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ON NEW EXAMPLES OF HYPOCRITICAL GROUPS

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Abstract. A group G is called *hypocritical* if whenever G lies in a locally finite variety generated by a section closed class of groups $\mathfrak{X}$, then G belongs to $\mathfrak{X}$. We prove that some coprime extensions of a p-group are hypocritical. The first example is given when such a p-group is nonabelian.

Keywords: locally finite varieties, finite groups, extraspecial p-groups.

1. Introduction

Recall that a group G is a *section* of a group F if there exists a subgroup H of F such that G is a homomorphic image of H. A class of groups $\mathfrak{X}$ is said to be *section closed* if sections of groups from $\mathfrak{X}$ also belong to it.

L.F. Harris [1] introduced the following definition: a group G is called *hypocritical* if whenever G lies in a locally finite variety generated by a section closed class of groups $\mathfrak{X}$, then G lies in $\mathfrak{X}$. It is easy to see that a hypocritical group is in fact critical, that is, does not lie in a variety generated by its proper sections.

A group with a unique minimal normal subgroup is called *monolithic* and the minimal normal subgroup is the *monolith* of the group. By the Remak theorem, every critical and thus hypocritical group must be monolithic. L.F. Harris [1] mentions that the converse is true when the monolith is nonabelian, but gives no reference. A restricted version of this result was established in [2, Theorem 4] and we prove this fact in general (see Proposition 1).

In this work we consider finite groups which are coprime extensions of finite p-groups. By the well-known Schur–Zassenhaus theorem, these extensions are split, so we actually work with semidirect products of the form $D = B \rtimes A$, where B is a p-group and A is a p'-group. Suppose that B is abelian. Then for D to be

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hypocritical (or at least critical), the group A must act indecomposably on B, which is possible only if B is homocyclic (see [3, Theorem 5.2.2]). In this situation, by [1, Theorem 1], if the group D is hypocritical and B is not cyclic or elementary abelian, then the exponent of B is bounded in terms of $|A|$ and p.

Consider the excluded possibilities for B. In the case when B is elementary abelian, it is easy to establish that D is hypocritical (see Proposition 2). Let now B be cyclic. In the same paper, L.F. Harris mentioned that the hypocriticality of D in this case was shown by R.M. Bryant and L.G. Kovács before, but we found no traces of the original proof. Moreover, the situation when B is cyclic is closely connected with the following problem due to L.G. Kovács (Problem 8.23 in the *Kourovka notebook* [4]): if the dihedral group D of order 18 is a section of a direct product $X \times Y$, must at least one of X and Y have a section isomorphic to D? In Lemma 1 we show that for finite groups X and Y that question is equivalent to checking hypocriticality of D. Note that some partial results on Kovács' problem were obtained in [5]. Here we provide the proof in the cyclic case in general.

Theorem 1. *Let p be a prime, and let $D = B \rtimes A$ be a semidirect product of its subgroups B and A, where B is a finite cyclic p-group and A is a p'-group acting faithfully on B. Then D is hypocritical.*

Theorem 1 implies that the main difficulty in Kovács' problem appears when groups X and Y are not locally finite.

Now we turn to the case when B is nonabelian. The smallest example occurs when D is equal to a semidirect product of the quaternion group Q_8 and the cyclic group C_3, that is, $D \simeq SL_2(3)$. Recall that a finite p-group E is called *extraspecial*, if its center $Z(E)$, Frattini subgroup $\Phi(E)$ and derived subgroup E' coincide and have order equal to p. Notice that Q_8 is a 2-generated extraspecial group and it turns out that we can prove our result in the general case of a 2-generated extraspecial group.

Theorem 2. *Let p be a prime, and let $D = E \rtimes A$ be a semidirect product of its subgroups E and A, where E is a 2-generated extraspecial p-group and A is a p'-group acting faithfully and irreducibly on $E/\Phi(E)$. Then D is hypocritical.*

As we will see, the proof of Theorem 2 does not depend on E being 2-generated until the last step, which gives the hope of proving the similar statement for a general extraspecial group. Having Harris' results in mind, one can conjecture that a necessary condition for a group D to be hypocritical is that $|E/\Phi(E)|$ is bounded by some function of p and $|A|$, though no examples of such behavior were found so far.

The structure of the rest of this paper is as follows. In Section 2 we reduce the problem to studying sections of direct products and obtain some general results in that setting. As a simple corollary we prove the proposition about groups with a nonabelian monolith and establish some additional constraints on groups involved in the case of finite groups. Section 3 provides the proof of Theorem 1, while Section 4 section deals with Theorem 2.

2. Reduction lemmata

Lemma 1. *A finite group D is hypocritical if and only if whenever it is a section of a direct product $X \times Y$ of finite groups X and Y, then D is a section of either X or Y.*

Proof. Suppose D is hypocritical. Obviously finite groups X and Y generate a locally finite variety and D belongs to it. By the definition of hypocriticality we get that D is a section of either X or Y.

Let $\mathfrak{V}$ be a locally finite variety generated by a section closed class of groups $\mathfrak{X}$ and suppose that D lies in $\mathfrak{V}$. By [6, Lemma 4.3], the group D is a section of a direct product of a finite number of groups from $\mathfrak{X}$, i.e. D is a section of $X_1 \times \cdots \times X_n$, $n \geq 1$, and X_i, $i = 1 \dots n$, lie in $\mathfrak{X}$. Note that the definition of a section implies the existence of a subgroup G of $X_1 \times \cdots \times X_n$ such that D is a quotient of G.

Let H be a normal subgroup of G with $D \simeq G/H$. For each element of D we can pick a corresponding preimage in G. Let M be a subgroup of G generated by those preimages. Obviously M is finitely generated and since it lies in a locally finite group, M is finite. By definition, $HM = G$, hence $D \simeq G/H \simeq M/(M \cap H)$. Replacing G by M, we may assume G to be finite.

Clearly we can replace groups X_i by projections of G on corresponding factors of this direct product, and as $\mathfrak{X}$ is section closed, we can assume groups X_i to be finite. Now, inducting on n and applying the hypothesis we derive that D is a section of one of X_i, thus it lies in $\mathfrak{X}$ as needed. □

Let G be a subgroup of a direct product $F = X \times Y$ of groups X and Y. Then let G_X and G_Y denote the projections of G on X and Y respectively. We think of G_X and G_Y as of subgroups of F.

Lemma 2. *Let G be a subgroup of $F = X \times Y$ and let $X = G_X$ and $Y = G_Y$. If H is normal in G, then H_X and H_Y are normal in F.*

Proof. A projection on X is a homomorphism from G into X. Under our conditions this map is surjective, thus H_X is a normal subgroup of X. The group Y centralizes X in F, so H_X is actually a normal subgroup of F. The proof is similar for H_Y. □

Lemma 3. *Let $F = X \times Y$ be a direct product of groups, and let G be a subgroup of F. Suppose also that H is a normal subgroup of G. Then there exist groups $\hat{X}$, $\hat{Y}$, $\hat{G}$ and $\hat{H}$ such that $\hat{G}$, $\hat{X}$, and $\hat{Y}$ are sections of G, X, and Y respectively, $\hat{H}$ is normal in $\hat{G}$ and the following hold:*

(i) $\hat{X} = \hat{G}_{\hat{X}}$ *and* $\hat{Y} = \hat{G}_{\hat{Y}}$;

(ii) $\hat{G}/\hat{H}$ *is isomorphic to* G/H;

(iii) if $\hat{K_1} = \hat{G} \cap \hat{G}_{\hat{X}}$ *and* $\hat{K_2} = \hat{G} \cap \hat{G}_{\hat{Y}}$, *then* $\hat{K_1} \cap \hat{H} = \hat{K_2} \cap \hat{H} = 1$.

Proof. In the following we are going to repeatedly replace groups G, X and Y by their sections, preserving the same symbols for brevity. The groups $\hat{G}$, $\hat{X}$, and $\hat{Y}$ can be thought of as the final "product" of this process.

It is easy to see that G is a subgroup of $G_X \times G_Y$, thus we can always assume that $X = G_X$ and $Y = G_Y$. By Lemma 2 the groups H_X and H_Y are normal in F. Then $L_1 = H \cap H_X$ and $L_2 = H \cap H_Y$ are normal in G, but obviously $(L_1)_X = L_1$ and $(L_2)_Y = L_2$, so L_i, $i = 1, 2$, are actually normal in F by another application of Lemma 2.

Consider the quotient group $F/(L_1 \times L_2)$:

$$F/(L_1 \times L_2) \simeq G_X/L_1 \times G_Y/L_2 \text{ and } (G/(L_1 \times L_2))/(H/(L_1 \times L_2)) \simeq G/H.$$

Therefore replacing our groups appropriately, we can assume that $L_1 = L_2 = 1$. As a consequence, if we take $K_1 = G \cap G_X$ then $H \cap K_1 = H \cap H_X = 1$. The similar equality is true for $K_2 = G \cap G_Y$. □

Lemma 4. *Let $F = X \times Y$ be a direct product of groups, and let D be a monolithic subgroup of F. Then D is a section of X or Y.*

Proof. Denote by K_1 and K_2 kernels of projection maps from D into Y and X respectively. Subgroups $K_i \cap D$, $i = 1, 2$ are normal in D and intersect trivially. Since D is monolithic, it follows that $K_1 \cap D$ or $K_2 \cap D$ is trivial, which gives us an embedding of D into Y or X, as wanted. □

Now we are ready to resolve the case of a group with a nonabelian monolith.

Proposition 1. *Let D be a finite monolithic group with a nonabelian monolith. Then D is hypocritical.*

Proof. By Lemma 1 it is sufficient to prove that for all finite groups X and Y, if D is a section of $F = X \times Y$ then D is a section of either X or Y. Suppose that D is a section of neither X nor Y. Let G be a subgroup of F and let H be a normal subgroup of G, such that $D \simeq G/H$. Denote by K_1 and K_2 kernels of projections of G into Y and X respectively and notice that $K_1 = G \cap G_X$ and $K_2 = G \cap G_Y$. By Lemma 3 we can choose groups X and Y in such a way that $K_1 \cap H = K_2 \cap H = 1$. Note also that $K_1 \cap K_2 = 1$.

Now set $L = K_1H \cap K_2H$. Obviously $H \leq L$. The groups K_i, $i = 1, 2$, and H are normal in G, and consequently $[K_1, K_2] = [K_1, H] = [K_2, H] = 1$. Therefore,

$$[L, L] \leq [K_1H, K_2H] = [K_1, K_2][K_1, H][H, K_2][H, H] = [H, H] \leq H.$$

It follows that L/H is an abelian normal subgroup of $G/H \simeq D$ and thus it cannot contain the monolith of D, so in fact $H = L$.

As $H = K_1H \cap K_2H$, we easily obtain an embedding of $D \simeq G/H$ into the direct product $G/K_1H \times G/K_2H$. The factors of this direct product do not have D as a section (because otherwise X or Y would), so we can now assume that D is a subgroup of F. Lemma 4 gives us the final contradiction. □

If a finite monolithic group has an abelian monolith, it must be an elementary abelian p-group for some prime p. In this situation and when groups involved are finite, we obtain a refinement of Lemma 3.

Lemma 5. *In the notation of Lemma 3, suppose that groups X and Y are finite, and the group $D \simeq G/H$ has an abelian monolith K. Then we can assume that $\hat{H}$ is a p-group, where p is a prime dividing $|K|$.*

Proof. As in the proof of Lemma 3, we will drop "hats" from our symbols.

Pick the group G to be of minimal order such that G satisfies the conclusion of Lemma 3. Let M be a maximal subgroup of G and suppose that H does not lie in M. Then $HM = G$ and $M/(M \cap H) \simeq G/H \simeq D$, so we can replace G by M. But M has a smaller order than G, which contradicts the minimality condition, thus actually M includes H.

The Frattini subgroup is an intersection of maximal subgroups of a group, and by the preceding paragraph H lies in all maximal subgroups of G. Therefore, H lies in the Frattini subgroup of G which is nilpotent by [3, Theorem 6.1.6], so H is nilpotent too.

As H is nilpotent, it contains a Hall p'-subgroup S. It is characteristic in H so it is normal in G. Set $L = K_1S \cap K_2S$. Then L is a normal subgroup of G and obviously $S \leq L$. Let R be a Sylow p-subgroup of L. We have $R \leq K_1S$. As S is a

p'-group and K_1 is a normal subgroup of K_1S, it contains all Sylow p-subgroups of K_1S and R in particular. In other words, $R \le K_1$ and by symmetry $R \le K_2$. But $K_1 \cap K_2 = 1$ because $G_X \cap G_Y = 1$. Thus, $R = 1$ and L is a p'-group.

Now we have that $|LH : H| = |L : L \cap H|$ is a p'-number, so LH/H is a normal p'-subgroup of $G/H \simeq D$. The monolith of D is a p-group and thus it intersects LH/H trivially, which implies that $LH/H = 1$. Consequently, $LH = H$ and $L \le H$.

Notice that D is a section of $\tilde{F} = G/K_1S \times G/K_2S$. Indeed, G/L is a subgroup of $\tilde{F}$ and $D \simeq G/H \simeq (G/L)/(H/L)$. Suppose $L \ne 1$. Applying Lemma 3 to the groups G/L, H/L, G/K_1S, and G/K_2S, we get a contradiction with the minimality of G. Then $S \le L$ must be trivial. It follows that H is a p-group. □

Proposition 2. *Let $D = B \rtimes A$ be a semidirect product of a finite elementary abelian p-group B and a p'-group A. Suppose also that A acts faithfully and irreducibly on B. Then D is hypocritical.*

Proof. By Lemma 1 we can consider the following situation: let X and Y be finite groups, and let G be a subgroup of $X \times Y$. Denote by H the normal subgroup of G, such that $D \simeq G/H$. By Lemma 5 we can assume that H is a p-group. Denote by P a Sylow p-subgroup of G. Notice that $H \le P$ and P is normal in G.

Since the action of A on B is faithful, B is a minimal normal subgroup of D. Now, let N be a normal subgroup of D, such that $N \cap B = 1$. Then N centralizes B, and by the conjugacy part of the Schur–Zassenhaus theorem, we have $N \le A$. The action of A on B is faithful, hence $N = 1$, so D is in fact monolithic.

The proof goes by contradiction. Let K_1 be a kernel of the projection from G into Y. If K_1 is trivial, then G (and thus D) is a section of Y, so we can assume $K_1 > 1$. Then K_1H/H is a nontrivial normal subgroup of G/H, and therefore $P \le K_1H$. We have

$$P \le P \cap K_1H = (P \cap K_1)H \le P,$$

so $(P \cap K_1)H = P$. But since P/H is elementary abelian, H lies in $\Phi(P)$, thus $P \cap K_1 = P$. Then $H \le P \le K_1$, and we derive $H = 1$ which gives us the contradiction with Lemma 4. □

3. Proof of Theorem 1

Recall that $D = B \rtimes A$ is a semidirect product of its subgroups, where B is a cyclic p-group of order p^n and A is a group of order coprime to p. In order to prove hypocriticality of D, we first need one simple lemma.

Lemma 6. *The group D is monolithic and its monolith has order p.*

Proof. Let K be the unique subgroup of order p in B. Obviously K is normal in D. We prove that K is the monolith of D.

Let N be a nontrivial normal subgroup of D. Suppose that N is a p'-group. Clearly $N \cap B = 1$ and therefore N centralizes B. Since AN is a p'-group, $|AN|$ divides $|D : B| = |A|$, hence $N \le A$. So A has nonidentity elements centralizing B contrary to our hypothesis. Thus, $|N|$ is divisible by p.

Since the group B is a normal Sylow p-subgroup of D, the groups B and N intersect nontrivially. Then $K \le B \cap N \le N$, which proves the claim. □

We need the following general lemma.

Lemma 7. *Let P be a finite p-group and let Q be a p'-group acting on P by automorphisms. Suppose that H is a normal Q-invariant subgroup of P, such that the quotient P/H is cyclic. Then there exists a cyclic Q-invariant subgroup C of P, such that $HC = P$.*

Proof. Note that the case $P = H$ is trivial, thus from now on we will suppose that $H < P$. Choose the smallest subgroup C of P, such that C is Q-invariant, and $HC = P$. We prove that C is cyclic.

Notice that $H \cap C$ is a normal subgroup of C and $C/(H \cap C) \simeq P/H$ is cyclic. Let $L = H \cap C$. Since $H < P$, we have $L < C$, and therefore $\Phi(C)L < C$. The group $C/(\Phi(C)L)$ is cyclic and elementary abelian, hence $|C/(\Phi(C)L)| = p$. Then the group $(\Phi(C)L)/\Phi(C)$ is a Q-invariant subgroup of index p in $C/\Phi(C)$, and by Maschke's theorem, there exists a Q-invariant group T, such that

$$T/\Phi(C) \times (\Phi(C)L)/\Phi(C) = C/\Phi(C),$$

and $T/\Phi(C)$ has order p. We derive that $T\Phi(C)L = C$, so $TL = C$. Then

$$TH = TLH = CH = P,$$

where the first equality holds because $L \leq H$. Since $T \leq C$, and C was chosen to be minimal satisfying the required properties, we obtain $T = C$. But $|T/\Phi(C)| = p$, so $|C/\Phi(C)| = p$, which proves that C is cyclic. □

The proof of the theorem goes by contradiction. By Lemma 1 we can assume that D is a section of a direct product of finite groups X and Y and as usual, denote by G a subgroup of $X \times Y$ and by H a normal subgroup of G such that $D \simeq G/H$. Also, let K_1 and K_2 be kernels of projections of G into Y and X respectively. By Lemma 6 and Lemma 5, we can assume H to be a p-group. Notice also that $K_1 \cap K_2 = 1$.

Let P be a Sylow p-subgroup of G. Obviously H lies in P, so P is normal in G. We have $|G : P| = |D : B| = |A|$, thus $|P|$ is coprime to $|G : P|$, and by the Schur–Zassenhaus theorem, there exists a p'-subgroup Q of G, such that $PQ = G$.

Lemma 7 implies the existence of a cyclic Q-invariant subgroup C of P, such that $HC = P$. Let $R = CQ$. Then we have

$$D \simeq G/H = PQ/H = HCQ/H \simeq R/(R \cap H).$$

It follows that D is a quotient of R, and the action of Q on C is faithful and nontrivial, since the action of A on B is so.

The group R is a semidirect product of a cyclic p-group C and a cyclic p'-group Q, thus Lemma 6 applies. Then R is monolithic, and by Lemma 4, the group R can be embedded in X or Y, which in its turn shows that D is a section of one of these groups. That contradicts the choice of X and Y, proving the theorem.

4. Proof of the Theorem 2

Recall that $D = E \rtimes A$ is a semidirect product of its subgroups, where E is an extraspecial 2-generated p-group, and A is a group of order coprime to p, acting faithfully and irreducibly on $E/\Phi(E)$.

Lemma 8. *Let N be a normal subgroup of D. Then either $E \leq N$ or $N \leq E$. As a consequence, D is monolithic.*

Proof. Let Q be a normal subgroup of A. Consider $N_E(Q)$. It is obviously an A-invariant subgroup of E and hence $\Phi(E)N_E(Q)$ is an A-invariant subgroup of E. The action of A on $E/\Phi(E)$ is irreducible, therefore we have either $\Phi(E)N_E(Q) = E$ or $\Phi(E)N_E(Q) = \Phi(E)$. In the first case, $N_E(Q) = E$, and it follows that Q is a normal subgroup of $D = EA$. Then Q centralizes E, and thus the action of A on $E/\Phi(E)$ is not faithful, which is a contradiction. Therefore, $\Phi(E)N_E(Q) = \Phi(E)$, so $N_E(Q) \leq \Phi(E)$ and finally $N_D(Q) = AN_E(Q) \leq A\Phi(E)$.

Let N be a normal subgroup of D and suppose that N is not contained in E. Then N is not a p-group, and hence N intersects A nontrivially. The group $Q = N \cap A$ is a normal subgroup of A, so by the result of the previous paragraph, $N_D(Q) \leq A\Phi(E)$. Notice that if N intersected E trivially, then N would lie in A and therefore $Q = N$. Then $D = N_D(N) \leq A\Phi(E)$, which gives us a contradiction. Thus, N intersects E nontrivially and hence contains $Z(E) = \Phi(E)$.

We have $|N : N \cap A| = |NA : A|$, and since NA is a subgroup of G, the latter index is a p-number. Hence, Q is a Hall p'-subgroup of N. Obviously $E \cap N$ is a Sylow p-subgroup of N, therefore $N = (E \cap N)Q$, and by the Schur–Zassenhaus theorem, all complements to $E \cap N$ in N are conjugate. Then the Frattini-type argument applies and we derive:

$$D = NN_D(Q) \leq N\Phi(E)A = NA,$$

where the last equality holds because $\Phi(E) \leq N$. Now $D = NA$ and as A is a p'-group, we have $E \leq N$ as claimed.

Let N be a minimal normal subgroup of D. Obviously a minimal normal subgroup cannot contain E, since otherwise $\Phi(E)$ lies in N and is a smaller normal subgroup of D. Therefore, the group N lies in E, so $\Phi(E) \leq N$ and hence by minimality $\Phi(E) = N$. □

The proof of Theorem 2 goes by contradiction and starts similarly to the proof of Theorem 1. By the consequential application of Lemma 1 and Lemma 5, we arrive at the following situation: D is isomorphic to G/H, where G is a subgroup of a direct product $X \times Y$, H is a normal subgroup of G, groups X and Y are finite and finally H is a p-group. As usual, denote by K_1 and K_2 kernels of projections of G onto Y and X respectively. By Lemma 3 we have $H \cap K_1 = H \cap K_2 = 1$. Clearly $K_1 \cap K_2 = 1$.

Suppose also that our groups are picked to be the minimal counterexample subject to $|G|$. We note that neither of K_i, $i = 1, 2$, can be trivial, since otherwise G would be a subgroup of X or Y, which obviously contradicts the assumption that those groups do not contain D as a section. Also by Lemma 4 the group H cannot be trivial as well.

Let P be a Sylow p-subgroup of G. Obviously P is normal in G, the group H is contained in P and by the Schur–Zassenhaus theorem there exists a complement Q, such that $G = PQ$ and $Q \simeq A$. Note also that $P/H \simeq E$.

Lemma 9. *We can assume that $P \nleq K_1H$.*

Proof. Suppose that $P \leq K_1H$. Set $M = K_1Q$. Then

$$M/(M \cap H) = MH/H = K_1QH/H = G/H \simeq D,$$

so $G = M = K_1Q$ by the minimality of G. In that situation the p-part of $|G|$ must divide $|K_1|$, hence $P \leq K_1$. But $H \leq P$, thus $H = H \cap P \leq H \cap K_1 = 1$, which gives us the contradiction. □

The group K_1 is normal in G, hence HK_1/H is normal in G/H. By Lemma 8, the group HK_1/H either contains P/H or is contained in it, therefore $HK_1 \leq P$ or $P \leq HK_1$. The second case does not occur by Lemma 9, so K_1 lies in P and is, therefore, a p-group. By the similar argument, K_2 is a p-group as well.

We prove that K_i, $i = 1, 2$, have order p. Notice first that since A acts irreducibly on E, the only normal A-invariant subgroups of E are the identity subgroup, $\Phi(E)$ and E itself. The action of Q on P/H is similar, thus the same applies to Q-invariant subgroups of P/H. Now, K_1H/H is a nontrivial Q-invariant subgroup of P/H. It cannot be equal to P/H, since by Lemma 9, the group P is not contained in K_1H. Then the only option is $K_1H/H \simeq \Phi(E)$, therefore $K_1 \simeq K_1H/H$ has order p. By the identical reasoning, K_2 also has order p.

Now we claim that $\Phi(P) = HK_1$.

Obviously $\Phi(P) \leq HK_1$, because P/HK_1 is elementary abelian. Suppose that $\Phi(P) < HK_1$. Denote by $\overline{} : P \to P/\Phi(P)$ the natural homomorphism from P onto $P/\Phi(P)$. Clearly $HK_1 < P$, thus $\overline{HK_1}$ is a nontrivial proper Q-invariant subgroup of $\overline{P}$. By Maschke's theorem there exists a nontrivial proper Q-invariant subgroup $\overline{L}$ of $\overline{P}$ such that $\overline{P} = \overline{HK_1} \times \overline{L}$. Notice that $\overline{HK_1} \cap \overline{L} = 1$.

Let L be the full preimage of $\overline{L}$. Then $\Phi(P) < L < P$ and L is a normal Q-invariant subgroup of P. Hence, HL/H is a normal Q-invariant subgroup of P/H and thus it is either trivial or equal to the whole of P/H or equal to HK_1/H (which is the derived subgroup of P/H). In the first case, $HL = H$, therefore $L \leq H$ and hence $\Phi(P) \leq H$. But P/H is nonabelian, so this case does not occur. In the second case $HL = P$. Set $R = LQ$. Then $RH = PQ = G$ and $R/(R \cap H) \simeq D$, thus by the minimality of G, we have $G = R$ and $L = P$ which is a contradiction. Therefore, the only option is $HL = HK_1$, so $L \leq HK_1$. It follows that $\overline{L} \leq \overline{HK_1}$ and together with $\overline{L} \cap \overline{HK_1} = 1$ we have $\overline{L} = 1$, which gives us a contradiction. Hence, $\Phi(P) = HK_1$ and the claim is proved.

We show now that P/P' is homocyclic. Denote by $\overline{} : P \to P/P'$ the natural homomorphism from P onto P/P'. Suppose $\overline{P}$ is not homocyclic and then by [3, Theorem 5.2.2] it decomposes into a nontrivial direct product of Q-invariant subgroups $\overline{L}$ and $\overline{M}$. Let L and M be the full preimages of these subgroups. Note that they are proper Q-invariant subgroups of P and that $P = LM$.

Notice that L and M cannot be subgroups of $\Phi(P)$ at the same time, since otherwise $P = \Phi(P)$ which is not the case. Without loss of generality, suppose that $L \not\leq \Phi(P)$. Therefore, $L\Phi(P)/\Phi(P)$ is a nontrivial proper Q-invariant subgroup of $P/\Phi(P)$ which is impossible, since Q acts irreducibly on $P/\Phi(P)$. This is a contradiction, thus P/P' is homocyclic.

Let T be a p-group. Denote by $\Omega_1(T)$ the subgroup generated by all elements of order dividing p in T, i.e. $\Omega_1(T) = \langle x \in T \mid x^p = 1 \rangle$.

Suppose that $K_1 \not\leq P'$. Again, let $\overline{} : P \to P/P'$ denote the natural homomorphism from P onto P/P'. Then $\overline{K_1}$ is a Q-invariant subgroup of $\overline{P}$ of order p. Therefore, $\overline{K_1}$ lies in $\Omega_1(\overline{P})$ and hence the action of Q on $\Omega_1(\overline{P})$ is not irreducible. Obviously the action of Q on $\Omega_1(\overline{P})$ is equivalent to the action of Q on $\overline{P}/\Phi(\overline{P}) \simeq P/\Phi(P)$, so the latter is not irreducible as well, giving us a contradiction. Thus, K_1 must be contained in P' and by the similar argument the same is true for K_2.

Now we use $\overline{} : P \to P/K_1$ to denote a natural homomorphism from P onto P/K_1. Notice that $K_1K_2 \leq P'$, so $\overline{P'}$ is nontrivial. Obviously $\overline{P'}$ is the derived subgroup of $\overline{P}$ and since $\overline{P}$ is nilpotent, we have $[\overline{P'}, \overline{P}] < \overline{P'}$. Clearly $\overline{[P', P]} = [\overline{P'}, \overline{P}]$ which yields $[P', P]K_1 < P'$.

Until now we have not used the fact that E is 2-generated. Applying it, we deduce that P is also 2-generated, because

$$P/\Phi(P) = P/HK_1 \simeq (P/H)/(HK_1/H) \simeq E/\Phi(E).$$

Therefore $P'/[P', P]$ is cyclic.

Since $P'/(H \cap P') \simeq HP'/H$ has order p, the intersection $H \cap P'$ is a maximal subgroup of P'. The quotient P/H is extraspecial, so $[P', P]$ lies in H. Hence $(H \cap P')/[P', P]$ is the maximal subgroup of a cyclic group $P'/[P', P]$. By above arguments, $[P', P]K_1/[P', P]$ is a proper subgroup of $P'/[P', P]$, so it is contained in $(H \cap P')/[P', P]$. Therefore $K_1 \leq H$. However $K_1 \cap H = 1$, thus $K_1 = 1$, which gives us the final contradiction.

References

[1] L.F. Harris, *Hypocritical and sincere groups*, Proc. Second Internat. Conf. Theory of Groups, **372** (1973), 333–336.

[2] L.G. Kovács, M.F. Newman, *Minimal verbal subgroups*, Proc. Camb. Phil. Soc., **62** (1966), 347–350. MR0210768

[3] D. Gorenstein, *Finite groups*, 2nd edition, New York: Chelsea Publishing Co., 1980. MR0569209

[4] *Unsolved problems in group theory. The Kourovka Notebook*, 18th edn (Eds. V. D. Mazurov and E. I. Khukhro), Novosibirsk, Sobolev Institute of Mathematics (2014).

[5] A.V. Vasil'ev, S.V. Skresanov, *On L.G. Kovàcs' problem*, Algebra and Logic, **55**:4 (2016), 512–518. MR3711128

[6] G. Higman, *Some remarks on varieties of groups*, Quart. J. Math. Oxford Ser., **10** (1959), 165–178. MR0113925

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