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ON $\mathcal{T}$-δ-NONCOSINGULAR MODULES

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Abstract. In this paper, we introduce and study the notion of $\mathcal{T}$-δ-noncosingular modules. The aim of this paper is to present some applications. Let R be a commutative ring. If R_R is $\mathcal{T}$-δ-noncosingular, we show right R_R is nonsingular. Also we prove that any singular regular module is an $\mathcal{T}$-δ-noncosingular module.

Keywords: $\mathcal{T}$-δ-noncosingular module, δ-lifting module.

1. Introduction

In this paper, R will present an associative ring with identity and all modules over R are unitary right modules. We denote $J(R)$ and $Rad(M)$ for the Jacobson radical of R and module M, respectively. For a submodule N of M, we use $N \leq M$ and $N \leq^{\oplus} M$ to mean that N is a submodule of M and N is a direct summand of M. A submodule N of M is called *small* in M (denoted by $N \ll M$) if, for every submodule K of M the equality $N + K = M$ implies $K = M$. A submodule N of M is called *essential* in M (denoted by $N \leq_e M$) if $N \cap K \neq 0$ for every nonzero submodule K of M. The singular submodule of a module M is $Z(M) = \{x \in M \mid xI = 0$ for some right ideal $I \leq_e R\}$. A module M is called *singular* (*nonsingular*) if $Z(M) = M$ (resp. $Z(M) = 0$). Following [16], a submodule N of a module M is said to be an δ-*small* submodule (denoted by $N \ll_\delta M$) if, whenever $M = N + X$ with M/X singular, implies $M = X$. The sum of all δ-small submodules of M is denoted by $\delta(M)$. Since every small submodule of a module M is δ-small in M, so $Rad(M) \subseteq \delta(M)$. If also M is singular, every submodule of M is small if and only if it is δ-small. It follows that $Rad(M) = \delta(M)$.

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In ([13]), Talebi and Vanaja defined $\overline{Z}(M)$ as follows:

$$\overline{Z}(M) = Rej(M, \mathsf{S}) = \bigcap\{Kerf \mid f : M \to L, L \in \mathsf{S}\} = \bigcap\{U \subseteq M \mid M/U \in \mathsf{S}\}$$

where S denotes the class of all small modules. If $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$), then M is called a *cosingular* (resp.,*noncosingular*) module. In [11], inspired by this definition Özcan defined the submodule $\overline{Z}_\delta(M)$ of M as $\overline{Z}_\delta(M) = \bigcap\{Kerg \mid g : M \to N, N$ is a δ-small module$\}$. Clearly, $\overline{Z}_\delta(M) \subseteq \overline{Z}(M)$. A module M is called a *δ-cosingular* (resp.,*δ-noncosingular*) module if $\overline{Z}_\delta(M) = 0$ ($\overline{Z}_\delta(M) = M$). Every cosingular module is δ-cosingular and every δ-noncosingular module is noncosingular.

In [5], Keskin and Tribak introduce $\mathcal{T}$-noncosingular modules. A module M is called $\mathcal{T}$-noncosingular relative to N if, for every nonzero homomorphism $f : M \longrightarrow N$, Imf is not small in N. If M is $\mathcal{T}$-noncosingular relative to M, then M is $\mathcal{T}$-noncosingular.

In Section 2, we introduce the concept of $\mathcal{T}$-δ-noncosingular modules. We show that in general a direct sum of $\mathcal{T}$-δ-noncosingular modules need not be a $\mathcal{T}$-δ-noncosingular module. Then we provide a necessary and sufficient condition for a direct sum of $\mathcal{T}$-δ-noncosingular modules to be $\mathcal{T}$-δ-noncosingular (see Proposition 3).

The focus of our investigations in Section 3 is on connections of an $\mathcal{T}$-δ-noncosingular module with other modules. We show that, if M is a module with a projective δ-cover P such that P is $\mathcal{T}$-δ-noncosingular. Then M is $\mathcal{T}$-δ-noncosingular (see Proposition 8). We also provide an equivalent condition for an $\mathcal{T}$-δ-noncosingular module (see Corollary 5).

For unexplained concepts and notations, we refer the reader to [1] and [14].

2. $\mathcal{T}$-δ-noncosingular modules

Let M and N be two R-modules. We say M is $\mathcal{T}$-δ-noncosingular relative to N if, for every nonzero homomorphism $f : M \longrightarrow N$, Imf is not δ-small in N. If M is $\mathcal{T}$-δ-noncosingular relative to M, we say that M is $\mathcal{T}$-δ-noncosingular. A ring R is said to be right (left) $\mathcal{T}$-δ-noncosingular if the right (left) R-module R is $\mathcal{T}$-δ-noncosingular. Clearly, every δ-noncosingular module is $\mathcal{T}$-δ-noncosingular and every $\mathcal{T}$-δ-noncosingular module is $\mathcal{T}$-noncosingular.

Let M and N be right R-modules. Denote $[M, N] := Hom_R(M, N)$. Beidar and Kasch [2] defined and studied the cosingular ideal $\nabla[M, N]$ such as: $\nabla[M, N] = \{f : M \to N | Im(f) \ll N\}$. Cosingular ideals were studied by some authors (see [10], [12]). Recall that $\alpha \in [M, N]$ is called *regular* if $\alpha = \alpha\beta\alpha$ for some $\beta \in [N, M]$. The module $[M, N]$ is said to be regular if each $\alpha \in [M, N]$ is regular (see [4], [10], [12]). We define δ-cosingular ideal $\nabla_\delta[M, N] = \{f : M \to N | Im(f) \ll_\delta N\}$. Note that M is $\mathcal{T}$-δ-noncosingular relative to N if and only if $\nabla_\delta[M, N] = 0$.

Let M be a module. The injective hull of M is denoted by $E(M)$.

Proposition 1. *Let M and N be right R-modules. If $\nabla_\delta[E(M), E(N)] = 0$, then $\nabla_\delta[M, N] = 0$.*

Proof. Assume that $f \in \nabla_\delta[M, N]$. Then $Im(f) \ll_\delta N$ and there exists $\bar{f} \in [E(M), E(N)]$ such that $\bar{f}|_M = f$. Since $Im(f) \ll_\delta N$, it follows that $Im(f) \ll_\delta E(N)$. Now $\nabla_\delta[E(M), E(N)] = 0$ implies that $\bar{f} = 0$ or $f = 0$. □

Next we will study direct sums of $\mathcal{T}$-δ-noncosingular modules.

Lemma 1. *Let M and N be modules and A a direct summand of M. If $\nabla_\delta[M,N] = 0$, then $\nabla_\delta[A,N] = 0$.*

Proof. Assume that $M = A \oplus A'$ for some A' of M. Let $\varphi \in \nabla_\delta[A,N]$. Then $Im(\varphi) \ll_\delta N$. We consider the homomorphism $\varphi \oplus 0 : A \oplus A' \to N$ defined by $(\varphi \oplus 0)(a + a') = \varphi(a)$ for all $a \in A$ and $a' \in A'$. Then $Im(\varphi \oplus 0) = Im(\varphi) \ll_\delta N$. It follows that $\varphi \oplus 0 = 0$ or $\varphi = 0$. □

Corollary 1. *Every direct summand of an $\mathcal{T}$-δ-noncosingular module is $\mathcal{T}$-δ-noncosingular.*

Theorem 1. *Let $M = \oplus_{i\in I} M_i$ and $N = \oplus_{j\in J} N_j$ be right R-modules, where I, J are arbitrary non-empty sets. Then $\nabla_\delta[M,N] = 0$ if and only if $\nabla_\delta[M_i,N_j] = 0$ for all $i \in I$, $j \in J$.*

Proof. Assume that $\nabla_\delta[M_i,N_j] = 0$ for all $i \in I$, $j \in J$. Let $f \in \nabla_\delta[M,N_j]$. We consider the inclusion $\iota_i : M_i \to M$. Then $Im(f\iota_i) \ll_\delta N_j$ for all $i \in I$, because $Im(f) \ll_\delta N_j$. By the hypothesis, we can obtain that $f\iota_i = 0$ for all $i \in I$. It follows that $f = 0$. Now, let $\varphi \in \nabla_\delta[M,N]$. Then $Im(\varphi) \ll_\delta N$. For each $i \in I$, we consider the projection $\pi_j : N \to N_j$. Let $\varphi_i := \pi_j\varphi : M \to N_j$ for each $j \in J$. Then $Im(\varphi_i) \ll_\delta N_j$. By the hypothesis, we can obtain that $\varphi_i = 0$. Hence $\varphi = 0$. The converse is clear by Lemma 1. □

Corollary 2. *Let $M = \oplus_{i\in I} M_i$ and $N = \oplus_{j\in J} N_j$ be right R-modules, where I, J are arbitrary non-empty sets. Then M is $\mathcal{T}$-δ-noncosingular relative to N if and only if M_i is $\mathcal{T}$-δ-noncosingular relative to N_j for all $i \in I$, $j \in J$.*

Proposition 2. *Let $(M_i)_{i\in I}$ be a family of modules. Then $M = \bigoplus_{i\in I} M_i$ is an $\mathcal{T}$-δ-noncosingular module if and only if M_i is an $\mathcal{T}$-δ-noncosingular module relative to M_j for all $i, j \in I$.*

Proof. The result follows from Corollary 2. □

The following example shows that class of $\mathcal{T}$-δ-noncosingular contains properly the class of δ-noncosingular.

Example 1. *Let $M = \mathbb{Q}\oplus\mathbb{Z}_2$ ba a $\mathbb{Z}$-module. We have $\mathbb{Q}$, $\mathbb{Z}_2$ are $\mathcal{T}$-δ-noncosingular $\mathbb{Z}$-modules and $Hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}_2) = Hom_{\mathbb{Z}}(\mathbb{Z}_2,\mathbb{Q}) = 0$. Then M is a $\mathcal{T}$-δ-noncosingular module by Proposition 2. On the other hand, by* [11, Proposition 2.5]

$$\overline{Z}_\delta(\mathbb{Q} \oplus \mathbb{Z}_2) = \overline{Z}_\delta(\mathbb{Q}) \oplus \overline{Z}_\delta(\mathbb{Z}_2) = \mathbb{Q} \oplus 0.$$

So M is not δ-noncosingular.

In general, a direct sum of $\mathcal{T}$-δ-noncosingular modules is not $\mathcal{T}$-δ-noncosingular, as the following example shows.

Example 2 ([5, Example 2.12])**.** *For any prime integer p, the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}/p\mathbb{Z}$ is not a $\mathcal{T}$-δ-noncosingular $\mathbb{Z}$-module.*

Next, we provide a characterization for an arbitrary direct sum of $\mathcal{T}$-δ-noncosingular modules to be an $\mathcal{T}$-δ-noncosingular module.

A submodule N of M is said to be fully invariant if $f(N)$ is contained in N for every $f \in End(M)$ and denoted by $N \trianglelefteq M$.

Proposition 3. *Let $M = \bigoplus_{i\in I} M_i$ be a direct sum of fully invariant submodules M_i. Then M is $\mathcal{T}$-δ-noncosingular if and only if M_i is $\mathcal{T}$-δ-noncosingular for all $i \in I$.*

Proof. The necessity follows from Corollary 1. Conversely, we need only to show that M_i is an $\mathcal{T}$-δ-noncosingular module relative to M_j for all $i, j \in I$ with $i \neq j$. Suppose that $f : M_i \longrightarrow M_j$ $(i \neq j)$ be a homomorphism. Let $\pi_i : M \longrightarrow M_i$ is the projection map and $\alpha_j : M_j \longrightarrow M$ is the inclusion map. Then $g := \alpha_j f \pi_i \in End_R(M)$ and $g(M) \subseteq M_j$. Since M_i is fully invariant in M, we have $g(M_i) \subseteq M_i$. So, $g(M_i) \subseteq M_i \cap M_j = 0$. Hence, $f = 0$. Consequently, M is $\mathcal{T}$-δ-noncosingular. □

It is easy to see that every module with $\delta(M) = 0$ is $\mathcal{T}$-δ-noncosingular and the converse is not true in general (see Example 1). The following proposition shows that when the converse is true.

Proposition 4. *The following conditions are equivalent for a ring R:*

(1) *Every right R-module is $\mathcal{T}$-δ-noncosingular;*
(2) *Every right R-module is δ-noncosingular;*
(3) *For any right R-module M, $\delta(M) = 0$.*

Proof. (1) $\Longrightarrow$ (2) Let M and N be two modules. Since $M \oplus N$ is $\mathcal{T}$-δ-noncosingular, M is $\mathcal{T}$-δ-noncosingular relative to N by Proposition 2. Therefore, M is δ-noncosingular.

(2) $\Longrightarrow$ (3) Let N be an δ-small submodule of M. By (2), N is δ-noncosingular and so $N = 0$.

(3) $\Longrightarrow$ (1) is obvious. □

Let M be a right R-module such that $S = End_R(M)$ is regular. By [5, Proposition 2.5] M is $\mathcal{T}$-noncosingular, but it is not $\mathcal{T}$-δ-noncosingular.

Proposition 5. *Let M be an R-module such that $S = End_R(M)$ is regular and $T(M) := \{L \subseteq M \mid \delta(L) = L\}$. If $T(M) = 0$, then M is $\mathcal{T}$-δ-noncosingular.*

Proof. Let $f \in End(M)$ with $Imf \ll_\delta M$. So $Imf \subseteq \delta(M)$. Since S is regular, there exists an endomorphism $g \in End(M)$ such that $f = fgf$. Now fg is an idempotent and so that $M = Imfg \oplus Kerfg$. From $Imfg \subseteq Imf \subseteq \delta(M)$ and applying modular law to $\delta(M) = \delta(Imfg \oplus Kerfg) = \delta(Imfg) \oplus \delta(Kerfg)$, we obtain that, $Imfg \cap \delta(M) = Imfg = \delta(Imfg) \oplus (Imfg \cap \delta(Kerfg))$ and so, $Imfg = \delta(Imfg)$. Therefore $Imfg \in T(M) = 0$, or $fg = 0$. Hence $f = 0$. This shows that M is an $\mathcal{T}$-δ-noncosingular. □

A module N is called *direct M-projective* if $M/K \cong P \leq^{\oplus} N$ implies that $K \leq^{\oplus} M$ (see [10]).

Theorem 2. *Let M and N be two modules. If N is direct M-projective and $T(N) = 0$. The following conditions are equivalent:*

(1) *$[M, N]$ is regular;*
(2) (*i*) *$\nabla_\delta[M, N] = 0$.*
    (*ii*) *For any $\alpha \in [M, N]$, there exists a direct decomposition $N = P \oplus Q$ with $P \leq Im(\alpha)$ and $Q \cap Im(\alpha) \ll_\delta N$.*

Proof. (1) $\Longrightarrow$ (2) Let $\alpha \in [M,N]$. Then $Im(\alpha) \leq^{\oplus} N$ (because α is regular). Moreover, if $\alpha \in \nabla_\delta[M,N]$, then by the proof of Proposition 5 we have $\alpha = 0$. The rest is obvious.

(2) $\Longrightarrow$ (1) Let $\alpha \in [M,N]$. By (2), N has a decomposition $N = P \oplus K$ such that $P \leq \alpha(M)$ and $\alpha(M) \cap K \ll_\delta K$. Let $\pi : N \to N$ be the homomorphism such that $\pi^2 = \pi$, $\pi(N) = P$ and $(1_N - \pi)(N) = K$. Then $\pi\alpha : M \to P$ is an epimorphism. Since N is direct M-projective and P is a direct summand of N, $\pi\alpha$ is a splits epimorphism. There exists a homomorphism $\theta : P \to M$ such that $(\pi\alpha)\theta = 1_P$. Let $\gamma = \theta\pi : N \to M$ and so $\pi\alpha\gamma = \pi$. Let $\beta = \gamma\pi$. We have $\beta\alpha\beta = \beta$ and

$$\begin{aligned}(\alpha - \alpha\beta\alpha)(M) &= (1_N - \alpha\beta)\alpha(M)\\ &= \alpha(M) \cap (1_N - \alpha\beta)(N)\\ &= \alpha(M) \cap K \ll_\delta N.\end{aligned}$$

Thus $\alpha - \alpha\beta\alpha \in \nabla_\delta[M,N] = 0$ and hence $\alpha = \alpha\beta\alpha$. □

We recall, a submodule V of an R-module M is an δ-supplement of a submodule U of M if and only if $U + V = M$ and $U \cap V \ll_\delta V$. An R-module M is δ-supplemented if every submodule of M has an δ-supplement in M (see [7]).

Theorem 3. *Let M and N be two modules. Assume that N is M-projective. If N is δ-supplemented and $T(N) = 0$, then the following conditions are equivalent:*

(1) *$[M,N]$ is regular;*
(2) *$\nabla_\delta[M,N] = 0$.*

Proof. (1) $\Longrightarrow$ (2) By Theorem 2.

(2) $\Longrightarrow$ (1) Assume that N is δ-supplemented and $\nabla_\delta[M,N] = 0$. Let $\alpha \in [M,N]$ and $\alpha \neq 0$. Then $Im(\alpha)$ is not δ-small in N. Since N is δ-supplemented, there exists a submodule L of N such that $N = Im(\alpha) + L$ and $Im(\alpha) \cap L \ll_\delta N$. We consider the canonical projection $\pi : N \longrightarrow N/L$. Then $\pi\alpha : M \longrightarrow N/L$ is an epimorphism. On the other hand, since N is M-projective, there exists a homomorphism $\beta \in [N,M]$ such that $\pi\alpha\beta = \pi$. It follows that $Im(\alpha - \alpha\beta\alpha) \leq Im(\alpha) \cap L$ and so $\alpha - \alpha\beta\alpha \in \nabla_\delta[M,N] = 0$. Thus $\alpha = \alpha\beta\alpha$. □

We will use the following notation, where M and N are R-modules:

$$Z_{\delta-M}(N) = \bigcap_{\varphi \in \nabla_\delta[M,N]} Ker(\varphi)$$

Theorem 4. *Let M and N be modules. Then:*

(1) *$\overline{Z}_\delta(M)$ is a submodule of $Z_{\delta-M}(N)$.*
(2) *$Z_{\delta-M}(N)$ is a fully invariant submodule of M.*
(3) *$\nabla_\delta[M,N] = 0$ if and only if $M/Z_{\delta-M}(N) = 0$.*

Proof. (1) It is clear.

(2) Let $\varphi \in \nabla_\delta[M,N]$. Then $Im(\varphi) \ll_\delta N$ and so, for all $f \in End_R(M)$, we can obtain that $Im(\varphi f) \leq Im(\varphi)$ and so $\varphi f \in \nabla_\delta[M,N]$. Therefore $Z_{\delta-M}(N)$ is a fully invariant submodule of M.

(3) $M/Z_{\delta-M}(N) = 0$ if and only if $M = Z_{\delta-M}(N)$ if and only if $\varphi = 0$ for all $\varphi \in [M,N]$ with $Im(\varphi) \ll_\delta N$. □

Corollary 3. *Let M and N be modules. Then M is $\mathcal{T}$-δ-noncosingular relative to N if and only if $Z_{\delta-M}(N) = M$.*

Theorem 5. *Let $M = \oplus_{i\in I}M_i$ and $N = \oplus_{j\in J}N_j$ be R-modules. Then $Z_{\delta-M}(N) = \bigoplus_{i\in I} Z_{\delta-M_i}(N)$*

Proof. By Theorem 4, $Z_{\delta-M}(N)$ is a fully invariant submodule of M. So $Z_{\delta-M}(N) = \bigoplus_{i\in I}[M_i \cap Z_{\delta-M}(N)]$. For $i \in I$, let $m \in Z_{\delta-M_i}(N)$. Then $\varphi(m) = 0$ for all $\varphi \in \nabla_\delta[M_i, N]$. Let $\iota_i : M_i \longrightarrow M$ be the inclusion map. For any $\alpha \in \nabla_\delta[M, N]$, we can obtain that $Im(\alpha\iota_i) \subseteq Im(\alpha) \ll_\delta N$ which implies that $Im(\alpha\iota_i) \ll_\delta N$ or $\alpha\iota_i \in \nabla_\delta[M, N]$. It follows that $\alpha(m) = \alpha\iota_i(m) = 0$, i.e, $m \in M_i \cap Z_{\delta-M_i}(N)$. This shows that $Z_{\delta-M}(N) \subseteq \bigoplus_{i\in I} Z_{\delta-M_i}(N)$. □

Proposition 6. *Let R be a ring and $x \in R$ such that $Ann(x)$, the right annihilator of x, is an ideal of R. Then $M = xR$ is $\mathcal{T}$-δ-noncosingular if and only if $\delta(M) = 0$.*

Proof. Suppose that M is $\mathcal{T}$-δ-noncosingular and $\delta(M) \neq 0$. Therefore there exists $a \in R$ such that $xa \neq 0$ and $xa \in \delta(M)$. Consider the endomorphism f of M defined by $f(x\beta) = xa\beta$ for every $\beta \in R$. So f is well defined since $Ann(x)$ is an ideal of R. Thus, $Imf \subseteq \delta(M)$ and $f \neq 0$. However, by [16, Lemma 1.5], $\delta(M) \ll_\delta M$. Then M is not $\mathcal{T}$-δ-noncosingular, which is a contradiction. The converse is clear. □

Corollary 4. *A ring R is right (left) $\mathcal{T}$-δ-noncosingular if and only if $\delta(R_R) = 0$.*

Example 3. (1) *Let $R = \mathbb{Z}_6$. Then $J(\mathbb{Z}_6) = 0$ and $\delta(\mathbb{Z}_6) = \mathbb{Z}_6$. By* [5, Corollary 2.7] *and Corollary 4, R is $\mathcal{T}$-noncosingular but not $\mathcal{T}$-δ-noncosingular.*

(2) *It is known that every regular ring has zero Jacobson radical. So by* [5, Corollary 2.7]*, each regular ring is $\mathcal{T}$-noncosingular, but there are regular rings R with $\delta(R) \neq 0$. Let $Q = \prod_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $\bigoplus_{i=1}^{\infty} F_i$ and 1_Q. Then R is regular and $\bigoplus_{i=1}^{\infty} F_i = \delta(R)$. Then by Corollary 4, R is not $\mathcal{T}$-δ-noncosingular.*

(3) *Let F be a field, $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Consider the ring*

$$R = \{(x_1, ..., x_n, x, x, ...) : n \in \mathbb{N}, x_i \in M_2(F), x \in I\}$$

with component-wise operations. By [9, Example 2.5]*, $J(R) = 0$ and so R is $\mathcal{T}$-noncosingular. But by* [16, Example 4.3]*, $\delta(R) = \{(x_1, ..., x_n, x, x, ...) : n \in \mathbb{N}, x_i \in M_2(F), x \in K\}$ where $K = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Thus R is not $\mathcal{T}$-δ-noncosingular.*

For a singular module, the concepts $\mathcal{T}$-noncosingular and $\mathcal{T}$-δ-noncosingular are the same.

Example 4 ([8, Example 4.1])**.** *Let $A = \prod_{n=1}^{\infty} \mathbb{Z}_2$. Consider*

$$T = \{(a_n)_{n=1}^{\infty} \in A \mid a_n \text{ is eventually constant } \},$$

and

$$I = \{(a_n)_{n=1}^{\infty} \in A \mid a_n = 0 \;\; \text{eventually } \} = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2.$$

Set the ring $R = \begin{pmatrix} T & T/I \\ 0 & T/I \end{pmatrix}$, and consider the idempotent

$$e = \begin{pmatrix} (1, 1, ...) & 0 + I \\ 0 & 0 + I \end{pmatrix} \in R.$$

Then $M = eR = \begin{pmatrix} T & T/I \\ 0 & 0 \end{pmatrix}$. *By* [5, Proposition 2.5], *M is an $\mathcal{T}$-noncosingular module, since* $S = End_R(M) = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ *is regular. But since M is not singular, M is not $\mathcal{T}$-δ-noncosingular.*

3. Applications

In this section, we consider some applications of $\mathcal{T}$-δ-noncosingular modules. Let M be a right R-module such that $S = End_R(M)$ the ring of all R-endomorphisms of M. Let I be a subset of S and a N subset of M. We denote

$$E_M(I) = \sum\{Im\phi \mid \phi \in I\},$$

and

$$D_S(N) = \{\phi \in S \mid Im\phi \subseteq N\}.$$

Proposition 7. *Let R be a commutative ring. If R is $\mathcal{T}$-δ-noncosingular, then R is nonsingular.*

Proof. Since R is $\mathcal{T}$-δ-noncosingular, by Corollary 4, we have $\delta(R) = 0$. So R is a semiprime ring. Therefore, $Z(R) = 0$ by [3, Proposition 1.27(b)]. □

The converse of Proposition 7 is not true in general. Because every regular ring is nonsingular. The ring in Example 3(2) is nonsingular, but is not $\mathcal{T}$-δ-noncosingular.

Theorem 6. *The following statements are equivalent for a commutative ring R:*

(1) *Every $\mathcal{T}$-δ-noncosingular R-module M with $\delta(M) \ll_\delta M$ is semisimple;*
(2) *Every finitely generated $\mathcal{T}$-δ-noncosingular module is semisimple;*
(3) *$R/\delta(R)$ is semisimple.*

Proof. (1) $\Longrightarrow$ (2) Clear.

(2) $\Longrightarrow$ (3) Since $\delta(R/\delta(R)) = 0$, $R/\delta(R)$ is $\mathcal{T}$-δ-noncosingular as an R-module. The result follows by (2).

(3) $\Longrightarrow$ (1) Let M be an $\mathcal{T}$-δ-noncosingular R-module with $\delta(M) \ll_\delta M$. Since $R/\delta(R)$ is a semisimple, $M/M\delta(R)$ is semisimple $R/\delta(R)$-module and $\delta(M) = M\delta(R)$ by [16, Theorem 1.8]. If $a \in \delta(R)$ and f is the endomorphism of M defined by $f(x) = xa$ for all $x \in M$, then we have $Imf = Ma \subseteq M\delta(R) \ll_\delta M$. Because M is $\mathcal{T}$-δ-noncosingular, then $Ma = 0$. Thus, $\delta(M) = 0$. This implies that M is semisimple. □

Let P and M be two modules. An epimorphism $p : P \longrightarrow M$ is called a projective δ-cover of M, in case P is projective and $Kerf \ll_\delta P$ (see [16]).

Lemma 2. *Let N be an δ-small submodule of a module M. If L is a submodule of M such that $(L + N)/N \ll_\delta M/N$, then $L \ll_\delta M$.*

Proof. Let X be a submodule of M such that $L + X = M$ where M/X is singular. Then $(L+N)/N+(X+N)/N = M/N$ and $M/(X+N)$ is singular. By hypothesis, we have $(X + N)/N = M/N$. Since $N \ll_\delta M$, then $X = M$. It follows that $L \ll_\delta M$. □

Proposition 8. *Let M be a module with a projective δ-cover P such that P is $\mathcal{T}$-δ-noncosingular. Then M is $\mathcal{T}$-δ-noncosingular.*

Proof. By hypothesis, $f : P \longrightarrow M$ is an epimorphism with $Kerf \ll_\delta P$. Thus $P/Kerf \cong M$. Let $\phi \in End(P/Kerf)$ such that $Im\phi \ll_\delta P/Kerf$. Consider the natural epimorphism $\pi : P \longrightarrow P/Kerf$. Since P is projective, there exists a homomorphism $h : P \longrightarrow P$ such that $\phi\pi = \pi h$. Therefore, $\pi h(P) = \phi(P/Kerf) \ll_\delta P/Kerf$. So $h(P) \ll_\delta P$ by Lemma 2. But P is $\mathcal{T}$-δ-noncosingular. Then $h = 0$, and hence $\phi\pi = 0$. This implies that $\phi = 0$. Thus, M is $\mathcal{T}$-δ-noncosingular. □

Proposition 9. *The following conditions are equivalent for a ring R:*

(1) *R_R is $\mathcal{T}$-δ-noncosingular;*
(2) *Every projective right R-module is $\mathcal{T}$-δ-noncosingular;*
(3) *Every right R-module having a projective δ-cover is $\mathcal{T}$-δ-noncosingular.*

Proof. (1) $\Longrightarrow$ (2) By Corollary 4, $\delta(R) = 0$. Let P be a projective right R-module, by [16, Lemma 1.9], $\delta(P) = P(\delta(R)) = 0$. So, P is $\mathcal{T}$-δ-noncosingular.

(2) $\Longrightarrow$ (3) This follows from Proposition 8.

(3) $\Longrightarrow$ (1) This is obvious. □

It follows from [15, Theorem 1.6] that every cyclic submodule of a regular module is a direct summand of that module.

Proposition 10. *Any singular regular module is a $\mathcal{T}$-δ-noncosingular module.*

Proof. Let M be a regular module. Then for any $x \in M$, we have $M = xR \oplus N$ for some $N \subseteq M$. Let $\delta(M) \neq 0$. Then there is a non-zero element $x \in \delta(M)$ and so $xR \ll_\delta M$. Since M is regular, $xR = 0$ for every $r \in R$, and $x = 0$. Thus $\delta(M) = 0$. Therefore M is $\mathcal{T}$-δ-noncosingular. □

The converse of Proposition 10, is not true, for example the $\mathbb{Z}$-module $\mathbb{Q}$ is $\mathcal{T}$-δ-noncosingular, but $\mathbb{Q}$ can not be a regular $\mathbb{Z}$-module since $Hom_Z(\mathbb{Q}, \mathbb{Z}) = 0$.

Lemma 3. *For $N \leq M, I \leq S, P \trianglelefteq M, L \trianglelefteq S$, the following conditions hold:*

(1) *$E_M(D_S(E_M(I))) = E_M(I)$.*
(2) *$D_S(E_M(D_S(N))) = D_S(N)$.*
(3) *$E_M(L) \trianglelefteq M$.*
(4) *$D_S(P) \trianglelefteq S$.*

Proof. (1) $E_M(D_S(E_M(I))) = \sum_{\phi\in D_S(E_M(I))} Im\phi \subseteq E_M(I)$. Conversely, since $I \subseteq D_S(E_M(I))$, then $E_M(I) \subseteq E_M(D_S(E_M(I)))$.

(2) Similar to (1).

(3) If $L \trianglelefteq S$, $f : M \to M, f(\sum_{\phi\in L} Im\phi) = \sum_{\phi\in L} Imf\phi \leq \sum_{\phi\in L} Im\phi$, (since $\phi \in L, L \trianglelefteq S$ then $f\phi \in L$). Therefore $E_M(L) \trianglelefteq M$.

(4) $D_S(P) \leq S_S$. On the other hands, if $P \trianglelefteq M, \phi \in D_S(P)$, then $\phi(M) \subseteq P$, and so $\psi\phi(M) \subseteq \psi(P) \subseteq P, \forall\psi \in S$. Hence $\psi\phi \in D_S(P)$. Therefore $D_S(P) \trianglelefteq S$. □

Recall that a module M is *dual Baer* if for any submodule N of M, there exists an idempotent e in $S = End(M)$ such that $D_S(N) = eS$ (see [6]). We call a module M is an *δ-$\mathcal{K}$-module* if, for all $N \leq M, D_S(N) = 0$, implies $N \ll_\delta M$ (equivalently, $Im\phi \nleq N$ for all $0 \neq \phi \in S$ implies $N \ll_\delta M$). A module M is called an *δ-lifting module* if, for all $N \leq M$, $N = A \oplus D$ where A is a direct summand of M and $D \ll_\delta M$ (see [7]).

Proposition 11. *Let M be an R-module and $S := End(M)$. Then:*

(1) *M is $\mathcal{T}$-δ-noncosingular if and only if, for all $I \leq S_S, E_M(I) = eM \oplus L$ for $e^2 = e \in S, L \ll_\delta M$, implies that $I \cap (1-e)S = 0$;*

(2) *M is an δ-$\mathcal{K}$-module if and only if, for all $N \leq M, E_M(D_S(N)) \leq^{\oplus} M$ that $N = E_M(D_S(N)) \oplus L$ such that $L \ll_\delta M$.*

Proof. (1) Let $I \leq S_S$ such that $E_M(I) = eM \oplus L$ for $e^2 = e \in S, L \ll_\delta M$. Then $E_M(I \cap (1-e)S) \subseteqq E_M(I) \cap (1-e)M = (eM \oplus L) \cap (1-e)M \subseteq (1-e)M \cap (1-e)L$. Since $L \ll_\delta M, (1-e)L \ll_\delta M$, therefore $(1-e)M \cap (1-e)L \ll_\delta M$. Hence $E_M(I \cap (1-e)S) \ll_\delta M$. Since M is $\mathcal{T}$-δ-noncosingular, $I \cap (1-e)S = 0$.

Conversely, let $I = \phi S \leq S$ with $Im(\phi) \ll_\delta M$. Then $E_M(I) \ll_\delta M$. By hypothesis we have that $I \cap S = 0$, thus $I = 0$.

(2) $E_M(D_S(N)) = eM$ for $e^2 = e \in S$ and so $eM \subseteq N$. By Lemma 3 $D_S(N) = D_S(eM)$. Since $D_S(eM) \cap D_S((1-e)M \cap N) = 0$ and $D_S((1-e)M \cap N) \subseteq D_S(N) = D_S(eM)$. $D_S((1-e)M \cap N) = 0$. Since M is an δ-$\mathcal{K}$-module, $(1-e)M \cap N \ll_\delta M$, which implies that $N = E_M(D_S(N)) \oplus ((1-e)M \cap N)$ with $(1-e)M \cap N \ll_\delta M$.

Conversely, let $N \leq M$ and $D_S(N) = 0$. Then $E_M(D_S(N)) = 0$. By our assumption, $N = E_M(D_S(N)) \oplus D$ with $L \ll_\delta M$. Therefore $N = L \ll_\delta M$. □

Corollary 5. *A module M is $\mathcal{T}$-δ-noncosingular if and only if for all $I \leq S$, $E_M(I) \ll_\delta M$ implies $I = 0$.*

Lemma 4. *An δ-$\mathcal{K}$-module dual Baer module M is an δ-lifting module.*

Proof. Let $N \leq M$ and $D_S(N) = eS$ (by dual Baer property). Hence $eM \subseteqq N$ and so $N = eM \oplus ((1-e)M) \cap N)$. Let $\phi \in End(M), Im\phi \leq (1-e)M \cap N$. Since $Im\phi \leq N$, then $\phi \in D_S(N) = eS$. Therefore $\phi = e\psi$ for $\psi \in End(M)$. Hence $\phi(M) \leq (1-e)M \cap eM = 0$, which implies that $\phi = 0$. So by δ-$\mathcal{K}$-module we will have $(1-e)M \cap N \ll_\delta M$. It follows that M is δ-lifting. □

Lemma 5. *An $\mathcal{T}$-δ-noncosingular δ-lifting module M is dual Baer.*

Proof. Assume that M is a $\mathcal{T}$-δ-noncosingular δ-lifting module. Let $N \leq M$. Since N is δ-lifting, $N = eM \oplus B$ such that $e^2 = e \in S$, $B \ll_\delta M$. Hence $eS = D_S(eM) \leq D_S(N)$. Assume that inclusion is strict. Then there exists $\phi \in D_S(N) \setminus eS$. Since $S = eS \oplus (1-e)S$, we have that $\phi = es_1 + (1-e)s_2$ for some $s_1, s_2 \in S$, with $s_2 \neq 0$. Replacing ϕ with $\phi - es_1 \in D_S(N)$, we can safely assume ϕ is in $(1-e)S$. We obtain that $Im\phi \leq N$ and $Im\phi \leq Im(1-e)$, and so $Im\phi \leq N \cap Im(1-e) = (eM \oplus B) \cap Im(1-e) \leq (1-e)M \cap (1-e)B$. Since $B \ll_\delta M$, then $(1-e)B \ll_\delta M$. Therefore $Im\phi \ll_\delta M$. Hence by $\mathcal{T}$-δ-noncosingularity of M, we conclude that $\phi = 0$ which contradicts our hypothesis. Therefore $D_S(N) = eS$, and so M is dual Baer. □

Lemma 6. *Let M be $\mathcal{T}$-δ-noncosingular and X is a fully invariant submodule of M. Let $X = N \oplus B$ with $B \ll_\delta M$ and N a direct summand of M. Then N is a fully invariant submodule of M.*

Proof. We have $M = N \oplus P$. Let $\phi \in S$. We will show $\phi(N) \subseteq N$. Take $\psi = \pi_P(\phi(\pi_N))$, with π_N and π_P the respective canonical projections. Let $x \in N$ so that $\phi(x) \notin N$, hence $\psi(x) \neq 0$. But $\pi_P\phi\pi_N(M) = \pi_P\phi\pi_N(N \oplus P) = \pi_P\phi(N) \subseteq \pi_P(X) = X \cap P$, since X is a fully invariant submodule of M. On the other hand $X = N \oplus (X \cap P) = N \oplus B$. Therefore $X \cap P \cong B$ and so $X \cap P \ll_\delta M$. Hence

$Im\psi = Im\pi_P\phi\pi_N \ll_\delta M$. This is a contradiction, since M is $\mathcal{T}$-δ-noncosingular. It follows that $\phi(N) \subseteq N$. □

Proposition 12. *For an $\mathcal{T}$-δ-noncosingular module M then the following conditions are equivalent:*

(1) *For any fully invariant submodule A of M, there exists a direct summand B of M such that $A/B \ll_\delta M/B$;*
(2) *For any fully invariant submodule A of M, there exists a fully invariant direct summand B of M such that $A/B \ll_\delta M/B$.*

Proof. (1) $\Longrightarrow$ (2) Let X be a fully invariant submodule of M. By (1), there exists a decomposition $X = N \oplus B$, where $B \ll_\delta M$ and N is direct summand of M. But by Lemma 6, N is fully invariant. This complete the proof.

(2) $\Longrightarrow$ (1) Trivial. □

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