S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 15, стр. 362–372 (2018) УДК 517.53
DOI 10.17377/semi.2018.15.033 MSC 30C45, 11M35

PARTIAL SUMS OF A GENERALIZED CLASS OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED SRIVASTAVA-ATTIYA OPERATOR

K.A. CHALLAB, M. DARUS, F. GHANIM

ABSTRACT. Let $f_n(z) = z+\sum_{k=2}^{n} a_k z^k$ be the sequence of partial sums of the analytic function $f(z) = z+\sum_{k=2}^{\infty} a_k z^k$. In this paper, we determine sharp lower bounds for $\Re\{f(z)/f_n(z)\}, \Re\{f_n(z)/f(z)\}, \Re\{f'(z)/f_n'(z)\}$ and $\Re\{f_n'(z)/f'(z)\}$. The efficiency of the main result not only provides the unification of the results discussed in the literature but also generates certain new results.

Keywords: analytic functions, Hadamard product (or convolution), generalized Hurwitz—Lerch zeta function, Srivastava-Attiya operator.

1. Introduction and Preliminaries

Let $A(U)$ denote a class of all analytic functions defined in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}, j \in \mathbb{N} = \{1, 2, ...\}$, let

$$A[a,j] = \left\{f \in A(U) : f(z) = a + a_j z^j + a_{j+1} z^{j+1} + \ldots\right\}.$$

We denote a subclass of $A[a,1]$ by A whose members are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in U). \tag{1}$$

We denote by C the class of convex (univalent) functions in U and satisfying

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad (z \in U).$$

Partial sums of a generalized class of analytic functions defined by a generalized Srivastava-Attiya operator.

The work is supported by UKM (grant GUP-2017-064).

Received December, 16, 2016, published March, 9, 2018.

Using a Schwarzian function w with the conditions $w(0) = 0$ and $|w(z)| < 1$ and analytic in U, with f and g as its analytic functions in U. Subsequently, f is a subordinate to g (symbolically $f \prec g$) if $f(z) = g(w(z))$ is satisfied.

Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The Srivastava-Attiya operator is defined as [23] (see also [[3],[17],[30]]):

$$J_{s,a}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+a}{k+a}\right)^s a_k z^k$$

Where $z \in U, a \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}$ and $f \in A[a,1]$.

In fact, the linear operator $J_{s,a}(f)$ can be written as

$$J_{s,a}(f)(z) := G_{s,a}(z) * f(z).$$

In terms of Hadamard product (or convolution) where $G_{s,a}(z)$ is given by

$$G_{s,a}(z) := (1+a)^s \left[\Phi(z,s,a) - a^{-s}\right] \qquad (z \in U), \tag{2}$$

the function $\Phi(z,s,a)$ involved in the right-hand side of (2) is the well known Hurwitz-Lerch zeta function defined by: (see, for example , [[5] and [24], p. 121 et seq.])

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}$$

$$(a \in \mathbb{C}\backslash\mathbb{Z}_0^- : s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1).$$

Recently, a new family of γ-generalized Hurwitz- Lerch zeta function was investigated by Srivastava [26] (see also [[3], [5],[25]]). Srivastava considered the following function:

$$\Phi^{\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q}_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}(z,s,a;b,\gamma)$$
$$= \frac{1}{\gamma\Gamma(s)} \cdot \sum_{k=0}^{\infty} \frac{\Pi_{j=1}^{p}(\gamma_j)_{n\rho_j}}{(a+k)^s \cdot \Pi_{j=1}^{q}(\mu_j)_{n\sigma_j}} H^{2,0}_{0,2}\left[(a+k)\, b^{\frac{1}{\gamma}} \middle| (s,1), \left(0, \frac{1}{\gamma}\right)\right] \frac{z^k}{k!}, \tag{3}$$

$$(\min\{\Re(a), \Re(s)\} > 0; \Re(b) > 0; \gamma > 0), \textit{where}$$

$$\begin{pmatrix} \gamma_j \in \mathbb{C}(j=1,...,p) \text{ and } \mu_j \in \mathbb{C}\backslash\mathbb{Z}_0^-(j=1,...,q);\ \rho_j > 0(j=1,...,p); \\ \sigma_j > 0(j=1,...,q); 1 + \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j \geqslant 0 \end{pmatrix}$$

and the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\Pi_{j=1}^{p}{\rho_j}^{-\rho_j}\right) \cdot \left(\Pi_{j=1}^{q}\sigma_j^{\sigma_j}\right).$$

Here, and for the remainder of this paper, $(\gamma)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$(\gamma)_k := \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} = \begin{cases} \gamma(\gamma+1)\ldots(\gamma+n-1) & (k=n\in\mathbb{N};\ \gamma\in\mathbb{C}) \\ 1 & (k=0;\ \gamma\in\mathbb{C}\backslash\{0\}) \end{cases}.$$

Definition 1. *The H-function involved in the right-hand side of* (3) *is the well-known Fox's H-Function* [19, Definition 1.1] (*see also* [29]) *defined by*

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\big|_{(b_1,B_1),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_p,A_p)}\right]$$

$$= \frac{1}{2\pi i}\int_{\ell}\Xi(s)z^{-s}ds \qquad (z\in\mathbb{C}\backslash\{0\};|\arg(z)|<\pi), \tag{4}$$

where

$$\Xi(s) = \frac{\Pi_{j=1}^{m}\Gamma(b_j+B_js)\cdot\Pi_{j=1}^{n}(1-a_j-A_Js)}{\Pi_{j=n+1}^{p}\Gamma(a_j+A_Js)\cdot\Pi_{j=m+1}^{q}\Gamma(1-b_j-B_Js)},$$

an empty product is interpreted as 1, m, n, p and q are integers such that

$$\begin{pmatrix} 1\leqslant m\leqslant q, 0\leqslant n\leqslant p, A_j>0(j=1,\ldots,p), B_j>0(j=1,\ldots,q), \\ a_j\in\mathbb{C}(j=1,\ldots,p),\ b_j\in\mathbb{C}(j=1,\ldots,q) \end{pmatrix}$$

and ℓ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\{\Gamma(b_j+B_js)\}_{j=1}^{m}$$

from the poles of gamma functions

$$\{\Gamma(1-a_j+A_js)\}_{j=1}^{m}.$$

It is worthy to mention that using the fact that [4] and [22, p.1496 , Remark 7]

$$\lim_{b\to 0}\left\{H_{0,2}^{2,0}\left[(a+k)\,b^{\frac{1}{\gamma}}\overline{|(s,1),\left(0,\frac{1}{\gamma}\right)}\right]\right\} = \gamma\,\Gamma(s) \quad (\lambda>0),$$

Eq. (4) reduces to

$$\Phi_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}(z,s,a;0,\gamma)$$

$$:= \Phi_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}(z,s,a) = \sum_{k=0}^{\infty}\frac{\Pi_{j=1}^{p}(\gamma_j)_{k\rho_j}}{(a+n)^s\cdot\Pi_{j=1}^{q}(\mu_j)_{k\sigma_j}}\frac{z^k}{k!}. \tag{5}$$

Definition 2. *The function* $\Phi_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}(z,s,a)$ *involved in (5) is the multiparameter extension and generalization of the Hurwtiz- Lerch zeta function* $\Phi(z,s,a)$ *introduced by Srivastava et al.*[[6] *and* [31], *p. 503, Eq.(6.2)*] *defined by*

$$\Phi_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p;\sigma_1,\ldots,\sigma_q)}(z,s,a) := \sum_{k=0}^{\infty}\frac{\Pi_{j=1}^{p}(\gamma_j)_{k\rho_j}}{(a+k)^s\cdot\Pi_{j=1}^{q}(\mu_j)_{k\sigma_j}}\frac{z^k}{k!}$$

$$\begin{pmatrix} p,q\in\mathbb{N}_0;\gamma_j\in\mathbb{C}\,(j=1,\ldots,p)\,;a,\mu_j\in\mathbb{C}\backslash\mathbb{Z}_0^-\,(j=1,\ldots,q)\,; \\ \rho_j;\ \sigma_k\in\mathbb{R}^+\,(j=1,\ldots,p;k=1,\ldots,q)\,;\ \ \Delta>-1\,\textit{when}\ s,z\in\mathbb{C}; \\ \Delta=-1\,\textit{and}\,s\in\mathbb{C}\,\textit{when}\,|z|<\nabla^*;\ \ \Delta=-1\,\textit{and}\,\Re(\Xi)>\frac{1}{2}\textit{when}\,|z|=\nabla^* \end{pmatrix}$$

with

$$\nabla^* := \left(\Pi_{j=1}^{p} \rho_j^{-\rho_j}\right) . \left(\Pi_{j=1}^{q} \sigma_j^{\sigma_j}\right),$$

and

$$\Delta := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \text{ and } \Xi := s + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \gamma_j + \frac{p-q}{2}.$$

The following linear operator was introduced by Srivastava and Gaboury [25]:

$$J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f) : A(U) \to A(U)$$

defined by

$$J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z) = G^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(z) * f(z), \tag{6}$$

where $*$ denotes the Hadamard product (or convolution) of analytic functions and function $G^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(z)$ is given by

$$G^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(z) := \frac{\gamma \, \Pi_{j=1}^{q}(\mu_j) \, \Gamma(s) \, (a+1)^s}{\Pi_{j=1}^{p}(\gamma_j)}$$

$$\times \Lambda(a+1,b,s,\gamma)^{-1} \left[\Phi^{(1,\ldots,1,1,\ldots,1)}_{\gamma_1,\ldots,\gamma_p;\mu_1,\ldots,\mu_q}(z,s,a;b,\gamma) - \frac{a^{-s}}{\gamma \, \Gamma(s)} \Lambda(a,b,s,\gamma)\right]$$

$$= z + \sum_{k=2}^{\infty} \frac{\Pi_{j=1}^{p}(\gamma_j+1)_{k-1}}{\Pi_{j=1}^{q}(\mu_j+1)_{k-1}} \left(\frac{a+1}{a+k}\right)^s \left(\frac{\Lambda(a+k,b,s,\gamma)}{\Lambda(a+1,b,s,\gamma)}\right) \frac{z^k}{k!} \tag{7}$$

with

$$\Lambda(a,b,s,\gamma) := H^{2,0}_{0,2}\left[ab^{\frac{1}{\gamma}} \middle| \overline{(s,1), \left(0, \frac{1}{\gamma}\right)}\right].$$

Combining (6) and (7), we obtain

$$J^{s,a,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z) = z + \sum_{k=2}^{\infty} \frac{\Pi_{j=1}^{p}(\gamma_j+1)_{k-1}}{\Pi_{j=1}^{q}(\mu_j+1)_{k-1}} \left(\frac{a+1}{a+k}\right)^s \left(\frac{\Lambda(a+k,b,s,\gamma)}{\Lambda(a+1,b,s,\gamma)}\right) a_k \frac{z^k}{k!}$$

$$\left(\gamma_j \in \mathbb{C} \, (j=1,...,p) \text{ and } \mu_j \in \mathbb{C}\backslash\mathbb{Z}_0^- \, (j=1,...,q) \, ; p \leqslant q+1; \; z \in U\right), \text{ with}$$

$$\min\{\Re(a), \Re(s)\} > 0; \gamma > 0 \text{ if } \Re(b) > 0$$

and

$$s \in \mathbb{C}; a \in \mathbb{C}\backslash\mathbb{Z}_0^- \text{ if } b = 0.$$

Motivated essentially by the Srivastava-Attiya operator, Xiang *et al.* [32] (see also Murugusundaramoorthy *et al.* [15]) introduced and investigated the integral operator $J^{\lambda,\delta}_{a,s} = z + \sum_{k=2}^{\infty} \left(\frac{1+k}{a+k}\right)^s \frac{\lambda!(k+\delta-2)!}{(\delta-2)!(k+\lambda-1)!} a_k z^k$.

Now we define a new integral operator

$$J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}(f)(z) = z + \sum_{k=2}^{\infty} \frac{\Pi_{j=1}^{p}(\gamma_j+1)_{k-1}}{\Pi_{j=1}^{q}(\mu_j+1)_{k-1}} \left(\frac{a+1}{a+k}\right)^s$$

$$\times \left(\frac{\Lambda(a+k,b,s,\gamma)}{\Lambda(a+1,b,s,\gamma)}\right) \frac{\lambda!(k+\delta-2)!}{(\delta-2)!(k+\lambda-1)!} a_k \frac{z^k}{k!} \quad (z \in U),$$

$$J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} = z + \sum_{k=2}^{\infty} \Omega_k^{\lambda}(a,s) a_k z^k = F(z),$$

and

$$\Omega_k^{\lambda}(a,s) = \frac{\Pi_{j=1}^{p}(\gamma_j+1)_{k-1}}{\Pi_{j=1}^{q}(\mu_j+1)_{k-1}} \left(\frac{a+1}{a+k}\right)^s \times \left(\frac{\Lambda(a+k,b,s,\gamma)}{\Lambda(a+1,b,s,\gamma)}\right) \frac{\lambda!(k+\delta-2)!}{(\delta-2)!(k+\lambda-1)!} \frac{1}{k!} \tag{8}$$

where (and throughout this paper unless otherwise mentioned) the parameters s, a, b, δ *and* λ are constrained as follows:

$$s \in \mathbb{C}; \quad a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad \delta > 2 \;\; and \;\; \lambda > -1.$$

We note that $J^{s,\,a,\,1,\,2,\,\gamma}_{(\gamma_p),(\mu_q),b}$ is the generalized Srivastava-Attiya operator [27]. Motivated by Murugusundaramoorthy ([12], [13], [14]) and making use of the generalized Srivastava-Attiya operator $J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b}$, we define the following new subclass of analytic functions with negative coefficients.

For $\alpha \geq 0$, $-1 \leq \eta < 1$, and $\beta \geq 0$, let $P_s^{\alpha}(\eta,\beta)$ be the subclass of A consisting of functions of the form (1) and satisfying the analytic criterion

$$\Re\left\{ \frac{z(J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} f(z))' + \alpha z^2((J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} f(z))''}{(1-\alpha)J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} f(z) + \alpha z(J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} f(z))'} - \eta \right\}$$

$$> \beta \left| \frac{z(J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} f(z))' + \alpha z^2(J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} f(z))''}{(1-\alpha)J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} f(z) + \alpha z(J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b} f(z))'} - 1 \right|, \tag{9}$$

where $z \in U$. Shortly we can state this condition by

$$\Re\left\{ \frac{zG'(z)}{G(z)} - \eta \right\} > \beta \left| \frac{zG'(z)}{G(z)} - 1 \right|,$$

where

$$G(z) = (1-\alpha)F(z) + \alpha z F'(z) = z + \sum_{k=2}^{\infty} [1+\alpha(k-1)] |\Omega_k^{\lambda}(a,s)| a_k z^k,$$

and $F(z) = J^{s,\,a,\,\lambda,\,\delta,\,\gamma}_{(\gamma_p),(\mu_q),b}$, with $\Omega_k^{\lambda}(a,s)$ given by (8).

Silverman [21] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums.

In the present paper and by following the earlier work by Silverman [21] (see [8], [18], [20]) and recent work (see [11], [16], [28]) on partial sums of analytic functions, we study the ratio of a function of the form (1) to its sequence of partial sums of the form

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k,$$

when the coefficients of $f(z)$ satisfy the condition (9) a function $f(z)$ is in $P_s^{\alpha}(\eta,\beta)$. Also, we will determine sharp lower bounds for $\Re\{f(z)/f_n(z)\}$, $\Re\{f_n(z)/f(z)\}$,

$\Re\{f'(z)/f_n'(z)\}$, and $\Re\{f_n'(z)/f'(z)\}$. It is seen that this study not only gives us a particular case, the results of Silverman [21], but also gives rise to several new results.

Before stating and proving our main results, we derive a sufficient condition giving the coefficient estimates for functions $f(z)$ to belong to this generalized function class.

Lemma 1. *A function $f(z)$ of the form (1) is in $P_s^{\alpha}(\eta,\beta)$ if*

$$\sum_{k=2}^{\infty}(1+\alpha(k-1))[k(1+\beta)-(\eta+\beta)]|a_k|\Omega_k^{\lambda}(a,s)\le 1-\eta, \tag{10}$$

where, for convenience,

$$\rho_k=\rho_k(\alpha,\eta,\delta)=(1+\alpha(k-1))[n(1+\beta)-(\eta+\beta)]\Omega_k^{\lambda}(a,s),$$

$0\le\alpha\le 1$, $-1\le\eta<1$, $\beta\ge 0$, *and* $\Omega_k^{\lambda}(a,s)$, *is given by (8).*

Proof.

$$\beta\left|\frac{z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'+\alpha z^2(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))''}{(1-\alpha)J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z)+\alpha z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'}-1\right|$$

$$-\Re\left\{\frac{z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'+\alpha z^2((J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))''}{(1-\alpha)J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z)+\alpha z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'}-1\right\}\le 1-\eta.$$

We have

$$\beta\left|\frac{z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'+\alpha z^2(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))''}{(1-\alpha)J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z)+\alpha z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'}-1\right|$$

$$-\Re\left\{\frac{z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'+\alpha z^2((J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))''}{(1-\alpha)J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z)+\alpha z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'}-1\right\}$$

$$\le(1+\beta)\left|\frac{z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'+\alpha z^2(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))''}{(1-\alpha)J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z)+\alpha z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'}-1\right|$$

and

$$\beta\left|\frac{z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'+\alpha z^2(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))''}{(1-\alpha)J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z)+\alpha z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'}-1\right|$$

$$-\Re\left\{\frac{z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'+\alpha z^2((J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))''}{(1-\alpha)J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z)+\alpha z(J^{s,a,\lambda,\delta,\gamma}_{(\gamma_p),(\mu_q),b}f(z))'}-1\right\}$$

$$\leq \frac{(1+\beta)\sum_{k=2}^{\infty}(k-1)[1+\alpha(k-1)]|a_k|\Omega_k^{\lambda}(a,s)}{1-\sum_{k=2}^{\infty}[1+\alpha(k-1)]|a_k|\Omega_k^{\lambda}(a,s)}.$$

This last expression is bounded by $1-\eta$ if

$$\sum_{k=2}^{\infty}(1+\alpha(k-1))[k(1+\beta)-(\eta+\beta)]|a_k|\Omega_k^{\lambda}(a,s)\leq 1-\eta.$$

□

2. Main Results

Theorem 1. *If f of the form (1) satisfies the condition (10), then*

$$\Re\left\{\frac{f(z)}{f_n(z)}\right\}\geq \frac{\rho_{n+1}(\alpha,\eta,\delta)-1+\eta}{\rho_{n+1}(\alpha,\eta,\delta)}\quad (z\in U), \tag{11}$$

where

$$\rho_k=\rho_k(\alpha,\eta,\delta)\geq\begin{cases}1-\eta & if\quad k=2,3,...,n,\\ \rho_{n+1} & if\quad k=n+1,n+2,n+3,...\end{cases}.$$

The result (11) is sharp with the function given by

$$f(z)=z+\frac{1-\eta}{\rho_{n+1}}z^{n+1}. \tag{12}$$

Proof. Define the function $\omega(z)$ by

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{\rho_{n+1}}{1-\eta}\left[\frac{f(z)}{f_n(z)}-\frac{\rho_{n+1}-1+\eta}{\rho_{n+1}}\right]$$

$$\frac{1+\omega(z)}{1-\omega(z)}=\frac{1+\sum_{k=2}^{n}a_kz^{k-1}+(\rho_{n+1}/(1-\eta))\sum_{k=n+1}^{\infty}a_kz^{k-1}}{1+\sum_{k=2}^{n}a_kz^{k-1}}. \tag{13}$$

It suffices to show that $|\omega(z)|\leq 1$. Now, from (13) we can write

$$\omega(z)=\frac{(\rho_{n+1}/(1-\eta))\sum_{k=n+1}^{\infty}a_kz^{k-1}}{2+2\sum_{k=2}^{n}a_kz^{k-1}+(\rho_{n+1}/(1-\eta))\sum_{k=n+1}^{\infty}a_kz^{k-1}}.$$

Hence, we obtain

$$|\omega(z)|\leq\frac{(\rho_{n+1}/(1-\eta))\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{n}|a_k|-(\rho_{n+1}/(1-\eta))\sum_{k=n+1}^{\infty}|a_k|}.$$

Now $|\omega(z)|\leq 1$ if and only if

$$2\left(\frac{\rho_{n+1}}{1-\eta}\right)\sum_{k=n+1}^{\infty}|a_k|\leq 2-2\sum_{k=2}^{n}|a_k|$$

or, equivalently,

$$\sum_{k=2}^{n}|a_k|+\sum_{k=n+1}^{\infty}\frac{\rho_{n+1}}{1-\eta}|a_k|\leq 1.$$

From the condition (10), it is sufficient to show that

$$\sum_{k=2}^{n}|a_k| + \sum_{k=n+1}^{\infty}\frac{\rho_{n+1}}{1-\eta}|a_k| \le \sum_{k=2}^{\infty}\frac{\rho_k}{1-\eta}|a_k|$$

which is equivalent to

$$\sum_{k=2}^{n}(\frac{\rho_k - 1+\eta}{1-\eta})|a_k| - \sum_{k=n+1}^{\infty}(\frac{\rho_{n+1}}{1-\eta})|a_k| \ge 0. \tag{14}$$

To see that the function given by (12) gives the sharp result, we observe that for $z = re^{\frac{i\pi}{n}}$,

$$\frac{f(z)}{f_n(z)} = 1 + \frac{1-\eta}{\rho_{n+1}}z^n \to 1 - \frac{1-\eta}{\rho_{n+1}}$$

$$= \frac{\rho_{n+1} - 1 + \eta}{\rho_{n+1}} \quad when\ \ r \to -1.$$

□

We next determine bounds for $f_n(z)/f(z)$.

Theorem 2. *If f of the form* (1) *satisfies the condition* (10)*, then*

$$\Re\left\{\frac{f_n(z)}{f(z)}\right\} \ge \frac{\rho_{n+1}}{\rho_{n+1}+1-\eta} \quad (z \in U), \tag{15}$$

where $\rho_{n+1} \ge 1-\eta$ *and*

$$\rho_k \ge \begin{cases} 1-\eta & if \quad k = 2,3,...,n, \\ \rho_{n+1} & if \quad k = n+1, n+2, n+3, ... \end{cases}.$$

The result (15) is sharp with the function given by (12).

Proof. We write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{\rho_{n+1}+1-\eta}{1-\eta}\left[\frac{f_n(z)}{f(z)} - \frac{\rho_{n+1}}{\rho_{n+1}+1-\eta}\right]$$

$$= \frac{1+\sum_{k=2}^{n} a_k z^{k-1} - (\rho_{n+1}/(1-\eta))\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum_{k=2}^{\infty} a_k z^{k-1}},$$

where

$$|\omega(z)| \le \frac{((\rho_{n+1+1-\eta})/(1-\eta))\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{n}|a_k| - ((\rho_{n+1}-1+\eta)/(1-\eta))\sum_{k=n+1}^{\infty}|a_k|} \le 1.$$

This last inequality is equivalent to

$$\sum_{k=2}^{n}|a_k| + \sum_{k=n+1}^{\infty}\frac{\rho_{n+1}}{1-\eta}|a_k| \le 1.$$

We are making use of (10) to get (14). Finally, equality holds in (15) for the extremal function $f(z)$ given by (12). □

We next turn to ratios involving derivatives.

Theorem 3. *If f of the form* (1) *satisfies the condition* (10), *then*

$$\Re\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{\rho_{n+1}-(n+1)(1-\eta)}{\rho_{n+1}} \quad (z \in U),$$

$$\Re\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{\rho_{n+1}}{\rho_{n+1}-(n+1)(1-\eta)} \quad (z \in U), \tag{16}$$

where $\rho_{n+1} \geq (n+1)(1-\eta)$ *and*

$$\rho_k \geq \begin{cases} k(1-\eta) & if \quad k=2,3,...,n, \\ k(\frac{\rho_{n+1}}{n+1}) & if \quad k=n+1,n+2,n+3,.... \end{cases}.$$

The results are sharp with the function given by (12).

Proof. We write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{\rho_{n+1}}{(n+1)(1-\eta)}\left[\frac{f'(z)}{f_n'(z)} - (\frac{\rho_{n+1}-(n+1)(1-\eta)}{\rho_{n+1}})\right],$$

where

$$\omega(z) = \frac{(\rho_{n+1}/((n+1)(1-\eta)))\sum_{k=n+1}^{\infty} ka_k z^{k-1}}{2+2\sum_{k=2}^{n} ka_k z^{k-1} + (\rho_{n+1}/((n+1)(1-\eta)))\sum_{k=n+1}^{\infty} ka_k z^{k-1}}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$\sum_{k=2}^{n} k|a_k| + \frac{\rho_{n+1}}{(n+1)(1-\eta)}\sum_{k=n+1}^{\infty} k|a_k| \leq 1.$$

From the condition (10). It is sufficient to show that

$$\sum_{k=2}^{n} k|a_k| + \frac{\rho_{n+1}}{(n+1)(1-\eta)}\sum_{k=n+1}^{\infty} k|a_k| \leq \sum_{k=2}^{\infty}\frac{\rho_k}{1-\eta}|a_k|$$

which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{\rho_k-(1-\eta)k}{1-\eta}\right)|a_k| + \sum_{k=n+1}^{\infty}\frac{(n+1)\rho_k - k\rho_{n+1}}{(n+1)(1-\eta)}|a_k| \geq 0.$$

To prove the result (16), define the function $\omega(z)$ by

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{(n+1)(1-\eta)+\rho_{n+1}}{(n+1)(1-\eta)}\left[\frac{f_n'(z)}{f'(z)} - \left(\frac{\rho_{n+1}}{\rho_{n+1}+(n+1)(1-\eta)}\right)\right],$$

where

$$\omega(z) = \frac{-(1+\rho_{n+1}/(n+1)(1-\eta))\sum_{k=n+1}^{\infty} ka_k z^{k-1}}{2+2\sum_{k=2}^{n} ka_k z^{k-1} + (1-\rho_{n+1}/(n+1)(1-\eta))\sum_{k=n+1}^{\infty} ka_k z^{k-1}}.$$

Now $|\omega(z)| \leq 1$ if and only if

$$\sum_{k=2}^{n} k|a_k| + (\frac{\rho_{n+1}}{(1-\eta)(n+1)})\sum_{k=n+1}^{\infty} k|a_k| \leq 1. \tag{17}$$

It suffices to show that the left hand side of (17) is bounded previously by the condition

$$\sum_{k=2}^{\infty}\frac{\rho_k}{1-\eta}|a_k|,$$

which is equivalent to

$$\sum_{k=2}^{\infty}(\frac{\rho_k}{1-\eta}-k)|a_k|+\sum_{k=n+1}^{\infty}(\frac{\rho_k}{1-\eta}-\frac{\rho_{n+1}}{(1-\eta)(n+1)})ka_k\geq 0.$$

□

Remark. As a special case of the previous theorems, we can determine new sharp lower bounds for $\Re f(z)/f_n(z)$, $\Re f_n(z)/f(z)$, $\Re f'(z)/f'_n(z)$, $\Re f'_n(z)/f'(z)$ for various function classes involving the Alexander integral operator [1], Bernardi integral operator [2], Jung-Kim-Srivastava integral operator [9] and Choi-Saigo-Srivastava operator (see [7] and [10]) on specializing the values of δ, λ, s *and* a.

References

[1] Alexander J.W., *Functions which map the interior of the unit circle upon simple regions*, The Annals of Mathematics, **17**:1 (1915), 12–22. JFM 45.0672.02

[2] Bernardi S.D., *Convex and starlike univalent functions*, Transactions of the American Mathematical Society, **135** (1969), 429–446. Zbl 0172.09703

[3] Challab K.A., Darus M., Ghanim F., *Certain problems related to generalized Srivastava-Attiya operator*, Asian-European Journal of Mathematics, **10**:2 (2017), 1–21. Zbl 1373.30016

[4] Challab K.A., Darus M., Ghanim F., *Inclusion properties of meromorphic functions associated with the extended Cho-Kwon-Srivastava operator by using hypergeometric function*, Nonlinear Functional Analysis and Applications, **22**:5 (2017), 925–936. Zbl 06855032

[5] Challab K.A., Darus M., Ghanim F., *On Certain Subclass of Meromorphic Functions Defined by New Linear Differential Operator*, Journal of Mathematical and Fundamental Sciences, **49**:3 (2017), 269–282.

[6] Challab K.A., Darus M., Ghanim F., *On subclass of meromorphically univalent functions defined by a linear operator associated with λ-generalized Hurwitz-Lerch zeta function and q-hypergeometric function*, Italian Journal of Pure and Applied Mathematics, (2017).(Accepted)

[7] Choi J.H., Saigo M., Srivastava H.M., *Some inclusion properties of a certain family of integral operators*, Journal of Mathematical Analysis and Applications, **276**:1 (2002), 432–445. Zbl 1035.30004

[8] Frasin B.A., *Generalization of partial sums of certain analytic and univalent functions*, Applied Mathematics Letters, **21**:7 (2008), 735–741. Zbl 1152.30308

[9] Jung I.B., Kim Y.C., Srivastava H.M., *The hardy space of analytic functions associated with certain one-parameter families of integral operators*, Journal of Mathematical Analysis and Applications, **176**:1 (1993), 138–147. Zbl 0774.30008

[10] Ling Y., Liu F., *The Choi–Saigo–Srivastava integral operator and a class of analytic functions*, Applied mathematics and computation, **165**:3 (2005), 613–621. Zbl 1093.30005

[11] Maharana S., Prajapat J.K., Srivastava H.M., *The radius of convexity of partial sums of convex functions in one direction*, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, **87**:2 (2017), 215–219. Zbl 1381.30013

[12] Murugusundaramoorthy G., *A subclass of analytic functions associated with the Hurwitz-Lerch zeta function*, Hacettepe Journal of Mathematics and Statistics, **39**:2 (2010), 265–272. Zbl 1200.30019

[13] Murugusundaramoorthy G., *Subordination results and integral means inequalities for k-uniformly starlike functions defined by convolution involving the Hurwitz-Lerch zeta function*, Studia Universitatis Babes-Bolyai Mathematica, **55**:4 (2010), 155–166. Zbl 1240.30058

[14] Murugusundaramoorthy G., Rosy T., Muthunagai K., *A unified class of analytic functions with negative coefficients*, Lobachevskii Journal of Mathematics, **29**:3 (2008), 175–185. Zbl 1166.30305

[15] Murugusundaramoorthy G., Uma K., Darus M., *Partial sums of generalized class of analytic functions involving Hurwitz-Lerch zeta function*, Abstract and Applied Analysis, **2011** (2011), 1–10. Zbl 1220.30025

[16] Owa S., Srivastava H.M., Saito N., *Partial sums of certain classes of analytic functions*, International Journal of Computer Mathematics, **81**:10 (2004), 1239–1256. Zbl 1060.30022

[17] Prajapat J.K., Goyal S.P., *Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions*, Journal of Mathematical Inequalities, **3**:1 (2009), 129–137. Zbl 1160.30325

[18] Rosy T., Subramanian K.G., Murugusundaramoorthy G., *Neighbourhoods and partial sums of starlike functions based on Ruscheweyh derivatives*, Journal of Inequalities in Pure and Applied Mathematics, **4**:4 (2003), 1–8. Zbl 1054.30014

[19] Saxena R.K., Mathai A.M., Haubold H.J., *The H-function: theory and applications*, Springer Science and Business Media, New York, Dordrecht, Heidelberg and London, 2009.

[20] Sheil-Small T., *A note on the partial sums of convex schlicht functions*, Bulletin of the London Mathematical Society, **2**:2 (1970), 165–168. Zbl 0217.09701

[21] Silverman H.M., *Partial sums of starlike and convex functions*, Journal of Mathematical Analysis and Applications, **209**:1 (1997), 221–227. Zbl 0894.30010

[22] Srivastava H.M., *A new family of the λ-generalized Hurwitz-Lerch zeta functions with applications*, Applied Mathematics & Information Sciences, **8**:4 (2014), 1485–1500.

[23] Srivastava H.M., Attiya A.A., *An integral operator associated with the Hurwitz–Lerch zeta function and differential subordination*, Integral Transforms and Special Functions, **18**:3 (2007), 207–216. Zbl 1112.30007

[24] Srivastava H.M., Choi J., *Series associated with the zeta and related functions*, Dordrecht: Kluwer Academic Publishers, 2001. Zbl 1014.33001

[25] Srivastava H.M., Gaboury S., *A new class of analytic functions defined by means of a generalization of the Srivastava-Attiya operator*, Journal of Inequalities and Applications, **2015**:1 (2015), 1–15. Zbl 1308.30024

[26] Srivastava H.M., Gaboury S., *New expansion formulas for a family of the-generalized Hurwitz-Lerch zeta functions*, International Journal of Mathematics and Mathematical Sciences, **2014** (2014), 1–13.

[27] Srivastava H.M., Gaboury S., Ghanim F., *A unified class of analytic functions involving a generalization of the Srivastava–Attiya operator*, Applied Mathematics and Computation, **251** (2015), 35–45. Zbl 1328.30012

[28] Srivastava H.M., Gaboury S., Ghanim F., *Partial sums of certain classes of meromorphic functions related to the Hurwitz-Lerch zeta function*, Moroccan Journal of Pure and Applied Analysis, **1**:1 (2015), 38–50.

[29] Srivastava H.M., Gupta K.C., Goyal S.P., *The H-functions of one and two variables, with applications*, New Delhi and Madras: South Asian Publishers, 1982. Zbl 0506.33007

[30] Srivastava H.M., Răducanu D., Sălăgean G.S., *A new class of generalized close-to-starlike functions defined by the Srivastava-Attiya operator*, Acta Mathematica Sinica, English Series, **29**:5 (2013), 833–840. Zbl 1293.30038

[31] Srivastava H.M., Saxena R.K., Pogány T., Saxena R., *Integral and computational representations of the extended Hurwitz–Lerch zeta function*, Integral Transforms and Special Functions, **22**:7 (2011), 487–506. Zbl 1242.11065

[32] Xiang R.G., Wang Z.G., Darus M., *A family of integral operators preserving subordination and superordination*, Bulletin of the Malaysian Mathematical Sciences Society, **33**:1 (2010), 121–131. Zbl 1182.30024

K.A. Challab, M. Darus
School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
43600, Bangi-Selangor D. Ehsan, Malaysia
E-mail address: `khalid_math1363@yahoo.com, maslina@ukm.edu.my`

F. Ghanim
Department of Mathematics,
College of Sciences,
University of Sharjah,
Sharjah, United Arab Emirates
E-mail address: `fgahmed@sharjah.ac.ae`