{cases} c, & x \in \{a+31, a+32, a+21\},\\ -c, & x \in \{a+23, a+12, a+13\},\\ 0, & \text{else.} \end{cases}$$

The function $f_{03,1}$ is shown in Fig. 2.

Lemma 2. *Let f be an eigenfunction of the Shrikhande graph with eigenvalue* 2. *Then* $|S(f)| \geq 6$. *Moreover, if* $|S(f)| = 6$, *then* $f = f_{a,c}$ *fore some vertex* $a \in \mathrm{vSh}$ *and some* $c \in \mathbb{R}$.

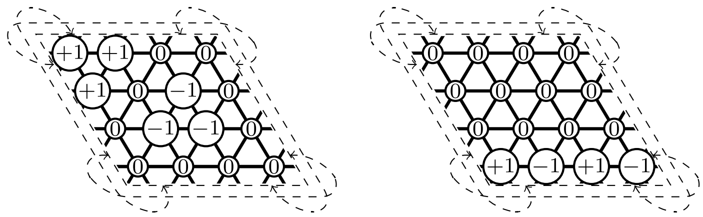

FIGURE 2. Eigenfunctions of the Shrikhande graph with the eigenvalues 2 and -2 and with minimum supports.

Proof. By Lemma 1(1) f has positive and negative values. Let for some vertex x the function f take the maximum value $w = f(x) > 0$, and for some vertex y the function f take the minimum value $u = f(y) < 0$. Then neighbourhood of x has 2 vertices with value w and 4 vertices with value 0 or at least 3 vertices with positive values. Analogously, neighbourhood of y has 2 vertices with value u and 4 vertices with value 0 or at least 3 vertices with negative values. Hence, $|S(f)| \geq 6$. Moreover, if $|S(f)| = 6$, then the support of f consists of three pairwise adjacent vertices with value w and three pairwise adjacent vertices with value $u = -w$, and there are no such pair of adjacent vertices z, t, that $f(z) = -f(t) = w$. Assume $|S(f)| = 6$, then for some $b \in \mathrm{v}Sh$ the set $\{b, b + 01, b + 02, b + 03\}$ has at least three vertices with value 0. It follows from Lemma 1(3), that all vertices from $\{b, b+01, b+02, b+03\}$ take the value 0. Analogously, for some vertex $d = b + s$, where $s \in \{00, 01, 02, 03\}$, the function f takes the value 0 on vertices d, $d + 10$, $d + 20$, $d + 30$. Hence, taking into account all the above, it is easy to see that $f = \pm f_{d,w}$. The fact that $f_{d,w}$ is an eigenfunction with the eigenvalue 2 is easy to check directly. $\square$

For $c \in \mathbb{R}$, $a \in \mathrm{vSh}$, $s \in \{01, 10, 11\}$ we define the following function on $\mathrm{v}D(1, 0)$:

$$u_{a,s,c}(x) = \begin{cases} c, & x \in \{a, a + 2s\} \\ -c, & x \in \{a + s, a + 3s\} \\ 0, & \text{else} \end{cases}$$

Lemma 3. *Let h be an eigenfunction of the Shrikhande graph corresponding to the eigenvalue -2. Then $|S(h)| \geq 4$. Moreover, if $|S(h)| = 4$, then $h = u_{a,s,c}$ for some $a \in \mathrm{vSh}$, $s \in \{01, 10, 11\}$ and $c \in \mathbb{R}$.*

Proof. Let for some vertex x the function h take a maximum in absolute magnitude value $w = f(x)$. Let without loss of generality $w > 0$. Then either there are at least 3 vertices with negative values or exactly 2 vertices with value $-w$ in the neighbourhood of x. In the first case $|S(h)| > 4$ (indeed, if $|S(h)| = 4$, then it is easy to see that there is a vertex y with value 0 such that there is a vertex with negative value and there are no vertex with positive value in the neighbourhood of y). In the second case, there is at least one more vertex with positive value, and if exactly one, then the vertices of support form the cycle $\{a, a + s, a + 2s, a + 3s\}$ for some $a \in \mathrm{vSh}$ and $s \in \{01, 10, 11\}$. $\square$

For functions $g : \mathrm{v}D(m, n) \to \mathbb{R}$ and $h : \mathrm{v}D(m', n') \to \mathbb{R}$ we define the product $f = gh : \mathrm{v}D(m + m', n + n') \to \mathbb{R}$ as $f(x, x', y, y') = g(x, y)h(x', y')$, where $x \in \mathrm{v}D(m, 0)$, $y \in \mathrm{v}D(0, n)$, $x' \in \mathrm{v}D(m', 0)$, $y' \in \mathrm{v}D(0, n')$. The next lemma follows from the definition.

Lemma 4. *Let* $g \in V_i^{m,n}$, $h \in V_j^{m',n'}$. *Then* $f = gh \in V_{i+j}^{m+m',n+n'}$.

The following lemma is well known in the different formulations (see for example [9](1.4.6) or [10] (Theorems 2.23, 2.24)), but for completeness we will prove it directly.

Lemma 5. *Let* G *and* H *be graphs with the eigenvalues* $\lambda_1, \ldots, \lambda_m$ *and* $\mu_1, \ldots, \mu_n$ *respectively. Let* $e_{i1}, \ldots, e_{is(i)}$ *be a basis of the eigensubspace* $V_i(G)$ *of dimension* $s(i)$ *corresponding to the eigenvalue* λ_i*, and let* $y_{j1}, \ldots, y_{jt(j)}$ *be a basis of the eigensubspace* $V_j(H)$ *of dimension* $t(j)$ *corresponding to the eigenvalue* μ_j.

Then graph $F = G\Box H$ *has the following eigenvalues:* $\{\theta_r : \theta_r = \lambda_i + \mu_j$ *for some* $i = 1, \ldots, m; j = 1, \ldots, n\}$*, and the set of functions* $\{e_{ip}y_{ju}$: *for all* i, j *such that* $\lambda_i + \mu_j = \theta_r, p = 1, \ldots, s(i), u = 1, \ldots, t(j)\}$ *form a basis of the eigensubspace* $V_r(F)$*. The dimension of* $V_r(F)$ *is equal to* $\sum\limits_{\substack{i,j \\ \lambda_i+\mu_j=\theta_r}} s(i)t(j)$

Proof. Let prove that the set of the functions

$$\{e_{ij}y_{kl} : i = 1, \ldots, m, j = 1, \ldots s(i), k = 1, \ldots, n, l = 1, \ldots, t(k)\}$$

form a basis of the real-valued functions on F. Consider a linear combination of functions from this set. Let

$$\sum_{i,j} e_{ij} \sum_{k,l} \lambda_{ijkl} \cdot y_{kl} \equiv 0.$$

Define the function f_{ij} on H in the following way: $f_{ij} = \sum_{k,l} \lambda_{ijkl} \cdot y_{kl}$. Since the set of the functions e_{ij} forms a basis on G, for any i, j we have $f_{ij} \equiv 0$. Since the functions from the set $\{y_{kl}\}$ are linearly independent, we have $\lambda_{ijkl} = 0$ for all i, j, k, l. Hence all functions $e_{ij}y_{kl}$ such that $\lambda_i + \mu_j = \theta_r$ form a basis of $V_r(F)$; whence the lemma follows. $\square$

Corollary 1. *The dimension of the eigensubspace* $V_i^{m,n}$ *equals* $C_{2m+n}^i \cdot 3^i$.

Proof. The Shrikhande graph has 3 eigenvalues -2, 2, 6 with multiplicities 9, 6, 1 respectively. The Hamming graph $H(2, 4)$ has the same eigenvalues with the same multiplicities. Then by Lemma 5 the graph $D(m, n)$ has the same eigenvalues with the same multiplicities as the Hamming graph $H(2m+n, 4)$. In the Hamming graph $H(N, q)$ the eigenvalue $\lambda_i = N(q-1) - iq$ has multiplicity $C_N^i(q-1)^i$ ([11], 9.2), whence the statement of the corollary follows. $\square$

Corollary 2. *Let* f *be an eigenfunction of* $D(m, n)$ *with the largest eigenvalue* $\lambda_0 = 6m + 3n$*. Then* $f \equiv c$ *for some* $c \in \mathbb{R}$.

Proof. It follows from the fact that dimension of $V_0^{m,n}$ equals to 1 and that $f \equiv c$ is eigenfunction with eigenvalue λ_0. $\square$

Corollary 3. *Let* f *be an eigenfunction on* $D(m, n)$ *corresponding to the minimum eigenvalue* $\lambda_{2m+n} = -2m - n$*. Then for any* $a_1, \ldots, a_k \in \mathrm{vSh}$, $b_1, \ldots, b_l \in \mathrm{v}K_4$ *and any set of coordinates* $(i_1, \ldots, i_k; j_1, \ldots, j_l)$ *the restriction*

$$f|_{x_{i_1}=a_1,\ldots,x_{i_k}=a_k;y_{j_1}=b_1,\ldots,y_{j_l}=b_l}$$

is an eigenfunction of the graph $D(m-k, n-l)$ *with the minimum eigenvalue* $-2(m-k) - (n-l)$.

Proof. It follows from Lemma 5 that there is a basis of $V^{m,n}_{2m+n}$ such that any function e from basis can be represented as $e_1 \dots e_m y_1 \dots y_n$, where $e_1, \dots, e_m$ are eigenfunctions of Sh with the minimum eigenvalue -2 and $y_1, \dots, y_n$ are eigenfunctions of K_4 with the minimum eigenvalue -1. Then, since the restriction

$$e|_{x_{i_1}=a_1,\dots,x_{i_k}=a_k;y_{j_1}=b_1,\dots,y_{j_l}=b_l}$$

is an eigenfunction of $D(m-k, n-l)$ with the minimum eigenvalue, the statement of the corollary is true. □

Denote by $I^{m,n}$ the function on $D(m,n)$ that is identically equal to 1.

Corollary 4. *Let f be an eigenfunction of the Doob graph $D(m,n)$ corresponding to the eigenvalue $\lambda_1 = 6m+3n-4$. Then f can be represented as $f = hI^{1,0} + I^{m-1,n}g$, where $h \in V_1^{m-1,n}$, $g \in V_1^{1,0}$.*

Proof. Let $e_1, \dots, e_k$ be a basis of eigensubspace $V_1^{m-1,n}$, where $k = 6(m-1)+3n$, and let $y_1, \dots, y_6$ be a basis of eigensubspace $V_1^{1,0}$. Then from Lemma 5 the set of functions $e_1 I^{1,0}, \dots, e_k I^{1,0}, I^{m-1,n}y_1, \dots, I^{m-1,n}y_6$ forms a basis of eigensubspace $V_1^{m,n}$. Hence, for any function f from $V_1^{m,n}$ there is a system of numbers μ_i, $i = 1, \dots, 6m+3n$, such that $f = \mu_1 e_1 I^{1,0} + \dots + \mu_k e_k I^{1,0} + \mu_{k+1} I^{m-1,n} y_1 + \dots + \mu_{k+6} I^{m-1,n} y_6$. Then $f = hI^{1,0} + I^{m-1,n}g$, where $h = \mu_1 e_1 + \dots + \mu_k e_k$, $g = \mu_{k+1} y_1 + \dots + \mu_{k+6} y_6$. □

Lemma 6. *Let $F = G \square H$ be the direct product of graphs G and H. Let f be an eigenfunction of F corresponding to an eigenvalue λ. Assume that for any vertex $y \in \mathrm{v}H$ the function $g_y(z) = f(z,y)$ is an eigenfunction corresponding to an eigenvalue λ_1 or $g_y \equiv 0$, and for any vertex $x \in \mathrm{v}G$ the function $h_x(z) = f(x,z)$ is an eigenfunction corresponding to an eigenvalue λ_2 or $h_x \equiv 0$, where $\lambda_1 + \lambda_2 = \lambda$. Let r is a size of minimum possible support of eigenfunction of G corresponding to an eigenvalue λ_1, and s is a size of minimum possible support of eigenfunction of H corresponding to an eigenvalue λ_2.*

Then $|S(f)| \geq rs$. Moreover, if $|S(f)| = rs$, then $f = gh$, where g is eigenfunction of G corresponding to eigenvalue λ_1, h is eigenfunction of H corresponding to eigenvalue λ_2, and $|S(g)| = r$, $|S(h)| = s$.

Proof. There are some vertices $x \in \mathrm{v}G$, $y \in \mathrm{v}H$ such that $f(x,y) \neq 0$. Then $|S(g_y)| \geq r$. Since $|S(h_a)| \geq s$ for any vertex a of $S(g_y)$, $|S(f)| \geq rs$, and the equality is attained, if $|S(g_y)| = r$, $|S(h_x)| = s$, $S(h_a) = S(h_x)$ for any a from $S(g_y)$, and for any vertex b from $\mathrm{v}G \backslash S(g_y)$ the set $S(h_b)$ is empty. Moreover, $h_a(z) = c_a h_x(z)$ for any vertex $a \in S(g_y)$ and any vertex $z \in \mathrm{v}H$, where $c_a = g_y(a)/f(x,y)$. Indeed, consider the function $p(z) = c_a \cdot h_x(z) - h_a(z)$. Since $c_a \cdot h_x(y) = h_a(y)$, $|S(p)| < s$ and hence, $p \equiv 0$. Then $f = gh$, where $g(z) = g_y(z)/f(x,y)$, $h(z') = h_x(z')$, $z \in \mathrm{v}G$, $z' \in \mathrm{v}H$. □

Lemma 7. *Let a function $f : \mathrm{v}D(1,0) \to \mathbb{R}$ can be represented as $f = g + h$, where $g \in V_1^{1,0}$, $h \in V_0^{1,0}$, and functions g, h are not identically equal to 0. Then $|S(f)| \geq 7$.*

Proof. Let f satisfy the hypothesis of lemma, and $|S(f)| \leq 7$. Hence, $|f^{-1}(0)| \geq 9$. By condition we have that $h(a) = -c$ for any vertex $a \in \mathrm{vSh}$ and for some $c \in \mathbb{R}$, $c \neq 0$. Then $|g^{-1}(c)| = |f^{-1}(0)| \geq 9$. Denote by $C(a,s)$ the set $\{a, a+s, a+2s, a+3s\}$, $a \in \mathrm{vSh}$, $s \in \{01, 10, 11\}$. Then for any $s \in \{01, 10, 11\}$ there is $C(a_s, s)$

for some vertex a_s that includes at least 3 vertices from the set $g^{-1}(c)$. It follows from Lemma 1(3) that all vertices of such cycle belong to the set $g^{-1}(c)$. Without loss of generality, we can assume that vertices $00, 01, 02, 03, 10, 20, 30$ belong to $g^{-1}(c)$. Then at least one of the vertices 21, 32, 31, 13, 12, 23 belong to the set $g^{-1}(c)$ (otherwise, the vertices $11, 22, 33$ belong to $g^{-1}(c)$, and for 00 we have a contradiction with the fact that $g \in V_1^{1,0}$).

Since g is an eigenfunction corresponding to the eigenvalue 2, using the Lemma 1, in each case we can uniquely reconstruct the values of g. All obtained functions are shown in Fig. 3 ($c = 1$). For each of them, the support of corresponding function f is equal to 7, which proves the lemma. □

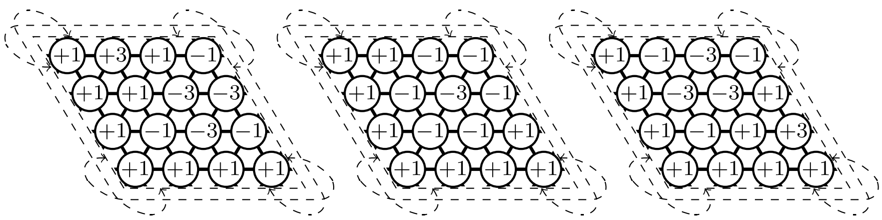

FIGURE 3. The cases from the proof of Lemma 7.

For $c \in \mathbb{R}$ and $k, l \in \{0, 1, 2, 3\}$, we define the following function on $\mathrm{v}D(0, 2)$:

$$h_{k,l,c}(x) = \begin{cases} c, & x \in \{(k+1, l), (k+2, l), (k+3, l)\} \\ -c, & x \in \{(k, l+1), (k, l+2), (k, l+3)\} \\ 0, & \text{else} \end{cases}$$

Lemma 8. [3] *Let h be an eigenfunction of the Hamming graph $H(n, 4)$ corresponding to the eigenvalue $\lambda_1 = 3n - 4$, with the minimum possible support. Then $|S(h)| = 6 \cdot 4^{n-2}$, and the function h can be represented as $h = h_1 \dots h_{n-1}$, where $h_i = h_{k,l,c}$ for some $k, l \in \{0, 1, 2, 3\}$, some $i = 1, \dots, n-1$ and some $c \in \mathbb{R}$, and $h_j = I^{0,1}$, $j \neq i$, $j = 1, \dots, n-1$.*

For $c \in \mathbb{R}$ and $k, l \in \{0, 1, 2, 3\}$, $k \neq l$, we define the following function on $\mathrm{v}D(0, 1)$:

$$r_{k,l,c}(x) = \begin{cases} c, & x = k \\ -c, & x = l \\ 0, & \text{else} \end{cases}$$

It is obvious that if the function f on $D(0, 1)$ is an eigenfunction with minimum eigenvalue and minimum possible support, then $f = r_{k,l,c}$ for some $c \in \mathbb{R}$ and $k, l \in \{0, 1, 2, 3\}$, $k \neq l$.

3. Main results

Now we can formulate and prove the main theorems.

Theorem 1. *Let f be an eigenfunction of the graph $D(m, n)$ corresponding to the eigenvalue λ_1. Then $|S(f)| \geq 6 \cdot 4^{2m+n-2}$. Moreover, if $|S(f)| = 6 \cdot 4^{2m+n-2}$, then one of the following statements holds:*

(1) $f = g_1 \dots g_m I^{0,n}$, *where* $g_i = f_{a,c}$ *for some* $i \in \{1, \dots, m\}$, $a \in \mathrm{v}\mathrm{Sh}$, $c \in \mathbb{R}$, *and* $g_j = I^{1,0}$ *for* $j \neq i$, $j = 1, \dots, m$

(2) $f = I^{m,0} h_1 \dots h_{n-1}$, *where* $h_i = h_{k,l,c}$ *for some* $i \in \{1, \dots, n-1\}$, $k, l \in \{0, 1, 2, 3\}$, $c \in \mathbb{R}$, $h_j = I^{0,1}$ *for* $j \neq i$, $j = 1, \dots, n-1$, .

Proof. We will prove this lemma by induction on m. For $m = 0$ the statement follows from Lemma 8. Let f be an eigenfunction with eigenvalue λ_1 and with minimum possible support. By Corollary 4 the function f can be represented as $f = hI^{1,0} + I^{m-1,n}g$, where $h \in V_1^{m-1,n}$, $g \in V_1^{1,0}$. If g is identically equal to 0, then $|S(f)| = 16 \cdot |S(h)|$, and by induction hypothesis $|S(f)| \geq 6 \cdot 4^{2m+n-2}$, and for h, and, hence for f, the statement is true. Assume that g is not identically equal to 0. Let us prove that h is identically equal to 0. Let x be an arbitrary vertex of $D(m-1, n)$. Denote by f_x the function such that $f_x(y) = f(x, y)$, $x \in \mathrm{v}D(m-1, n)$, $y \in \mathrm{v}\mathrm{Sh}$. Denote $v_x = h(x)I^{1,0}$. Then

$$|S(f)| = \sum_{x \in \mathrm{v}D(m-1,n)} |S(f_x)| = \sum_{x \in \mathrm{v}D(m-1,n)} |S(v_x + g)|.$$

By Lemma 7 if $h(x) \neq 0$, then $|S(v_x + g)| \geq 7$. If $h(x) = 0$, then $|S(v_x + g)| = |S(g)| \geq 6$. Next, if $h(x) \neq 0$ for some vertex x of $D(m-1, n)$, then $|S(f)| > 6 \cdot 4^{2m+n-2}$, and we have a contradiction with the size of the minimum support. Hence, h is identically equal to 0, and the statement follows from Lemma 2. □

Theorem 2. *Let f be an eigenfunction of $D(m, n)$ with the minimum eigenvalue $\lambda_{2m+n} = -2m - n$. Then $|S(f)| \geq 2^{2m+n}$. Moreover, if $|S(f)| = 2^{2m+n}$, then $f = c \cdot g_1 \dots g_m h_1 \dots h_n$, where $g_i = u_{a_i, s_i, 1}$, $h_j = r_{k_j, l_j, 1}$, $a_i \in \mathrm{v}\mathrm{Sh}$, $s_i \in \{01, 10, 11\}$, $k_j, l_j \in \{0, 1, 2, 3\}$, $k_j \neq l_j$, $i = 1, \dots, m$, $j = 1, \dots, n$, $c \in \mathbb{R}$.*

Proof. The statement follows from Lemma 3, Lemma 6 and Corollary 3. □

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