PARTIAL SUMS OF A GENERALIZED CLASS OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED SRIVASTAVA-ATTIYA OPERATOR

K.A. CHALLAB, M. DARUS, F. GHANIM

Abstract. Let \( f_n(z) = z + \sum_{k=2}^{n} a_k z^k \) be the sequence of partial sums of the analytic function \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \). In this paper, we determine sharp lower bounds for \( \Re\{ f(z)/f_n(z) \} \), \( \Re\{ f_n(z)/f(z) \} \), \( \Re\{ f'(z)/f'_n(z) \} \) and \( \Re\{ f''(z)/f'(z) \} \). The efficiency of the main result not only provides the unification of the results discussed in the literature but also generates certain new results.

Keywords: analytic functions, Hadamard product (or convolution), generalized Hurwitz—Lerch zeta function, Srivastava-Attiya operator.

1. Introduction and Preliminaries

Let \( A(U) \) denote a class of all analytic functions defined in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). For \( a \in \mathbb{C}, j \in \mathbb{N} = \{1, 2, \ldots\} \), let

\[
A[a, j] = \{ f \in A(U) : f(z) = a + a_j z^j + a_{j+1} z^{j+1} + \ldots \}.
\]

We denote a subclass of \( A[a, 1] \) by \( A \) whose members are of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U).
\]

We denote by \( C \) the class of convex (univalent) functions in \( U \) and satisfying

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in U).
\]
Using a Schwarzian function \( w \) with the conditions \( w(0) = 0 \) and \( |w(z)| < 1 \) and analytic in \( U \), with \( f \) and \( g \) as its analytic functions in \( U \). Subsequently, \( f \) is a subordinate to \( g \) (symbolically \( f < g \)) if \( f(z) = g(w(z)) \) is satisfied.

Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence:

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

The Srivastava-Attiya operator is defined as \([23]\) (see also \([3],[17],[30]\)):

\[
J_{s,a}(f)(z) = z + \sum_{k=2}^{\infty} \left( \frac{1 + a}{k + a} \right)^s a_k z^k
\]

Where \( z \in U, a \in C \setminus Z_0, s \in C \) and \( f \in A[a,1] \).

In fact, the linear operator \( J_{s,a}(f) \) can be written as

\[
J_{s,a}(f)(z) := G_{s,a}(z) \ast f(z).
\]

In terms of Hadamard product (or convolution) where \( G_{s,a}(z) \) is given by

\[
(2) \quad G_{s,a}(z) := (1 + a)^s [\Phi(z,s,a) - a^{-s}] \quad (z \in U),
\]

the function \( \Phi(z,s,a) \) involved in the right-hand side of (2) is the well known Hurwitz-Lerch zeta function defined by: (see, for example, \([5],[24]\), p. 121 et seq.)

\[
\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s}
\]

\((a \in C \setminus Z_0^+ : s \in C \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1).\)

Recently, a new family of \( \gamma \)-generalized Hurwitz-Lerch zeta function was investigated by Srivastava \([26]\) (see also \([3],[5],[25]\)). Srivastava considered the following function:

\[
\Phi_{\gamma_1,\ldots,\gamma_p;\sigma_1,\ldots,\sigma_q}(z,s,a;b,\gamma)
\]

\[
= \frac{1}{\Gamma(s)} \cdot \sum_{k=0}^{\infty} \frac{\Pi_{j=1}^{p} \gamma_j \cdot \Pi_{j=1}^{q} \mu_j}{(a + k)^s} \cdot H_{2,0}^{2,0} \left[ (a + k) b^\frac{1}{\gamma} |(s,1), \left( \frac{0,1}{\gamma} \right) \right] \frac{z^k}{k!},
\]

\((\min \{ \Re(a), \Re(s) \} > 0; \Re(b) > 0; \gamma > 0),\)

where

\[
\gamma_j \in C(j = 1,\ldots,p) \text{ and } \mu_j \in C \setminus Z_0(j = 1,\ldots,q); \rho_j > 0(j = 1,\ldots,p); \sigma_j > 0(j = 1,\ldots,q); 1 + \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \geq 0
\]

and the equality in the convergence condition holds true for suitably bounded values of \(|z|\) given by

\[
|z| < \nabla := \left( \Pi_{j=1}^{p} \rho_j^{-\rho_j} \right) \cdot \left( \Pi_{j=1}^{q} \sigma_j^{\sigma_j} \right).
\]
Here, and for the remainder of this paper, \((\gamma)_k\) denotes the Pochhammer symbol defined, in terms of Gamma function, by

\[
(\gamma)_k := \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} = \begin{cases} 
\gamma(\gamma+1)\ldots(\gamma+n-1) & (k = n \in \mathbb{N}; \ \gamma \in \mathbb{C}) \\
1 & (k = 0; \ \gamma \in \mathbb{C}\setminus\{0\})
\end{cases}.
\]

Definition 1. The H-function involved in the right-hand side of (3) is the well-known Fox's H-Function [19, Definition 1.1] (see also [29]) defined by

\[
H^{m,n}_{p,q}(z) = H^{m,n}_{p,q} \left[ \left( \begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p) \\ \{b_1, B_1}, \ldots, \{b_q, B_q} \end{array} \right); z \right]
\]

\[
(4) \quad = \frac{1}{2\pi i} \int_{\ell} \Xi(s) z^{-s} ds \quad (z \in \mathbb{C}\setminus\{0\}; \ |\arg(z)| < \pi),
\]

where

\[
\Xi(s) = \prod_{j=1}^{p} \Gamma(b_j + B_j, s) \cdot \prod_{j=1}^{q} (1 - a_j - A_j, s) \cdot \prod_{j=m+1}^{p} (1 - b_j - B_j, s),
\]

an empty product is interpreted as 1, \(m, n, \ p, \ q\) are integers such that

\[
\begin{align*}
1 & \leq m \leq q, \ 0 \leq n \leq p, \ A_j > 0(j = 1, \ldots, p), \ B_j > 0(j = 1, \ldots, q), \\
& a_j \in \mathbb{C}(j = 1, \ldots, p), \ b_j \in \mathbb{C}(j = 1, \ldots, q)
\end{align*}
\]

and \(\ell\) is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

\[
\{\Gamma(b_j + B_j, s)\}_{j=1}^{m}
\]

from the poles of gamma functions

\[
\{\Gamma(1 - a_j + A_j, s)\}_{j=1}^{m}.
\]

It is worthy to mention that using the fact that [4] and [22, p.1496, Remark 7]

\[
\lim_{b \to 0} \left\{ H^{0,2}_{0,2} \left[ \left( a + k \right) b^+ \left( s, 1 \right), \left( 0, \frac{1}{\gamma} \right) \right] \right\} = \gamma \Gamma(s) \quad (\lambda > 0),
\]

Eq. (4) reduces to

\[
\Phi(k_{1, \ldots, p, \gamma_{1, \ldots, q}, \mu_{1, \ldots, q}})(z, s, a; 0, \gamma) := \Phi(k_{1, \ldots, p, \gamma_{1, \ldots, q}, \mu_{1, \ldots, q}})(z, s, a) = \sum_{k=0}^{\infty} \prod_{j=1}^{p} (\gamma_j)^{k_{\rho_j}} \left( a + n \right)^{k} \cdot \prod_{j=1}^{q} (\mu_j)^{k_{\sigma_j}} \frac{z^k}{k!}.
\]

Definition 2. The function \(\Phi(k_{1, \ldots, p, \gamma_{1, \ldots, q}, \mu_{1, \ldots, q}})(z, s, a)\) involved in (5) is the multiparameter extension and generalization of the Hurwtiz- Lerch zeta function \(\Phi(z, s, a)\) introduced by Srivastava et al.[6] and [31], p. 503, Eq.(6.2) defined by

\[
\Phi(k_{1, \ldots, p, \gamma_{1, \ldots, q}, \mu_{1, \ldots, q}})(z, s, a) := \sum_{k=0}^{\infty} \prod_{j=1}^{p} (\gamma_j)^{k_{\rho_j}} \frac{z^k}{k!}.
\]
with
\[ \nabla^* := \left( \prod_{j=1}^{p} \rho_j^{-\rho_j} \right) \cdot \left( \prod_{j=1}^{q} \sigma_j^{-\sigma_j} \right), \]
and
\[ \Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \text{ and } \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \gamma_j + \frac{p - q}{2}. \]

The following linear operator was introduced by Srivastava and Gaboury [25]:
\[ J_{(\gamma_p), (\mu_q), b}^s, a, \gamma (f) : A(U) \to A(U) \]
defined by
\[ (6) \quad J_{(\gamma_p), (\mu_q), b}^s, a, \gamma (f) (z) = G_{(\gamma_p), (\mu_q), b}^{s, a, \gamma} (z) * f (z), \]
where * denotes the Hadamard product (or convolution) of analytic functions and function \( G_{(\gamma_p), (\mu_q), b}^{s, a, \gamma} (z) \) is given by
\[ (7) \quad G_{(\gamma_p), (\mu_q), b}^{s, a, \gamma} (z) := \frac{\gamma \prod_{j=1}^{p} \left( \mu_j \right)}{\prod_{j=1}^{p} \left( \gamma_j \right)} \times \Lambda (a + b, s, \gamma, \gamma) \left[ \Phi_{1, \gamma_1, \gamma_2, \ldots, \gamma_p, \mu_1, \ldots, \mu_q} (z, s, a; b, \gamma) - \frac{a - s}{\gamma} \Lambda (a + b, s, \gamma) \right], \]
with
\[ \Lambda (a, b, s, \gamma) := H_{0, 2}^2 \left[ ab^\gamma | (s, 1), \left( \frac{0}{1}, \frac{1}{\gamma} \right) \right]. \]

Combining (6) and (7), we obtain
\[ J_{(\gamma_p), (\mu_q), b}^{s, a, \gamma} (f) (z) = z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^{p} \left( \gamma_j + 1 \right)}{\prod_{j=1}^{p} \left( \mu_j + 1 \right)} \left( a + 1 \right)^s \left( a + k \right) \left( \frac{\Lambda (a + k, b, s, \gamma)}{\Lambda (a + b, s, \gamma)} \right) \frac{z^k}{k!} \]

\( (\gamma_j \in \mathbb{C} (j = 1, ..., p) \text{ and } \mu_j \in \mathbb{C} \setminus \mathbb{Z} \quad (j = 1, ..., q); \quad p \leq q + 1; \quad z \in U) \), with
\[ \min \{ \Re (a), \Re (s) \} > 0; \quad \gamma > 0 \text{ if } \Re (b) > 0 \]
and
\[ s \in \mathbb{C}; \quad a \in \mathbb{C} \setminus \mathbb{Z} \quad \text{if } b = 0. \]

Motivated essentially by the Srivastava-Attiya operator, Xiang et al. [32] (see also Murugusundaramoorthy et al. [15]) introduced and investigated the integral operator
\[ J_{a, \gamma}^{a, s} = z + \sum_{k=2}^{\infty} \left( \frac{1 + k}{a + k} \right)^s \frac{\lambda (k + \delta - 2)!}{(\sigma - 2)! (k + \lambda - 1)!} a_k z^k. \]

Now we define a new integral operator
\[ J_{(\gamma_p), (\mu_q), b}^{s, a, \lambda, \delta, \gamma} (f) (z) = z + \sum_{k=2}^{\infty} \frac{\prod_{j=1}^{p} \left( \gamma_j + 1 \right)}{\prod_{j=1}^{p} \left( \mu_j + 1 \right)} \left( a + 1 \right)^s \left( a + k \right) \left( \frac{\Lambda (a + k, b, s, \gamma)}{\Lambda (a + b, s, \gamma)} \right) \frac{z^k}{k!} \]
\[ \times \left( \frac{\lambda (k + \delta - 2)!}{(\delta - 2)! (k + \lambda - 1)!} \frac{a_k z^k}{k!} \right) (z \in U), \]
Also, we will determine sharp lower bounds for \( \Re\{\) 

We study the ratio of a function of the form (1) to its sequence of partial sums of \([18], [20]) and recent work (see [11], [16], [28]) on partial sums of analytic functions, sums.

when the coefficients of \( f \) and \( F \) where

\[
J^{s,a,\lambda,\delta,\gamma}_{(\gamma p),(\mu q),b} = z + \sum_{k=2}^{\infty} \Omega_k^\beta(a,s) a_k z^k = F(z),
\]

and

\[
\Omega_k^\beta(a,s) = \frac{\prod_{j=1}^{p}(\gamma_j + 1)k_{k-1}}{\prod_{j=1}^{q}(\mu_j + 1)k_{k-1}} \left( \frac{a + 1}{a + k} \right)^s \\
\times \left( \frac{\Lambda(a + k, b, s, \gamma)}{\Lambda(a + 1, b, s, \gamma)} \right)^{\frac{\lambda(k + \delta - 2)}{(\delta - 2)!}k_{k-1}^0} 1
\]

(8)

where (and throughout this paper unless otherwise mentioned) the parameters \( s, a, b, \delta \) and \( \lambda \) are constrained as follows:

\( s \in \mathbb{C}; \ a \in \mathbb{C} \setminus \mathbb{Z}_0^+; \ \delta > 2 \) and \( \lambda > -1. \)

We note that \( J^{s,a,1,1,\lambda,\gamma}_{(\gamma p),(\mu q),b} \) is the generalized Srivastava-Attiya operator [27]. Motivated by Murugusundaramoorthy ([12], [13], [14]) and making use of the generalized Srivastava-Attiya operator \( J^{s,a,\lambda,\delta,\gamma}_{(\gamma p),(\mu q),b} \), we define the following new subclass of analytic functions with negative coefficients.

For \( \alpha \geq 0, -1 \leq \eta < 1, \) and \( \beta \geq 0, \) let \( P_{\alpha}^\eta(\eta, \beta) \) be the subclass of \( A \) consisting of functions of the form (1) and satisfying the analytic criterion

\[
\Re \left\{ \frac{z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma p),(\mu q),b} f(z))^\prime + \alpha z^2(J^{s,a,\lambda,\delta,\gamma}_{(\gamma p),(\mu q),b} f(z))^\prime^\prime}{(1 - \alpha)J^{s,a,\lambda,\delta,\gamma}_{(\gamma p),(\mu q),b} f(z) + \alpha z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma p),(\mu q),b} f(z))^\prime)} - \eta \right\} > \beta \left[ \frac{zG(z)}{G(z)} - \eta \right] > \beta \left| \frac{zG(z)}{G(z)} - 1 \right|
\]

(9)

where \( z \in U. \) Shortly we can state this condition by

\[
\Re \left\{ \frac{zG^\prime(z)}{G(z)} - \eta \right\} > \beta \left| \frac{zG^\prime(z)}{G(z)} - 1 \right|
\]

where

\[
G(z) = (1 - \alpha)F(z) + \alpha zF^\prime(z) = z + \sum_{k=2}^{\infty} [1 + \alpha(k - 1)]\Omega_k^\beta(a,s) a_k z^k,
\]

and \( F(z) = J^{s,a,\lambda,\delta,\gamma}_{(\gamma p),(\mu q),b} \) with \( \Omega_k^\beta(a,s) \) given by (8).

Silverman [21] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums.

In the present paper and following the earlier work by Silverman [21] (see [8], [18], [20]) and recent work (see [11], [16], [28]) on partial sums of analytic functions, we study the ratio of a function of the form (1) to its sequence of partial sums of the form

\[
f_n(z) = z + \sum_{k=2}^{n} a_k z^k,
\]

when the coefficients of \( f(z) \) satisfy the condition (9) a function \( f(z) \) is in \( P_{\alpha}^\eta(\eta, \beta) \).

Also, we will determine sharp lower bounds for \( \Re\{f(z)/f_n(z)\}, \ Re\{f_n(z)/f(z)\}, \)
\[ \Re\{f'(z)/f''(z)\}, \quad \text{and} \quad \Re\{f''(z)/f'(z)\}. \] It is seen that this study not only gives us a particular case, the results of Silverman [21], but also gives rise to several new results.

Before stating and proving our main results, we derive a sufficient condition giving the coefficient estimates for functions \( f(z) \) to belong to this generalized function class.

**Lemma 1.** A function \( f(z) \) of the form (1) is in \( P_\alpha^\beta(\eta) \) if

\[
(10) \quad \sum_{k=2}^{\infty} (1 + \alpha(k-1))[k(1 + \beta) - (\eta + \beta)] |a_k| \Omega_k^n(a, s) \leq 1 - \eta,
\]

where, for convenience,

\[
\rho_k = \rho_0(\alpha, \eta, \delta) = (1 + \alpha(k-1))[n(1 + \beta) - (\eta + \beta)] \Omega_k^n(a, s),
\]

\[
0 \leq \alpha \leq 1, \quad -1 \leq \eta < 1, \quad \beta \geq 0, \quad \text{and} \quad \Omega_k^n(a, s), \text{is given by (8)}.
\]

**Proof.**

\[
\beta \left\{ \frac{z(J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b)f(z)'}{(1 - \alpha)J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b,f(z)'} + \alpha z^2(J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b,f(z))'' - 1 \right\} \leq 1 - \eta.
\]

We have

\[
\beta \left\{ \frac{z(J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b,f(z))'}{(1 - \alpha)J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b,f(z)'} + \alpha z^2(J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b,f(z))'' - 1 \right\} \leq 1 - \eta.
\]

and

\[
\beta \left\{ \frac{z(J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b,f(z))'}{(1 - \alpha)J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b,f(z)'} + \alpha z^2(J^{s,a,\lambda,\delta,\gamma}_0(\zeta_\rho,\mu_\delta),b,f(z))'' - 1 \right\} \leq 1 - \eta.
\]
\[
\leq \frac{(1 + \beta) \sum_{k=2}^{\infty} (k - 1)[1 + \alpha(k - 1)]|a_k|\Omega_k^a(a, s)}{1 - \sum_{k=2}^{\infty} [1 + \alpha(k - 1)]|a_k|\Omega_k^a(a, s)}.
\]

This last expression is bounded by \(1 - \eta\) if
\[
\sum_{k=2}^{\infty} (1 + \alpha(k - 1))[k(1 + \beta) - (\eta + \beta)]|a_k|\Omega_k^a(a, s) \leq 1 - \eta.
\]

2. Main Results

**Theorem 1.** If \(f\) of the form (1) satisfies the condition (10), then
\[
\Re \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{\rho_{n+1}(\alpha, \eta, \delta) - 1 + \eta}{\rho_{n+1}(\alpha, \eta, \delta)} \quad (z \in U),
\]
where
\[
\rho_k = \rho_k(\alpha, \eta, \delta) = \begin{cases} 1 - \eta & \text{if } k = 2, 3, \ldots, n, \\ \rho_{n+1} & \text{if } k = n+1, n+2, n+3, \ldots. \end{cases}
\]
The result (11) is sharp with the function given by
\[
f(z) = z + \frac{1 - \eta}{\rho_{n+1}} z^{n+1}.
\]

**Proof.** Define the function \(\omega(z)\) by
\[
\frac{1 + \omega(z)}{1 - \omega(z)} = \frac{\rho_{n+1}}{1 - \eta} \left[ \frac{f(z)}{f_n(z)} - \frac{\rho_{n+1} - 1 + \eta}{\rho_{n+1}} \right].
\]
It suffices to show that \(|\omega(z)| \leq 1\). Now, from (13) we can write
\[
\omega(z) = \frac{1 + \sum_{k=2}^{n} a_k z^{k-1} + (\rho_{n+1}/(1 - \eta)) \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1} + (\rho_{n+1}/(1 - \eta)) \sum_{k=n+1}^{\infty} a_k z^{k-1}}.
\]
Hence, we obtain
\[
|\omega(z)| \leq \frac{(\rho_{n+1}/(1 - \eta)) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^{n} |a_k| - (\rho_{n+1}/(1 - \eta)) \sum_{k=n+1}^{\infty} |a_k|}.
\]
Now \(|\omega(z)| \leq 1\) if and only if
\[
2 \left( \frac{\rho_{n+1}}{1 - \eta} \right) \sum_{k=n+1}^{\infty} |a_k| \leq 2 - 2 \sum_{k=2}^{n} |a_k|
\]
or, equivalently,
\[
\sum_{k=2}^{n} |a_k| + \sum_{k=n+1}^{\infty} \frac{\rho_{n+1}}{1 - \eta} |a_k| \leq 1.
\]
From the condition (10), it is sufficient to show that
\[ \sum_{k=2}^{n} |a_k| + \sum_{k=n+1}^{\infty} \frac{\rho_{k+1}}{1-\eta} |a_k| \leq \sum_{k=2}^{\infty} \frac{\rho_k}{1-\eta} |a_k| \]
which is equivalent to
\[ \sum_{k=2}^{n} \left( \frac{\rho_k - 1 + \eta}{1-\eta} \right) |a_k| - \sum_{k=n+1}^{\infty} \left( \frac{\rho_{k+1}}{1-\eta} \right) |a_k| \geq 0. \]
(14)

To see that the function given by (12) gives the sharp result, we observe that for \( z = re^{i\pi} \),
\[ \frac{f(z)}{f_n(z)} = 1 + \frac{1-\eta}{\rho_{n+1}} z^n \rightarrow 1 - \frac{1-\eta}{\rho_{n+1}} \]
\[ = \frac{\rho_{n+1} - 1 + \eta}{\rho_{n+1}} \text{ when } r \rightarrow -1. \]

We next determine bounds for \( f_n(z)/f(z) \).

**Theorem 2.** If \( f \) of the form (1) satisfies the condition (10), then
\[ \Re \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{\rho_{n+1}}{\rho_{n+1} + 1 - \eta} \quad (z \in U), \]
(15)
where \( \rho_{n+1} \geq 1 - \eta \) and
\[ \rho_k \geq \begin{cases} 1 - \eta \quad & \text{if } k = 2, 3, \ldots, n, \\ \rho_{n+1} \quad & \text{if } k = n + 1, n + 2, n + 3, \ldots. \end{cases} \]
The result (15) is sharp with the function given by (12).

**Proof.** We write
\[ \frac{1+\omega(z)}{1-\omega(z)} = \frac{\rho_{n+1} + 1 - \eta}{1-\eta} \left[ \frac{f_n(z)}{f(z)} - \frac{\rho_{n+1}}{\rho_{n+1} + 1 - \eta} \right] \]
\[ = 1 + \sum_{k=2}^{n} a_k z^{k-1} - \left( \frac{\rho_{n+1}}{1-\eta} \right) \sum_{k=n+1}^{\infty} a_k z^{k-1} \]
\[ \frac{1}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}}, \]
where
\[ \omega(z) \leq \frac{(\rho_{n+1}+1-\eta)/(1-\eta) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^{n} |a_k| - ((\rho_{n+1} - 1 + \eta)/(1-\eta)) \sum_{k=n+1}^{\infty} |a_k|} \leq 1. \]
This last inequality is equivalent to
\[ \sum_{k=2}^{n} |a_k| + \sum_{k=n+1}^{\infty} \frac{\rho_{n+1}}{1-\eta} |a_k| \leq 1. \]
We are making use of (10) to get (14). Finally, equality holds in (15) for the extremal function \( f(z) \) given by (12).

We next turn to ratios involving derivatives.
Theorem 3. If \( f \) of the form (1) satisfies the condition (10), then
\[
\Re \left\{ \frac{f'(z)}{f_n'(z)} \right\} \geq \frac{\rho_{n+1} - (n + 1)(1 - \eta)}{\rho_{n+1}} \quad (z \in U),
\]
where \( \rho_{n+1} \geq (n + 1)(1 - \eta) \) and
\[
\rho_k \geq \begin{cases} k(1 - \eta) & \text{if } k = 2, 3, \ldots, n, \\ k\left(\frac{\rho_{n+1}}{n+1}\right) & \text{if } k = n + 1, n + 2, n + 3, \ldots. \end{cases}
\]
The results are sharp with the function given by (12).

Proof. We write
\[
\frac{1 + \omega(z)}{1 - \omega(z)} = \frac{\rho_{n+1}}{(n + 1)(1 - \eta)} \left[ \frac{f'(z)}{f_n'(z)} - \left( \frac{\rho_{n+1} - (n + 1)(1 - \eta)}{\rho_{n+1}} \right) \right],
\]
where
\[
\omega(z) = \frac{\rho_{n+1}/((n + 1)(1 - \eta))) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{2 + 2 \sum_{k=2}^{n} k a_k z^{k-1} + (\rho_{n+1}/((n + 1)(1 - \eta))) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}.
\]
Now \( |\omega(z)| \leq 1 \) if and only if
\[
\sum_{k=2}^{n} k |a_k| + \frac{\rho_{n+1}}{(n + 1)(1 - \eta)} \sum_{k=n+1}^{\infty} k |a_k| \leq 1.
\]
From the condition (10). It is sufficient to show that
\[
\sum_{k=2}^{n} k |a_k| + \frac{\rho_{n+1}}{(n + 1)(1 - \eta)} \sum_{k=n+1}^{\infty} k |a_k| \leq \sum_{k=2}^{\infty} \frac{\rho_{k}}{1 - \eta} |a_k|
\]
which is equivalent to
\[
\sum_{k=2}^{n} \left( \frac{\rho_k}{1 - \eta} \right) |a_k| + \sum_{k=n+1}^{\infty} \left( \frac{(n + 1)\rho_k - k\rho_{n+1}}{(n + 1)(1 - \eta)} \right) |a_k| \geq 0.
\]
To prove the result (16), define the function \( \omega(z) \) by
\[
\frac{1 + \omega(z)}{1 - \omega(z)} = \frac{(n + 1)(1 - \eta) + \rho_{n+1}}{(n + 1)(1 - \eta)} \left[ \frac{f_n'(z)}{f'(z)} - \left( \frac{\rho_{n+1}}{\rho_{n+1} + (n + 1)(1 - \eta)} \right) \right],
\]
where
\[
\omega(z) = \frac{-(1 + \rho_{n+1}/(n + 1)(1 - \eta))) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{2 + 2 \sum_{k=2}^{n} k a_k z^{k-1} + (1 - \rho_{n+1}/(n + 1)(1 - \eta))) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}.
\]
Now \( |\omega(z)| \leq 1 \) if and only if
\[
\sum_{k=2}^{n} k |a_k| + \frac{\rho_{n+1}}{(1 - \eta)(n + 1)} \sum_{k=n+1}^{\infty} k |a_k| \leq 1.
\]
It suffices to show that the left hand side of (17) is bounded previously by the condition
\[
\sum_{k=2}^{\infty} \frac{\rho_{k}}{1 - \eta} |a_k|,
\]
which is equivalent to

\[
\sum_{k=2}^{\infty} \left( \frac{\rho_k}{1-\eta} - k \right) |a_k| + \sum_{k=n+1}^{\infty} \left( \frac{\rho_k}{1-\eta} - \frac{\rho_{n+1}}{(1-\eta)(n+1)} \right) k a_k \geq 0.
\]

**Remark.** As a special case of the previous theorems, we can determine new sharp lower bounds for \( Rf(z)/f_n(z) \), \( Rf_n(z)/f(z) \), \( Rf''(z)/f'_n(z) \), \( Rf'_n(z)/f''(z) \) for various function classes involving the Alexander integral operator \([1]\), Bernardi integral operator \([2]\), Jung-Kim-Srivastava integral operator \([9]\) and Choi-Saigo-Srivastava operator (see \([7]\) and \([10]\)) on specializing the values of \( \delta, \lambda, s \) and \( a \).

**References**


K.A. Challab, M. Darus, F. Ghanim

School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
43600, Bangi-Selangor D. Ehsan, Malaysia
E-mail address: khalid_math1363@yahoo.com, maslina@ukm.edu.my

F. Ghanim
Department of Mathematics,
College of Sciences,
University of Sharjah,
Sharjah, United Arab Emirates
E-mail address: fgahmed@sharjah.ac.ae