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ON ORDERED GROUPS OF MORLEY O-RANK 1

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Abstract. Given a cut s in an ordered structure $\mathcal{M}$ we can define a localization of Morley rank—Morley o-rank, replacing each formula in definition of Morley rank with the following partial types: the cut s extended with this formula. We prove in the paper that any ordered group of Morley o-rank 1 with boundedly many definable convex subgroups is weakly o-minimal and construct an example of an ordered group of Morley o-rank 1 and Morley o-degree at most 4.

Keywords: ordered group, weak o-minimality, o-stability, rank.

1. Introduction

Since A. Pillay and Ch. Steinhorn started investigating o-minimality in [4], ordered structures are in the focus of investigations in Model Theory. D. Macpherson, D. Marker and Ch. Steinhorn considered in [3] a generalization of o-minimality—weak o-minimality.

It is well-known that in o-minimal structures any cut in this structure defines a complete type over the structure. In [2] B. Kulpeshov proved that any cut in a weakly o-minimal structure has at most two extensions up to complete types over the structure and the set of realizations of these types are convex.

So, it became clear, that the number of extensions of a cut plays an important role in investigations of ordered structure, after that B. Baizhanov and V. Verbovskiy suggested in [1] notion of o-stability—each cut has a few extensions up to complete types over the considered structures. In this paper they considered basic properties of o-stable theories and proved, in particular, that such theories do not have the independence property.

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At the same time V. Verbovskiy in [6] applied o-stability to investigation of such classical algebraic structures as ordered groups: he considered o-stable ordered groups, proved that they are Abelian and gave some description of definable subsets.

It has been proved in [1] that the class of weakly o-minimal theories is a proper subclass of the class of o-stable theories. The simplest example of an o-stable ordered group which is not weakly o-minimal is $(\mathbb{R}, <, +, 0, Q)$, where the unary predicate Q names the subgroup of rational numbers [6]. But since the index $|\mathbb{R} : \mathbb{Q}|$ is infinite, Morley o-rank of this structure is 2. So, it is an interesting question whether there exists an o-stable ordered group of Morley o-rank 1, which is not weakly o-minimal. We prove in Section 2 that any ordered group of Morley o-rank 1 with boundedly many definable subgroups is weakly o-minimal and construct in Section 3 an example of an ordered group of Morley o-rank 1 and Morley o-degree at most 4. Below we give necessary definitions and facts.

Let $\mathcal{M} = (M, <, \dots)$ be a totally ordered structure, a is an element of M and A, B subsets. As usually we write

$$a < A, \quad \text{if } a < b \text{ for any } b \in A,$$
$$A < B, \quad \text{if } a < b \text{ for any } a \in A \text{ and } b \in B.$$

A partition $\langle C, D \rangle$ of M is called a *cut* if $C < D$. Given a cut $\langle C, D \rangle$ one can construct a partial type $\{c < x < d : c \in C,\ d \in D\}$, which we also call a cut and use the same notation $\langle C, D \rangle$.

A subset A of a totally ordered set M is called *convex* if for any a and $b \in A$ the interval $[a, b]$ is a subset of A. The *length* of a convex set A is defined as $\sup\{a - b : a, b \in A\}$. A *convex component* of a set A is a maximal convex subset of A. The *convex hull* A^c of a set A is defined as

$$A^c = \big\{ b \in M : \exists a_1, a_2 \in A \, (a_1 \le b \le a_2) \big\},$$

that is it is the least convex set containing the set A. An ordered structure is called *weakly o-minimal* if any its definable subset consists of finitely many convex components, and a theory is *weakly o-minimal*, if all its models are weakly o-minimal [3].

Let P be some property. We say that the property P holds *eventually in* A if there is an element $a \in M$ such that $a < \sup A$ and the property P holds on the intersection $(a, \infty) \cap A$. If $A = M$, we just write that the property P *holds eventually.* If the property P is equality of two sets B and C, then we say that the sets B and C are eventually equal in A we denote this by $B \overset{\infty}{=}_A C$. Let $B \subseteq A \subseteq M$. The set B is said to be *dense in* A if for any $a_1 < a_2$ from A there is $b \in B$ with $a_1 < b < a_2$. If $A = M$ I omit A and write just B is dense. A *dense component* of B in A is a maximal subset B_0 of B which is dense in $A \cap (\inf B_0, \sup B_0)$.

We say that an ordered group $\mathcal{M}$ contains *boundedly many definable convex subgroups* if there is a cardinal λ, such that in any group which is elementary equivalent to $\mathcal{M}$ the number of convex definable subgroups does not exceed λ. Otherwise we say that $\mathcal{M}$ has *unboundedly many definable convex subgroups.*

We say that a convex set A is *coset-infinite*, if it is not a finite union of cosets of definable subgroups.

Let s be a partial n-type, A a set. Then

$$S^n_s(A) \triangleq \big\{ p \in S^n(A) : p \cup s \ \text{ is consistent} \big\}.$$

Note, s need not to be a partial type over the set A.

Definition 1 (B. Baizhanov, V. Verbovskiy [1]).

(1) *An ordered structure* $\mathcal{M}$ *is* o-stable in λ *if for any* $A \subseteq M$ *with* $|A| \leq \lambda$ *and for any cut* $\langle C, D\rangle$ *in* $\mathcal{M}$ *there are at most* λ *1-types over* A *which are consistent with the cut* $\langle C, D\rangle$*, i.e.*
$$\left|S^1_{\langle C,D\rangle}(A)\right| \leq \lambda.$$
(2) *A theory* T *is* o-stable in λ *if every model of* T *is. Sometimes I write* T *is o-*λ*-stable.*

(3) *A theory* T *is* o-stable *if there exists an infinite cardinal* λ *in which* T *is o-stable.*

(4) *A theory* T *is* o-superstable *if there exists a cardinal* λ *such that* T *is o-stable in all* $\mu \geq \lambda$.

(5) *A theory* T *is* strongly o-stable *if in addition to its o-stability any definable cut in any model* $\mathcal{M}$ *of* T *is definable in the language of pure ordering, or, equivalently, if* $\sup A \in M$ *for any definable subset* A *of* $\mathcal{M}$.

Definition 2 (V. Verbovskiy, [6]).

(1) *We say that Morley o-rank of a formula* $\phi(x)$ *inside a cut* $\langle C, D\rangle$ *in* $\mathcal{M}$ *is equal to or greater than* 1 *and write* $RM_{\langle C,D\rangle,\mathcal{M}}(\phi)\geq 1$ *for this, if* $\{\phi(x)\} \cup \langle C, D\rangle$ *is consistent.*

(2) $RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \alpha + 1$ *if there are infinitely many pairwise inconsistent formulae* $\psi_i(x)$ *such that* $RM_{\langle C,D\rangle,\mathcal{M}}\big(\phi(x) \wedge \psi_i(x)\big) \geq \alpha$.

(3) *If* α *is a limit ordinal, then* $RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \alpha$ *if* $RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \beta$ *for all* $\beta < \alpha$.

(4) $RM_{\langle C,D\rangle,\mathcal{M}}(\phi) = \alpha$ *if* $RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \geq \alpha$ *and* $RM_{\langle C,D\rangle,\mathcal{M}}(\phi) \not\geq \alpha + 1$.

In a similar way one can define Morley o-degree of a formula inside a cut. As usual one can define Morley o-rank of a type.

In the previous definition we have given Morley o-rank for a fixed model. In the next one we consider Morley o-rank of a formula in an arbitrary elementary extension of a given model.

Definition 3.

(1) *We say that Morley o-rank of a formula* $\phi(x)$ *inside a cut* $\langle C, D\rangle$ *is equal to or greater than* 1 *and write* $RM_{\langle C,D\rangle}(\phi)\geq 1$ *for this, if* $\{\phi(x)\} \cup \langle C, D\rangle$ *is consistent.*

(2) $RM_{\langle C,D\rangle}(\phi) \geq \alpha + 1$ *if there is an elementary extension* $\mathcal{N}$ *of* $\mathcal{M}$*, a cut* $\langle C_1, D_1\rangle$ *in* $\mathcal{N}$ *containing the cut* $\langle C, D\rangle$ *and there are infinitely many pairwise inconsistent formulae* $\psi_i(x)$ *with parameters in* N *such that the inequality* $RM_{\langle C_1,D_1\rangle,\mathcal{N}}\big(\phi(x) \wedge \psi_i(x)\big) \geq \alpha$ *holds.*

(3) *If* α *is a limit ordinal, then* $RM_{\langle C,D\rangle}(\phi) \geq \alpha$ *if* $RM_{\langle C,D\rangle}(\phi) \geq \beta$ *for all* $\beta < \alpha$.

(4) $RM_{\langle C,D\rangle}(\phi) = \alpha$ *if* $RM_{\langle C,D\rangle}(\phi) \geq \alpha$ *and* $RM_{\langle C,D\rangle}(\phi) \not\geq \alpha + 1$.

Let T be a theory of a language $\mathcal{L}$, and $\mathcal{M} \prec \mathcal{N}$ two models of T such that $\mathcal{N}$ is $|M|^+$-saturated. For any formula $\phi(\bar{x}, \bar{\alpha})$ with parameters $\bar{\alpha}$ in N I add a new relational symbol $P_{\phi(\bar{x},\bar{\alpha})}(\bar{x})$ interpreted by $P_{\phi(\bar{x},\bar{\alpha})}(M) = \phi(N, \bar{\alpha}) \cap M^k$ in order to form language $\mathcal{L}^*$. The set $P_{\phi(\bar{x},\bar{\alpha})}(M)$ is called *externally definable.*

Theorem 1. [6] *Let* T *be an o-stable theory of a language* $\mathcal{L}$*, and* $\mathcal{M} \prec \mathcal{N}$ *two models of* T *such that* $\mathcal{N}$ *is* $|M|^+$*-saturated. Then the elementary theory* T^* *of the expansion* $\mathcal{M}^*$ *of* $\mathcal{M}$ *is o-stable.*

Theorem 2. [6] *Let G be densely ordered. Then for any infinite definable set there is an interval where this set is dense. By other words, no infinite nowhere dense subset is definable.*

Theorem 3. [6] *Let G be an ordered group with boundedly many definable convex subgroups whose elementary theory is o-stable. Assume that G is not weakly o-minimal, that is there is a definable subset A consisting of infinitely many convex components, and a non-rational cut $\langle C, D\rangle$ such that both A and the complement of A are consistent with this cut. Then there is an externally definable unbounded proper subgroup K of the group H_C^+ (where H_C^+ is the stabilizer of the set C). If in addition the cut $\langle C, D\rangle$ is definable then the group K is definable.*

2. The main result

Throughout the paper $\mathcal{M} = (M, <, +, 0, \dots)$ is an ordered group whose elementary theory is o-stable. Moreover, we assume that $\mathcal{M}$ is sufficiently saturated.

By Theorem 2.8 from [6] we know that this group is Abelian, that is why we use the additive notation for the group operation.

The main result in this paper is the following theorem.

Theorem 4. *Let $\mathcal{M}$ be an ordered group whose elementary theory is o-stable, moreover, $RM_s(x = x) = 1$ for any cut $s = \langle C, D\rangle$ in $\mathcal{M}$. Assume that $\mathcal{M}$ contains boundedly many definable convex subgroups. Then the elementary theory of $\mathcal{M}$ is weakly o-minimal.*

Proof. Clearly, that since $RM_s(x = x) = 1$ for any cut $s = \langle C, D\rangle$ in $\mathcal{M}$, so $Th(\mathcal{M})$ is o-ω-stable. Lemma 2.12 in [6] states that if the elementary theory of $\mathcal{M}$ is o-ω-stable then $\mathcal{M}$ is elementary equivalent to the ordered group of rationals as a pure ordered group, i.e., in the language $\{<, +, 0\}$. Thus, $\mathcal{M}$ is divisible.

Lemma 1.11 in [6] says that if T be o-λ-stable, $\mathcal{M} = (M, <, \dots)$ is a model of T, and A a definable subset of $\mathcal{M}$, then the elementary theory of A with the full induced structure is o-λ-stable. So, any definable subgroup of $\mathcal{M}$ is o-ω-stable and is divisible.

Assume the contrary, that the elementary theory of $\mathcal{M}$ is not weakly o-minimal. Since we have supposed that $\mathcal{M}$ is sufficiently saturated, so $\mathcal{M}$ is not weakly o-minimal. By definition there exists a definable subset A of $\mathcal{M}$, which consists of infinitely many convex components.

Lemma 1. [6] *If for some formula $\varphi(x)$ the number of coset-infinite convex components of the set $\varphi(G)$ is infinite, then the group G has unboundedly many definable convex subgroups.*

Since any convex component of a definable set is definable, we may assume that each convex components of A is not coset-infinite. So, each convex component of A is a finite union of cosets of definable convex subgroups.

Let a convex component B of A be a finite union of cosets $a_i + H_i$ of definable convex subgroups H_i, for $i < n$. Since these subgroups are convex, they are linearly ordered by the relation $\subseteq$. Then if two of these cosets have non-empty intersection, one of them is a subset of the other one. Thus we may assume that any pair of these cosets have an empty intersection.

Let $a_1 < a_2 < \dots < a_n$. Since the group is divisible, there exists an element b such that $b + b = a_1 + a_2$. Note that $b \notin a_1 + H_1$, because otherwise $a_2 = b + b - a_1$

belongs to $a_1 + H_1$, and $(a_1 + H_1) \cap (a_2 + H_2) \neq \emptyset$. Similar arguments show that $b \notin a_2 + H_2$. But then B is not convex.

Consequently, any convex component of A is a coset of some definable convex subgroup. Since the group $\mathcal{M}$ has boundedly many definable convex subgroups, the number of convex subgroups, whose cosets form convex components of A is finite, that is why we may assume that each convex component of A is a coset of some definable convex subgroup H.

Now we consider the quotient-group $\mathcal{M}/H$ with the full induced structure. As it was shown in [6] this structure is ordered and o-ω-stable. It is easy to see that $RM_s(x = x) = 1$ for any cut $s = \langle C, D\rangle$ in $\mathcal{M}/H$, that is why we may assume that $H = \{0\}$.

By Theorem 2 for any infinite definable set there exists an infinite interval, where this set is dense. Since A contains infinitely many convex component, there is an interval (a, b) in which both A and its complement are dense. Then both A and its complement are consistent with each cut in the interval (a, b).

Then there exists a cut $\langle C, D\rangle$ such that both A and the complement of A are consistent with this cut and the sets of all realizations of A and the complement of A in the cut $\langle C, D\rangle$ are not convex. Moreover, both A and its complement are not bounded above in C.

Since we may consider $\mathcal{M}$ to be sufficiently saturated, we can find such a cut $\langle C, D\rangle$, that $H_C \neq \{0\}$.

Now we recall Theorem 3 from [6], where it has been proved that the stabilizer of a set X inside the cut $\langle C, D\rangle$, which is the following subgroup: $H_X = \{g \in G \mid g + X \overset{\infty}{\cong}_C X\}$, is not bounded in H_C.

Case 1. H_C is a non-zero definable subgroup. Then K is a proper unbounded in H_C subgroup of H_C. Since K is definable, it is divisible, as well as H_C. Then the index $|H_C : K| = \infty$, so infinitely many cosets of K are consistent with the cut $\langle C, D\rangle$, which implies that the Morley o-rank of $x = x$ in $\langle C, D\rangle$ is at least 2, for a contradiction.

Case 2. H_C is a non-zero undefinable subgroup. Since H_C is convex it is externally definable. Indeed, let α realize the cut $\sup H_C$. Then $H_C = (-\alpha, \alpha) \cap M$.

We know by Theorem 1 that expansion by externally definable subsets preserves o-stability.

In order to define the group K we need just to define H_C, which can be done by the formula $P_{-\alpha<x<\alpha}$. Let us consider the elementary extension $\mathcal{N}$ of $\mathcal{M}$ and in the formula of the language $\mathcal{L}^*$ we replace $P_{-\alpha<x<\alpha}$ with $-\alpha < x < \alpha$ and obtain the formula $K_1(x)$ of the language $\mathcal{L}$.

Note that $K_1(\mathcal{N}) \cap M = K$. Since infinitely many cosets of K are consistent with the cut $\langle C, D\rangle$ in $\mathcal{M}$, so there exist infinitely many $a \in M$ such that $K_1(x-a)$ is consistent with the cut defined by $\sup C$ in the model $\mathcal{N}$, contradicting the fact that Morley o-rank of $x = x$ is equal to 1.

Hence, our assumption that $\mathcal{M}$ is not weakly o-minimal is wrong. □

3. An example

In this section we give an example of an ordered group, which is not weakly o-minimal, but which has Morley o-rank of the formula $x = x$ to be equal 1, and Morley degree at most 4.

First, we need some fact from [5].

Fact 1. [5] *Let T_E be a theory in the language $\mathcal{L}_E = \{<, E, +, -, 0\}$ with the following set of axioms:*

1. *the axioms for a linearly ordered Abelian divisible group;*
2. *E is an equivalence relation with convex classes;*
3. *order induced on E-classes is dense without endpoints;*
4. *$E(x, nx)$, for any positive $n < \omega$;*
5. *$E(x, y) \leftrightarrow E(-x, -y)$;*
6. *$E(x, 0) \to x = 0$.*

Then T_E admits quantifier elimination.

Fact 2. [5] *T_E is weakly o-minimal.*

Let G be a countable model of T_E and $\mathcal{M} = (M, <, \mathcal{L})$ be a countable densely ordered structure without endpoints. Let f be an arbitrary isomorphism between the structures $(G^+/E, <)$ and $(M, <)$, and $\mathcal{L}^+ = \mathcal{L} \cup \{<, +, -, E, 0\}$. Define $(G, \mathcal{L}^+)$ by the following: for any $R \in \mathcal{L}$ we put

$$G \models R(x_1, \ldots, x_n) \text{ iff } G \models \bigwedge_i 0 < x_i \text{ and } M \models R(f([x_1]_E), \ldots, f([x_n]_E)).$$

We may assume that the elementary theory of the restriction $\mathcal{G}_L$ of $\mathcal{G}^+ = (G, \mathcal{L}^+)$ to the language $\mathcal{L}_L = \mathcal{L} \cup \{<, E, 0\}$ admits quantifier elimination. Let $T_L = \mathrm{Th}(\mathcal{G}_L)$.

Fact 3. [5] *The theory $T^+ = \mathrm{Th}(\mathcal{G}^+)$ admits quantifier elimination.*

Theorem 5. *There exists an ordered group, which is not weakly o-minimal, but which has Morley o-rank of the formula $x = x$ to be equal 1, and Morley degree at most 4.*

Proof. Consider the following group $(G, <, +, 0, E, P)$, where the set G consists of all functions with a finite support from the set of real numbers $\mathbb{R}$ to itself (that is the set $\{r \in \mathbb{R} \mid f(r) \neq 0\}$ is finite for each function f), the addition is coordinate-wise (that is $(f + g)(r) = f(r) + g(r)$), and the order is lexicographical. Let E be the equivalence relation, whose classes are Archimedean classes, that is two elements are equivalent if the most left non-zero coordinates are equal. So, the factor-set $(G/E, <)$ is isomorphic to the ordered set of reals. We denote this isomorphism by τ. The predicate P is interpreted in the following way: $\models P(f)$ iff $\tau([f])$ is rational, where $[f]$ is the E-class of $f \in G$.

Since the elementary theory of an Abelian ordered divisible group admits quantifier elimination as well as the elementary theory of the ordered set of reals with the named subset of rationals, by the general construction, described in [5], the elementary theory of the given ordered group $(G, <, +, 0, E, P)$ admits quantifier elimination.

Hence, any formula in one free variable is a Boolean combination of intervals, moved E-equivalence classes $[h] + g$, and $P(G) + g$, or $G \setminus P(G) + g$ for some g's in G.

It was proved in [5] that the elementary theory of $(G, <, +, 0, E)$ is weakly o-minimal, so any cut has at most two completions up to complete types.

Consider a cut $\langle C, D\rangle$. It defines a convex subgroup $H = \{h \in G \mid h + C = C\}$. Obviously,

$$(P(G) + g) \cap (\sup[g], +\infty) = P(G) \cap (\sup[g], +\infty)$$

as well as

$$(P(G) + g) \cap (-\infty, \inf[-g]) = P(G) \cap (-\infty, \inf[-g])$$

The set $(P(G) + g) \cap (\sup[-g], \inf[g])$ is either empty (if the first non-zero coordinate is irrational), or equal to $(\sup[-g], \inf[g])$ (otherwise).

The last two cases are $(P(G) + g) \cap [g]$, and $(P(G) + g) \cap [-g]$.

If the first coordinate in g is irrational, then

$$(P(G) + g) \cap [g] = A + g,$$

and

$$(P(G) + g) \cap [-g] = A + (-g), \text{ where } A = P(G) \cap (\sup[-g], \inf[g])$$

If the first coordinate in g is rational, then

$$(P(G) + g) \cap [g] = [g] \text{ and } (P(G) + g) \cap [-g] = [-g]$$

Consider two positive elements $g_1 < g_2$.

If these elements are not E-equivalent, then for any cut $\langle C, D\rangle$ either $P(G) + g_1$ or $P(G) + g_2$ is equal to $P(G)$ in some interval (c, d) for some $c \in C$ and $d \in D$.

If these elements are E-equivalent and the first non-zero coordinate is rational, then

$$(P(G) + g_1) \cap [g_1] = [g_1] = [g_2] = P(G) + g_2 \cap [g_1]$$

If these elements are E-equivalent but the first non-zero coordinate is irrational, then inside $[g_1]$ the sets $P(G) + g_1$ and $P(G) + g_2$ have an empty intersection if the meanings of the first non-zero coordinate are different, and equal if the meaning of the first non-zero coordinate are equal.

So, as we can see for any cut there are at most 4 extensions by $h + [g]$ or the complement of this set and by $P(G) + g$ or the complement of this set. Thus in this group the Morley o-rank of $x = x$ is equal to 1 and Morley degree is at most 4 in any cut. □

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