ON NEW EXAMPLES OF HYPOCRITICAL GROUPS

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Abstract. A group $G$ is called hypocritical if whenever $G$ lies in a locally finite variety generated by a section closed class of groups $X$, then $G$ belongs to $X$. We prove that some coprime extensions of a $p$-group are hypocritical. The first example is given when such a $p$-group is nonabelian.

Keywords: locally finite varieties, finite groups, extraspecial $p$-groups.

1. Introduction

Recall that a group $G$ is a section of a group $F$ if there exists a subgroup $H$ of $F$ such that $G$ is a homomorphic image of $H$. A class of groups $X$ is said to be section closed if sections of groups from $X$ also belong to it.

L.F. Harris [1] introduced the following definition: a group $G$ is called hypocritical if whenever $G$ lies in a locally finite variety generated by a section closed class of groups $X$, then $G$ lies in $X$. It is easy to see that a hypocritical group is in fact critical, that is, does not lie in a variety generated by its proper sections.

A group with a unique minimal normal subgroup is called monolithic and the minimal normal subgroup is the monolith of the group. By the Remak theorem, every critical and thus hypocritical group must be monolithic. L.F. Harris [1] mentions that the converse is true when the monolith is nonabelian, but gives no reference. A restricted version of this result was established in [2, Theorem 4] and we prove this fact in general (see Proposition 1).

In this work we consider finite groups which are coprime extensions of finite $p$-groups. By the well-known Schur–Zassenhaus theorem, these extensions are split, so we actually work with semidirect products of the form $D = B \rtimes A$, where $B$ is a $p$-group and $A$ is a $p'$-group. Suppose that $B$ is abelian. Then for $D$ to be
hypocritical (or at least critical), the group $A$ must act indecomposably on $B$, which is possible only if $B$ is homocyclic (see [3, Theorem 5.2.2]). In this situation, by [1, Theorem 1], if the group $D$ is hypocritical and $B$ is not cyclic or elementary abelian, then the exponent of $B$ is bounded in terms of $|A|$ and $p$.

Consider the excluded possibilities for $B$. In the case when $B$ is elementary abelian, it is easy to establish that $D$ is hypocritical (see Proposition 2). Let now $B$ be cyclic. In the same paper, L.F. Harris mentioned that the hypocriticality of $D$ in this case was shown by R.M. Bryant and L.G. Kovács before, but we found no traces of the original proof. Moreover, the situation when $B$ is cyclic is closely connected with the following problem due to L.G. Kovács (Problem 8.23 in the Kourovka notebook [4]): if the dihedral group $D$ of order 18 is a section of a direct product $X \times Y$, must at least one of $X$ and $Y$ have a section isomorphic to $D$? In Lemma 1 we show that for finite groups $X$ and $Y$ that question is equivalent to checking hypocriticality of $D$. Note that some partial results on Kovács’ problem were obtained in [5]. Here we provide the proof in the cyclic case in general.

Theorem 1. Let $p$ be a prime, and let $D = B \rtimes A$ be a semidirect product of its subgroups $B$ and $A$, where $B$ is a finite cyclic $p$-group and $A$ is a $p'$-group acting faithfully on $B$. Then $D$ is hypocritical.

Theorem 1 implies that the main difficulty in Kovács’ problem appears when groups $X$ and $Y$ are not locally finite.

Now we turn to the case when $B$ is nonabelian. The smallest example occurs when $D$ is equal to a semidirect product of the quaternion group $Q_8$ and the cyclic group $C_3$, that is, $D \cong SL_2(3)$. Recall that a finite $p$-group $E$ is called extraspecial, if its center $Z(E)$, Frattini subgroup $\Phi(E)$ and derived subgroup $E'$ coincide and have order equal to $p$. Notice that $Q_8$ is a 2-generated extraspecial group and it turns out that we can prove our result in the general case of a 2-generated extraspecial group.

Theorem 2. Let $p$ be a prime, and let $D = E \rtimes A$ be a semidirect product of its subgroups $E$ and $A$, where $E$ is a 2-generated extraspecial $p$-group and $A$ is a $p'$-group acting faithfully and irreducibly on $E/\Phi(E)$. Then $D$ is hypocritical.

As we will see, the proof of Theorem 2 does not depend on $E$ being 2-generated until the last step, which gives the hope of proving the similar statement for a general extraspecial group. Having Harris’ results in mind, one can conjecture that a necessary condition for a group $D$ to be hypocritical is that $|E/\Phi(E)|$ is bounded by some function of $p$ and $|A|$, though no examples of such behavior were found so far.

The structure of the rest of this paper is as follows. In Section 2 we reduce the problem to studying sections of direct products and obtain some general results in that setting. As a simple corollary we prove the proposition about groups with a nonabelian monolith and establish some additional constraints on groups involved in the case of finite groups. Section 3 provides the proof of Theorem 1, while Section 4 section deals with Theorem 2.

2. Reduction lemmata

Lemma 1. A finite group $D$ is hypocritical if and only if whenever it is a section of a direct product $X \times Y$ of finite groups $X$ and $Y$, then $D$ is a section of either $X$ or $Y$. 

Suppose $D$ is hypocritical. Obviously finite groups $X$ and $Y$ generate a locally finite variety and $D$ belongs to it. By the definition of hypocriticality we get that $D$ is a section of either $X$ or $Y$.

Let $\mathfrak{X}$ be a locally finite variety generated by a section closed class of groups $\mathfrak{X}$ and suppose that $D$ lies in $\mathfrak{X}$. By [6, Lemma 4.3], the group $D$ is a section of a direct product of a finite number of groups from $\mathfrak{X}$, i.e. $D$ is a section of $X_1 \times \cdots \times X_n$, $n \geq 1$, and $X_i$, $i = 1 \ldots n$, lie in $\mathfrak{X}$. Note that the definition of a section implies the existence of a subgroup $G$ of $X_1 \times \cdots \times X_n$ such that $D$ is a quotient of $G$.

Let $H$ be a normal subgroup of $G$ with $D \simeq G/H$. For each element of $D$ we can pick a corresponding preimage in $G$. Let $M$ be a subgroup of $G$ generated by those preimages. Obviously $M$ is finitely generated and since it lies in a locally finite group, $M$ is finite. By definition, $HM = G$, hence $D \simeq G/H \simeq M/(M \cap H)$. Replacing $G$ by $M$, we may assume $G$ to be finite.

Clearly we can replace groups $X_i$ by projections of $G$ on corresponding factors of this direct product, and as $\mathfrak{X}$ is section closed, we can assume groups $X_i$ to be finite. Now, inducting on $n$ and applying the hypothesis we derive that $D$ is a section of one of $X_i$, thus it lies in $\mathfrak{X}$ as needed.

Let $G$ be a subgroup of a direct product $F = X \times Y$ of groups $X$ and $Y$. Then let $G_X$ and $G_Y$ denote the projections of $G$ on $X$ and $Y$ respectively. We think of $G_X$ and $G_Y$ as of subgroups of $F$.

**Lemma 2.** Let $G$ be a subgroup of $F = X \times Y$ and let $X = G_X$ and $Y = G_Y$. If $H$ is normal in $G$, then $H_X$ and $H_Y$ are normal in $F$.

**Proof.** A projection on $X$ is a homomorphism from $G$ into $X$. Under our conditions this map is surjective, thus $H_X$ is a normal subgroup of $X$. The group $Y$ centralizes $X$ in $F$, so $H_X$ is actually a normal subgroup of $F$. The proof is similar for $H_Y$. \qed

**Lemma 3.** Let $F = X \times Y$ be a direct product of groups, and let $G$ be a subgroup of $F$. Suppose also that $H$ is a normal subgroup of $G$. Then there exist groups $\hat{X}$, $\hat{Y}$, $\hat{G}$ and $\hat{H}$ such that $\hat{G}$, $\hat{X}$, and $\hat{Y}$ are sections of $G$, $X$, and $Y$ respectively, $H$ is normal in $\hat{G}$ and the following hold:

(i) $\hat{X} = \hat{G}_X$ and $\hat{Y} = \hat{G}_Y$;

(ii) $\hat{G}/\hat{H}$ is isomorphic to $G/H$;

(iii) if $K_1 = \hat{G} \cap \hat{G}_X$ and $K_2 = \hat{G} \cap \hat{G}_Y$, then $K_1 \cap \hat{H} = K_2 \cap \hat{H} = 1$.

**Proof.** In the following we are going to repeatedly replace groups $G$, $X$ and $Y$ by their sections, preserving the same symbols for brevity. The groups $\hat{G}$, $\hat{X}$, and $\hat{Y}$ can be thought of as the final “product” of this process.

It is easy to see that $G$ is a subgroup of $G_X \times G_Y$, thus we can always assume that $X = G_X$ and $Y = G_Y$. By Lemma 2 the groups $H_X$ and $H_Y$ are normal in $F$. Then $L_1 = H \cap H_X$ and $L_2 = H \cap H_Y$ are normal in $G$, but obviously $(L_1)_X = L_1$ and $(L_2)_Y = L_2$, so $L_i$, $i = 1,2$, are actually normal in $F$ by another application of Lemma 2.

Consider the quotient group $F/(L_1 \times L_2)$:

$$F/(L_1 \times L_2) \simeq G_X/L_1 \times G_Y/L_2$$ and $$(G/(L_1 \times L_2))/(H/(L_1 \times L_2)) \simeq G/H.$$ Therefore replacing our groups appropriately, we can assume that $L_1 = L_2 = 1$. As a consequence, if we take $K_1 = G \cap G_X$ then $H \cap K_1 = H \cap H_X = 1$. The similar equality is true for $K_2 = G \cap G_Y$. \qed
Lemma 4. Let $F = X \times Y$ be a direct product of groups, and let $D$ be a monolithic subgroup of $F$. Then $D$ is a section of $X$ or $Y$.

Proof. Denote by $K_1$ and $K_2$ kernels of projection maps from $D$ into $Y$ and $X$ respectively. Subgroups $K_i \cap D$, $i = 1, 2$ are normal in $D$ and intersect trivially. Since $D$ is monolithic, it follows that $K_1 \cap D$ or $K_2 \cap D$ is trivial, which gives us an embedding of $D$ into $Y$ or $X$, as wanted.

Now we are ready to resolve the case of a group with a nonabelian monolith.

Proposition 1. Let $D$ be a finite monolithic group with a nonabelian monolith. Then $D$ is hypocritical.

Proof. By Lemma 1 it is sufficient to prove that for all finite groups $X$ and $Y$, if $D$ is a section of $F = X \times Y$ then $D$ is a section of either $X$ or $Y$. Suppose that $D$ is a section of neither $X$ nor $Y$. Let $G$ be a subgroup of $F$ and let $H$ be a normal subgroup of $G$, such that $D \simeq G/H$. Denote by $K_1$ and $K_2$ kernels of projections of $G$ onto $Y$ and $X$ respectively and notice that $K_1 = G \cap G_X$ and $K_2 = G \cap G_Y$. By Lemma 3 we can choose groups $X$ and $Y$ in such a way that $K_1 \cap H = K_2 \cap H = 1$. Note also that $K_1 \cap K_2 = 1$.

Now set $L = K_1H \cap K_2H$. Obviously $H \leq L$. The groups $K_i$, $i = 1, 2$, and $H$ are normal in $G$, and consequently $[K_1, K_2] = [K_1, H] = [K_2, H] = 1$. Therefore,

$$[L, L] \leq [K_1H, K_2H] = [K_1, K_2][K_1, H][H, K_2][H, H] = [H, H] \leq H.$$ 

It follows that $L/H$ is an abelian normal subgroup of $G/H \simeq D$ and thus it cannot contain the monolith of $D$, so in fact $H = L$.

As $H = K_1H \cap K_2H$, we easily obtain an embedding of $D \simeq G/H$ into the direct product $G/K_1H \times G/K_2H$. The factors of this direct product do not have $D$ as a section (because otherwise $X$ or $Y$ would), so we can now assume that $D$ is a subgroup of $F$. Lemma 4 gives us the final contradiction.

If a finite monolithic group has an abelian monolith, it must be an elementary abelian $p$-group for some prime $p$. In this situation and when groups involved are finite, we obtain a refinement of Lemma 3.

Lemma 5. In the notation of Lemma 3, suppose that groups $X$ and $Y$ are finite, and the group $D \simeq G/H$ has an abelian monolith $K$. Then we can assume that $H$ is a $p$-group, where $p$ is a prime dividing $|K|$.

Proof. As in the proof of Lemma 3, we will drop “hats” from our symbols.

Pick the group $G$ to be of minimal order such that $G$ satisfies the conclusion of Lemma 3. Let $M$ be a maximal subgroup of $G$ and suppose that $H$ does not lie in $M$. Then $HM = G$ and $M/(M \cap H) \simeq G/H \simeq D$, so we can replace $G$ by $M$. But $M$ has a smaller order than $G$, which contradicts the minimality condition, thus actually $M$ includes $H$.

The Frattini subgroup is an intersection of maximal subgroups of a group, and by the preceding paragraph $H$ lies in all maximal subgroups of $G$. Therefore, $H$ lies in the Frattini subgroup of $G$ which is nilpotent by [3, Theorem 6.1.6], so $H$ is nilpotent too.

As $H$ is nilpotent, it contains a Hall $p'$-subgroup $S$. It is characteristic in $H$ so it is normal in $G$. Set $L = K_1S \cap K_2S$. Then $L$ is a normal subgroup of $G$ and obviously $S \leq L$. Let $R$ be a Sylow $p'$-subgroup of $L$. We have $R \leq K_1S$. As $S$ is a
$p'$-group and $K_1$ is a normal subgroup of $K_1 S$, it contains all Sylow $p$-subgroups of $K_1 S$ and $R$ in particular. In other words, $R \leq K_1$ and by symmetry $R \leq K_2$. But $K_1 \cap K_2 = 1$ because $G_X \cap G_Y = 1$. Thus, $R = 1$ and $L$ is a $p'$-group.

Now we have that $|LH : H| = |L : L \cap H|$ is a $p'$-number, so $LH/H$ is a normal $p'$-subgroup of $G/H \simeq D$. The monolith of $D$ is a $p'$-group and thus it intersects $LH/H$ trivially, which implies that $LH/H = 1$. Consequently, $LH = H$ and $L \leq H$.

Notice that $D$ is a section of $\bar{F} = G/K_1 S \times G/K_2 S$. Indeed, $G/L$ is a subgroup of $\bar{F}$ and $D \simeq G/H \simeq (G/L)/(H/L)$. Suppose $L \neq 1$. Applying Lemma 3 to the groups $G/L, H/L, G/K_1 S$, and $G/K_2 S$, we get a contradiction with the minimality of $G$. Then $S \leq L$ must be trivial. It follows that $H$ is a $p$-group.

**Proposition 2.** Let $D = B \rtimes A$ be a semidirect product of a finite elementary abelian $p'$-group $B$ and a $p'$-group $A$. Suppose also that $A$ acts faithfully and irreducibly on $B$. Then $D$ is hypocritical.

**Proof.** By Lemma 1 we can consider the following situation: let $X$ and $Y$ be finite groups, and let $G$ be a subgroup of $X \times Y$. Denote by $H$ the normal subgroup of $G$, such that $D \simeq G/H$. By Lemma 5 we can assume that $H$ is a $p$-group. Denote by $P$ a Sylow $p$-subgroup of $G$. Notice that $H \leq P$ and $P$ is normal in $G$.

Since the action of $A$ on $B$ is faithful, $B$ is a minimal normal subgroup of $D$. Now, let $N$ be a normal subgroup of $D$, such that $N \cap B = 1$. Then $N$ centralizes $B$, and by the conjugacy part of the Schur–Zassenhaus theorem, we have $N \leq A$. The action of $A$ on $B$ is faithful, hence $N = 1$, so $D$ is in fact monolithic.

The proof goes by contradiction. Let $K_1$ be a kernel of the projection from $G$ into $Y$. If $K_1$ is trivial, then $G$ (and thus $D$) is a section of $Y$, so we can assume $K_1 > 1$. Then $K_1 H/H$ is a nontrivial normal subgroup of $G/H$, and therefore $P \leq K_1 H$. We have

$$P \leq P \cap K_1 H = (P \cap K_1) H \leq P,$$

so $(P \cap K_1) H = P$. But since $P/H$ is elementary abelian, $H$ lies in $\Phi(P)$, thus $P \cap K_1 = P$. Then $H \leq P \leq K_1$, and we derive $H = 1$ which gives us the contradiction with Lemma 4. \hfill \Box

### 3. Proof of Theorem 1

Recall that $D = B \rtimes A$ is a semidirect product of its subgroups, where $B$ is a cyclic $p$-group of order $p^n$ and $A$ is a group of order coprime to $p$. In order to prove hypocriticality of $D$, we first need one simple lemma.

**Lemma 6.** The group $D$ is monolithic and its monolith has order $p$.

**Proof.** Let $K$ be the unique subgroup of order $p$ in $B$. Obviously $K$ is normal in $D$. We prove that $K$ is the monolith of $D$.

Let $N$ be a nontrivial normal subgroup of $D$. Suppose that $N$ is a $p'$-group. Clearly $N \cap B = 1$ and therefore $N$ centralizes $B$. Since $AN$ is a $p'$-group, $|AN|$ divides $|D : B| = |A|$, hence $N \leq A$. So $A$ has nonidentity elements centralizing $B$ contrary to our hypothesis. Thus, $|N|$ is divisible by $p$.

Since the group $B$ is a normal Sylow $p$-subgroup of $D$, the groups $B$ and $N$ intersect nontrivially. Then $K \leq B \cap N \leq N$, which proves the claim. \hfill \Box

We need the following general lemma.
Lemma 7. Let $P$ be a finite $p$-group and let $Q$ be a $p'$-group acting on $P$ by automorphisms. Suppose that $H$ is a normal $Q$-invariant subgroup of $P$, such that the quotient $P/H$ is cyclic. Then there exists a cyclic $Q$-invariant subgroup $C$ of $P$, such that $HC = P$.

Proof. Note that the case $P = H$ is trivial, thus from now on we will suppose that $H < P$. Choose the smallest subgroup $C$ of $P$, such that $C$ is $Q$-invariant, and $HC = P$. We prove that $C$ is cyclic.

Notice that $H \cap C$ is a normal subgroup of $C$ and $C/(H \cap C) \simeq P/H$ is cyclic. Let $L = H \cap C$. Since $H < P$, we have $L < C$, and therefore $\Phi(C)L < C$. The group $C/(\Phi(C)L)$ is cyclic and elementary abelian, hence $|C/(\Phi(C)L)| = p$. Then the group $(\Phi(C)L)/\Phi(C)$ is a $Q$-invariant subgroup of index $p$ in $C/\Phi(C)$, and by Maschke’s theorem, there exists a $Q$-invariant group $T$, such that

$$T/\Phi(C) \times (\Phi(C)L)/\Phi(C) = C/\Phi(C),$$

and $T/\Phi(C)$ has order $p$. We derive that $T\Phi(C)L = C$, so $TL = C$. Then

$$TH = TLH = CH = P,$$

where the first equality holds because $L \leq H$. Since $T \leq C$, and $C$ was chosen to be minimal satisfying the required properties, we obtain $T = C$. But $|T/\Phi(C)| = p$, so $|C/\Phi(C)| = p$, which proves that $C$ is cyclic. \qed

The proof of the theorem goes by contradiction. By Lemma 1 we can assume that $D$ is a section of a direct product of finite groups $X$ and $Y$ and as usual, denote by $G$ a subgroup of $X \times Y$ and by $H$ a normal subgroup of $G$ such that $D \simeq G/H$. Also, let $K_1$ and $K_2$ be kernels of projections of $G$ into $Y$ and $X$ respectively. By Lemma 6 and Lemma 5, we can assume $H$ to be a $p$-group. Notice also that $K_1 \cap K_2 = 1$.

Let $P$ be a Sylow $p$-subgroup of $G$. Obviously $H$ lies in $P$, so $P$ is normal in $G$. We have $|G : P| = |D : B| = |A|$, thus $|P|$ is coprime to $|G : P|$, and by the Schur–Zassenhaus theorem, there exists a $p'$-subgroup $Q$ of $G$, such that $PQ = G$.

Lemma 7 implies the existence of a cyclic $Q$-invariant subgroup $C$ of $P$, such that $HC = P$. Let $R = CQ$. Then we have

$$D \simeq G/H = PQ/H = HCQ/H \simeq R/(R \cap H).$$

It follows that $D$ is a quotient of $R$, and the action of $Q$ on $C$ is faithful and nontrivial, since the action of $A$ on $B$ is so.

The group $R$ is a semidirect product of a cyclic $p$-group $C$ and a cyclic $p'$-group $Q$, thus Lemma 6 applies. Then $R$ is monolithic, and by Lemma 4, the group $R$ can be embedded in $X$ or $Y$, which in its turn shows that $D$ is a section of one of these groups. That contradicts the choice of $X$ and $Y$, proving the theorem.

4. Proof of the Theorem 2

Recall that $D = E \rtimes A$ is a semidirect product of its subgroups, where $E$ is an extraspecial $2$-generated $p$-group, and $A$ is a group of order coprime to $p$, acting faithfully and irreducibly on $E/\Phi(E)$.

Lemma 8. Let $N$ be a normal subgroup of $D$. Then either $E \leq N$ or $N \leq E$. As a consequence, $D$ is monolithic.
Let \( K \subseteq G \) of \( p \)-index is a Hall \( p' \)-subgroup of \( G \) and finally the argument applies and we derive: the theorem, all complements to \( E \) is nontrivial, and hence contains \( Z(G) \). Notice that if \( N \) intersected \( E \) trivially, then \( N \) would lie in \( A \) and therefore \( N = 1 \). Then \( D = N_D(N) \leq A\Phi(E) \), which gives us a contradiction. Thus, \( N \) intersects \( E \) nontrivially and hence contains \( Z(E) = \Phi(E) \).

We have \(|N : N \cap A| = |NA : A|\), and since \( NA \) is a subgroup of \( G \), the latter index is a \( p' \)-number. Hence, \( Q \) is a Hall \( p' \)-subgroup of \( N \). Obviously \( E \cap N \) is a Sylow \( p' \)-subgroup of \( N \), therefore \( N = (E \cap N)Q \), and by the Schur–Zassenhaus theorem, all complements to \( E \cap N \) in \( N \) are conjugate. Then the Frattini-type argument applies and we derive:

\[
D = NN_D(Q) \leq N\Phi(E)A = NA,
\]

where the last equality holds because \( \Phi(E) \leq N \). Now \( D = NA \) and as \( A \) is a \( p' \)-group, we have \( E \leq N \) as claimed.

Let \( N \) be a minimal normal subgroup of \( D \). Obviously a minimal normal subgroup cannot contain \( E \), since otherwise \( \Phi(E) \) lies in \( N \) and is a smaller normal subgroup of \( D \). Therefore, the group \( N \) lies in \( E \), so \( \Phi(E) \leq N \) and hence by minimality \( \Phi(E) = N \).

The proof of Theorem 2 goes by contradiction and starts similarly to the proof of Theorem 1. By the consequent application of Lemma 1 and Lemma 5, we arrive at the following situation: \( D \) is isomorphic to \( G/H \), where \( G \) is a subgroup of a direct product \( X \times Y \), \( H \) is a normal subgroup of \( G \), groups \( X \) and \( Y \) are finite and finally \( H \) is a \( p' \)-group. As usual, denote by \( K_1 \) and \( K_2 \) kernels of projections of \( G \) onto \( Y \) and \( X \) respectively. By Lemma 3 we have \( H \cap K_1 = H \cap K_2 = 1 \). Clearly \( K_1 \cap K_2 = 1 \).

Suppose also that our groups are picked to be the minimal counterexample subject to \( |G| \). We note that neither of \( K_i \), \( i = 1, 2 \), can be trivial, since otherwise \( G \) would be a subgroup of \( X \) or \( Y \), which obviously contradicts the assumption that those groups do not contain \( D \) as a section. Also by Lemma 4 the group \( H \) cannot be trivial as well.

Let \( P \) be a Sylow \( p' \)-subgroup of \( G \). Obviously \( P \) is normal in \( G \), the group \( H \) is contained in \( P \) and by the Schur–Zassenhaus theorem there exists a complement \( Q \), such that \( G = PQ \) and \( Q \nsimeq A \). Note also that \( P/H \simeq E \).

**Lemma 9.** We can assume that \( P \nleq K_1H \).

**Proof.** Suppose that \( P \leq K_1H \). Set \( M = K_1Q \). Then

\[
M/(M \cap H) = MH/H = K_1QH/H = G/H \simeq D,
\]

so \( G = M = K_1Q \) by the minimality of \( G \). In that situation the \( p' \)-part of \( |G| \) must divide \( |K_1| \), hence \( P \leq K_1 \). But \( H \leq P \), thus \( H = H \cap P \leq H \cap K_1 = 1 \), which gives us the contradiction. \( \square \)
The group $K_1$ is normal in $G$, hence $HK_1/H$ is normal in $G/H$. By Lemma 8, the group $HK_1/H$ either contains $P/H$ or is contained in it, therefore $HK_1 \leq P$ or $P \leq HK_1$. The second case does not occur by Lemma 9, so $K_1$ lies in $P$ and is, therefore, a $p$-group. By the similar argument, $K_2$ is a $p$-group as well.

We prove that $K_i, i = 1, 2$, have order $p$. Notice first that since $A$ acts irreducibly on $E$, the only normal $A$-invariant subgroups of $E$ are the identity subgroup, $\Phi(E)$ and $E$ itself. The action of $Q$ on $P/H$ is similar, thus the same applies to $Q$-invariant subgroups of $P/H$. Now, $K_1/H$ is a nontrivial $Q$-invariant subgroup of $P/H$. It cannot be equal to $P/H$, since by Lemma 9, the group $P$ is not contained in $K_1H$. Then the only option is $K_1H/H \simeq \Phi(E)$, therefore $K_1 \simeq K_1H/H$ has order $p$. By the identical reasoning, $K_2$ also has order $p$.

Now we claim that $\Phi(P) = HK_1$.

Obviously $\Phi(P) \leq HK_1$, because $P/HK_1$ is elementary abelian. Suppose that $\Phi(P) < HK_1$. Denote by $\overline{\cdot} : P \rightarrow P/\Phi(P)$ the natural homomorphism from $P$ onto $P/\Phi(P)$. Clearly $HK_1 < P$, thus $HK_1$ is a nontrivial proper $Q$-invariant subgroup of $\overline{P}$. By Maschke’s theorem there exists a nontrivial proper $Q$-invariant subgroup $\overline{L}$ of $\overline{P}$ such that $\overline{P} = \overline{HK_1} \times \overline{L}$. Notice that $\overline{HK_1} \cap \overline{L} = 1$.

Let $L$ be the full preimage of $\overline{L}$. Then $\Phi(P) < L < P$ and $L$ is a normal $Q$-invariant subgroup of $P$. Hence, $HL/H$ is a normal $Q$-invariant subgroup of $P/H$ and thus it is either trivial or equal to the whole of $P/H$ or equal to $HK_1/H$ (which is the derived subgroup of $P/H$). In the first case, $HL = H$, therefore $L \leq H$ and hence $\Phi(P) \leq H$. But $P/H$ is nonabelian, so this case does not occur. In the second case $HL = P$. Set $R = LQ$. Then $RH = PQ = G$ and $R/(R \cap H) \simeq D$, thus by the minimality of $G$, we have $G = R$ and $L = P$ which is a contradiction. Therefore, the only option is $HL = HK_1$, so $L \leq HK_1$. It follows that $\overline{L} \leq \overline{HK_1}$ and together with $\overline{L} \cap \overline{HK_1} = 1$ we have $\overline{L} = 1$, which gives us a contradiction. Hence, $\Phi(P) = HK_1$ and the claim is proved.

We show now that $P/P'$ is homocyclic. Denote by $\overline{\cdot} : P \rightarrow P/P'$ the natural homomorphism from $P$ onto $P/P'$. Suppose $\overline{P}$ is not homocyclic and then by [3, Theorem 5.2.2] it decomposes into a nontrivial direct product of $Q$-invariant subgroups $\overline{L}$ and $\overline{M}$. Let $L$ and $M$ be the full preimages of these subgroups. Note that they are proper $Q$-invariant subgroups of $P$ and that $P = LM$.

Notice that $L$ and $M$ cannot be subgroups of $\Phi(P)$ at the same time, since otherwise $P = \Phi(P)$ which is not the case. Without loss of generality, suppose that $L \not\leq \Phi(P)$. Therefore, $L\Phi(P)/\Phi(P)$ is a nontrivial proper $Q$-invariant subgroup of $P/\Phi(P)$ which is impossible, since $Q$ acts irreducibly on $P/\Phi(P)$. This is a contradiction, thus $P/P'$ is homocyclic.

Let $T$ be a $p$-group. Denote by $\Omega_1(T)$ the subgroup generated by all elements of order dividing $p$ in $T$, i.e. $\Omega_1(T) = \langle x \in T : x^p = 1 \rangle$.

Suppose that $K_1 \not\leq P'$. Again, let $\overline{\cdot} : P \rightarrow P/P'$ denote the natural homomorphism from $P$ onto $P/P'$. Then $\overline{K_1}$ is a $Q$-invariant subgroup of $\overline{P}$ of order $p$. Therefore, $\overline{K_1}$ lies in $\Omega_1(\overline{P})$ and hence the action of $Q$ on $\Omega_1(\overline{P})$ is not irreducible. Obviously the action of $Q$ on $\Omega_1(\overline{P})$ is equivalent to the action of $Q$ on $\overline{P}/\Phi(P) \simeq P/\Phi(P)$, so the latter is not irreducible as well, giving us a contradiction. Thus, $K_1$ must be contained in $P'$ and by the similar argument the same is true for $K_2$.

Now we use $\overline{\cdot} : P \rightarrow P/K_1$ to denote a natural homomorphism from $P$ onto $P/K_1$. Notice that $K_1K_2 \leq P'$, so $P'$ is nontrivial. Obviously $P'$ is the derived
subgroup of $P$ and since $P$ is nilpotent, we have $[P', P] < P'$. Clearly $[P', P] = [P', P] < P'$. Until now we have not used the fact that $E$ is 2-generated. Applying it, we deduce that $P$ is also 2-generated, because

$$P/\Phi(P) = P/HK_1 \simeq (P/H)/(HK_1/H) \simeq E/\Phi(E).$$

Therefore $P'/[P', P]$ is cyclic.

Since $P'/((H \cap P') \simeq HP'/H$ has order $p$, the intersection $H \cap P'$ is a maximal subgroup of $P'$. The quotient $P/H$ is extraspecial, so $[P', P]$ lies in $H$. Hence $(H \cap P')/[P', P]$ is the maximal subgroup of a cyclic group $P'/[P', P]$. By above arguments, $[P', P][K_1]/[P', P]$ is a proper subgroup of $P'/[P', P]$, so it is contained in $(H \cap P')/[P', P]$. Therefore $K_1 \leq H$. However $K_1 \cap H = 1$, thus $K_1 = 1$, which gives us the final contradiction.

References


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