ABELIAN SCHUR GROUPS OF ODD ORDER

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Abstract. A finite group $G$ is called a Schur group if any Schur ring over $G$ is associated in a natural way with a subgroup of $\text{Sym}(G)$ that contains all right translations. It is proved that the group $C_3 \times C_3 \times C_p$ is Schur for any prime $p$. Together with earlier results, this completes a classification of the abelian Schur groups of odd order.

Keywords: Schur rings, Schur groups, permutation groups.

1. Introduction

A Schur ring or $S$-ring over a finite group $G$ can be defined as a subring of the group ring $\mathbb{Z}G$ that is a free $\mathbb{Z}$-module spanned by a partition of $G$ closed under taking inverse and containing the identity element $e$ of $G$ as a class (see Section 2 for details). An important example of such a partition is given by the orbits of the point stabilizer $K_e$ of a permutation group $K$ such that

$$G_{\text{right}} \leq K \leq \text{Sym}(G),$$

where $G_{\text{right}}$ is the group induced by the right translations of $G$. The corresponding $S$-rings are said to be schurian in honor of I. Schur who studied the $S$-rings of this type.

In fact, there are a lot of non-shurian $S$-rings. An infinite family of them can be found in paper of R. Pöschel [10], where he introduced a concept of a Schur group.
the group $G$ is Schur if every S-ring over $G$ is schurian. A motivation for being interested in the Schur groups comes from the problem of testing isomorphism of Cayley graphs, see [6, 9].

In [4], it was proved that every finite abelian Schur group belongs to one of several explicitly given families. Recent results [8, 11] show that two of them are indeed consist of Schur groups. The main result of the present paper is given in the theorem below concerning the next family.

**Theorem 1.1.** For any prime $p$, all S-rings over a group $G = E_9 \times C_p$ are schurian. In particular, $G$ is a Schur group.

All cyclic Schur groups were classified in [3]. Therefore, as an immediate consequence of this theorem and the above mentioned results, we obtain a classification of all abelian Schur groups of odd order.

**Theorem 1.2.** A noncyclic abelian group of odd order is Schur if and only if it is isomorphic to $C_3 \times C_{3^k}$ for an integer $k \geq 1$, or $E_9 \times C_p$ for a prime $p \geq 3$.

In Section 6, we deduce Theorem 1.1 from Theorem 6.1 stating that any S-ring $A$ over the group $G = E_9 \times C_p$ is either obtained from two S-rings over smaller groups (and then the schurity of $A$ is under control) or is a cyclotomic S-ring (and then $A$ is schurian by definition). The proof of Theorem 6.1 is mainly based on Theorem 5.1 giving a sufficient condition for $A$ to be cyclotomic. This carried out in three steps. First, the S-rings over $E_9$ are completely described (Section 3). This enables us to prove that any class of the partition of $G$ associated with $A$ is an orbit of a suitable subgroup of $\text{Aut}(G)$ (Section 4). At the last step, we show that this subgroup can be chosen the same for all classes (Section 5). For reader’s convenience, we cite basic facts on S-rings in Section 2.

**Notation.**

As usual by $\mathbb{Z}$ we denote the ring of rational integers.

A finite field of order $q$ is denoted by $\text{GF}_q$.

The projections of $X \subseteq A \times B$ to $A$ and $B$ are denoted by $X_A$ and $X_B$, respectively.

The set of non-identity elements of a group $G$ is denoted by $G^\#$.

The center of a group $G$ is denoted by $Z(G)$.

Let $X \subseteq G$. The subgroup of $G$ generated by $X$ is denoted by $\langle X \rangle$; we also set $\text{rad}(X) = \{ g \in G : gX = Xg = X \}$.

Let $\sigma \in \text{Aut}(G)$. The element $\sum_{x \in X} x^\sigma$ of the group ring $\mathbb{Z}G$ is denoted by $X^\sigma$, and by $X$ if $\sigma$ is the identity.

The componentwise multiplication in the ring $\mathbb{Z}G$ is denoted by $\circ$.

The group of all permutations of $G$ is denoted by $\text{Sym}(G)$.

The induced action of $G \triangleleft \text{Sym}(\Omega)$ on an invariant set $\Delta \subseteq \Omega$ is denoted by $G^\Delta$.

The cyclic group of order $n$ is denoted by $C_n$.

The elementary abelian group of order $p^k$ is denoted by $E_{p^k}$.

2. Preliminaries

2.1. Definitions. Let $G$ be a finite group. A subring $\mathcal{A}$ of the group ring $\mathbb{Z}G$ is called a Schur ring (S-ring, for short) over $G$ if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of $G$ such that

(S1) $\{1_G\} \in \mathcal{S}$,
(S2) \( X \in S \Rightarrow X^{-1} \in S \).
(S3) \( \mathcal{A} = \text{Span}\{X : X \in S\} \).

In particular, condition (S3) implies that \( \mathcal{A} \) is closed with respect to the componentwise multiplication \( \circ \). A group isomorphism \( f : G \rightarrow G' \) is called a Cayley isomorphism from an S-ring \( \mathcal{A} \) over \( G \) onto an S-ring \( \mathcal{A}' \) over \( G' \) if \( S(A)^f = S(A') \).

The set of Cayley isomorphisms from \( \mathcal{A} \) to itself is denoted by \( \text{Iso}_{\text{cay}}(\mathcal{A}) \). Up to notation, the following statement is known as the Schur theorem on multipliers (see [15, statement (a) of Theorem 23.9]).

**Lemma 2.1.** Let \( \mathcal{A} \) be an S-ring over an abelian group \( G \). Then

\[
Z(\text{Aut}(G)) \leq \text{Iso}_{\text{cay}}(\mathcal{A}).
\]

The elements of \( S \) and the number \( \text{rk}(A) = |S| \) are called, respectively, the basic sets and rank of the S-ring \( A \). Any union of basic sets is called an \( \mathcal{A} \)-subset of \( G \) or \( \mathcal{A} \)-set; the set of all of them is denoted by \( S(A)^\mathcal{A} \). The latter set is closed with respect to taking inverse and product. Given \( X \in S(A)^\mathcal{A} \) the submodule of \( \mathcal{A} \) spanned by the set

\[
S(\mathcal{A})_X = \{Y \in S(\mathcal{A}) : Y \subset X\}
\]

is denoted by \( A_X \).

Any subgroup of \( G \) that is an \( \mathcal{A} \)-set is called an \( \mathcal{A} \)-subgroup of \( G \) or \( \mathcal{A} \)-group; the set of all of them is denoted by \( \mathfrak{S}(\mathcal{A}) \). With each \( \mathcal{A} \)-set \( X \), one can naturally associate two \( \mathcal{A} \)-groups, namely \( (X) \) and \( \text{rad}(X) \) (see Propositions 23.6 and 23.5 in [15], respectively). The following useful lemma was proved in [7, Lemma 1.2] (see also [4, Lemma 2.1]).

**Lemma 2.2.** Let \( \mathcal{A} \) be an S-ring over a group \( G \), \( H \in \mathfrak{S}(\mathcal{A}) \) and \( X \in S(\mathcal{A}) \). Then the cardinality of the set \( X \cap Hx \) does not depend on \( x \in X \).

Let \( S = U/L \) be a section of \( G \). It is called an \( \mathcal{A} \)-section if both \( U \) and \( L \) are \( \mathcal{A} \)-groups. Given \( X \in S(\mathcal{A})_{U/L} \), the module

\[
\mathcal{A}_S = \text{Span}\{\pi_S(X) : X \in S(A)_{U/L}\}
\]

is an S-ring over the group \( S \), where \( \pi_S : U \rightarrow S \) is the natural epimorphism. The basic sets of \( \mathcal{A}_S \) are exactly the sets from the right-hand side of the formula.

**2.2. Wreath and tensor products.** Let \( S = U/L \) be an \( \mathcal{A} \)-section. The S-ring \( \mathcal{A} \) is called an S-wreath product if \( L \leq G \) and \( L \leq \text{rad}(X) \) for all basic sets \( X \) outside \( U \); in this case, we write

\[
\mathcal{A} = \mathcal{A}_U \uparrow_S \mathcal{A}_{G/L},
\]

and omit \( S \) when \( U = L \). When the explicit indication of the section \( S \) is not important, we use the term generalized wreath product and omit \( S \) in the previous notation. The S-wreath product is nontrivial or proper if \( 1 \neq L \) and \( U \neq G \).

If \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are S-rings over groups \( G_1 \) and \( G_2 \) respectively, then the subring \( \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \) of the ring \( \mathbb{Z}G_1 \otimes \mathbb{Z}G_2 = \mathbb{Z}G \), where \( G = G_1 \times G_2 \), is an S-ring over the group \( G \) with

\[
S(\mathcal{A}) = \{X_1 \times X_2 : X_1 \in S(\mathcal{A}_1), X_2 \in S(\mathcal{A}_2)\}.
\]

It is called the tensor product of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \). The following statement was proved in [4, Lemma 2.3]
4.00 I.N. PONOMARENKO, G.K. RYABOV

Lemma 2.3. Let $\mathcal{A}$ be an $S$-ring over an abelian group $G = C \times D$. Assume that $C$ and $D$ are $\mathcal{A}$-groups. Then

1. $X_C$ and $X_D$ are basic sets of $\mathcal{A}$ for all $X \in \mathcal{S}(\mathcal{A})$.
2. $\mathcal{A} \supseteq \mathcal{A}_C \otimes \mathcal{A}_D$, and the equality holds if $\mathcal{A}_C$ or $\mathcal{A}_D$ is the group ring.

2.3. Cyclotomic $S$-rings. An $S$-ring $\mathcal{A}$ over a group $G$ is said to be cyclotomic if there exists $M \leq \text{Aut}(G)$ such that $\mathcal{S}(\mathcal{A}) = \text{Orb}(M, G)$. In this case, $\mathcal{A}$ is denoted by $\text{Cyc}(M, G)$. Obviously, the group $K = G_{\text{right}}M$ satisfies condition (1). Thus, any cyclotomic $S$-ring is schurian. When $M$ is a multiplicative subgroup of a finite field $F$, we say that $\mathcal{A}$ is a cyclotomic $S$-ring over $F$. For such a ring, the group $\text{Iso}_cay(\mathcal{A})$ contains the Frobenius automorphism of the field $F$. The following lemma has been reproved many times in various situations, see for example [10]. Below we formulate it in the form which is a special case of [1, Theorem 5.1].

Lemma 2.4. For every prime $p$, each $S$-ring over a group $C_p$ is cyclotomic.

2.4. Duality. Let $G$ be an abelian group. Denote by $\hat{G}$ the group dual to $G$, i.e., the group of all irreducible complex characters of $G$. It is well known that there is a uniquely determined lattice antiisomorphism between the subgroups of $G$ and $\hat{G}$ [13]. The image of the group $H$ with respect to this antiisomorphism is denoted by $H^\perp$.

The duality theory of $S$-rings over abelian groups goes back to [14]. Namely, for any $S$-ring $\mathcal{A}$ over the group $G$, the dual $S$-ring $\hat{\mathcal{A}}$ over $\hat{G}$ can be defined as follows: two irreducible characters of $G$ belong to the same basic set of $\hat{\mathcal{A}}$ if they have the same value on each basic set of $\mathcal{A}$ (for the exact definition, we refer to [1, 2]). One can prove that $\text{rk}(\hat{\mathcal{A}}) = \text{rk}(\mathcal{A})$ and the $S$-ring dual to $\hat{\mathcal{A}}$ is equal to $\mathcal{A}$. The following statement collect some facts on the dual $S$-rings proved in [2, Sec. 2.3].

Lemma 2.5. Let $\mathcal{A}$ be an $S$-ring over an abelian group $G$. Then

1. the mapping $\mathfrak{g}(\mathcal{A}) \rightarrow \mathfrak{g}(\hat{\mathcal{A}})$, $H \mapsto H^\perp$ is a lattice antiisomorphism,
2. $\hat{\mathcal{A}}_H = \hat{\mathcal{A}}_{G/H^\perp}^\perp$ and $\hat{\mathcal{A}}_{G/H} = \hat{\mathcal{A}}_{H^\perp}$ for every $H \in \mathfrak{g}(\mathcal{A})$,
3. $\mathcal{A} = \text{Cyc}(K, G)$ for $K \leq \text{Aut}(G)$ if and only if $\hat{\mathcal{A}} = \text{Cyc}(K, \hat{G})$,
4. $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ if and only if $\hat{\mathcal{A}} = \hat{\mathcal{A}}_1 \otimes \hat{\mathcal{A}}_2$,
5. $\mathcal{A}$ is the $U/L$-wreath product if and only if $\hat{\mathcal{A}}$ is the $L^\perp/U^\perp$-wreath product.

2.5. Subdirect product. Let $U$ and $V$ be groups. Assume that $\varphi$ and $\psi$ be homomorphisms from $U$ and from $V$ onto isomorphic groups, i.e., there exists $f \in \text{Iso}(\text{im}(\varphi), \text{im}(\psi))$.

In this situation, one can define the subdirect product of the groups $U$ and $V$ with respect to the homomorphisms $\varphi$, $\psi$, and the isomorphism $f$ as the following subgroup of $U \times V$:

$$U \prod_f V = \{(u, v) \in U \times V : f(\varphi(u)) = \psi(v)\}.$$
It is easily seen that if the groups $\text{im}(\varphi)$ and $\text{im}(\psi)$ are trivial, then the subdirect product equals $U \times V$.

In what follows, we are interested in the orbits of subdirect products of permutation groups, in which the homomorphisms $\varphi$ and $\psi$ are induced by the actions of the groups on imprimitivity systems associated with normal subgroups. More exactly, let $A$ and $P$ be groups, and we are given the following data:

1. $U_0 \leq U \leq \text{Aut}(A)$ and $V_0 \leq V \leq \text{Aut}(P)$,
2. $X_A \in \text{Orb}(U, A)$ and $X_P \in \text{Orb}(V, P)$,
3. A bijection $f_0 : \text{Orb}(U_0, X_A) \to \text{Orb}(V_0, X_P)$.

Under these conditions, the sets $\Pi_A = \text{Orb}(U_0, A)$ and $\Pi_P = \text{Orb}(V_0, P)$ form imprimitivity systems of the (transitive) groups $U^{X_A} \leq \text{Sym}(X_A)$ and $V^{X_P} \leq \text{Sym}(X_P)$.

Denote by $\varphi$ and $\psi$ the natural epimorphisms from $U$ onto $U^{\Pi_A}$ and from $V$ onto $V^{\Pi_P}$, respectively. Note that the permutation groups $U^{\Pi_A}$ and $V^{\Pi_P}$ are regular (this follows from the normality of the groups $U_0$ and $V_0$). Therefore, each isomorphism from $U^{\Pi_A}$ onto $V^{\Pi_P}$ is induced by a certain bijection $f_0$ of the form given in condition (P3).

**Lemma 2.6.** In the above notation, let $G = A \times P$. Assuming $f \in \text{Iso}(U^{\Pi_A}, V^{\Pi_P})$, denote by $K$ the subdirect product of the groups $U$ and $V$ with respect to the homomorphisms $\varphi$, $\psi$, and $f$. Then $K \leq \text{Aut}(A) \times \text{Aut}(P)$ and

$$\bigcup_{Y \in \Pi_A} Y \times Y_f \in \text{Orb}(K, G).$$

### 3. S-rings over $E_9$

Up to Cayley isomorphism, there are exactly ten S-rings over $E_9$. This can be checked in a straightforward way or with the help of the GAP package COCO2 [5]. In this section, we cite relevant properties of these S-ring to be used in Sections 4 and 5. The first statement can be established by inspecting the above ten S-rings one after the other.

**Theorem 3.1.** Every S-ring over a group $E_9$ is Cayley isomorphic to one of the S-rings listed below:

1. $\text{Cyc}(M, \mathbb{F})$, where $\mathbb{F} = \text{GF}(9)$ and $1 < M \leq \mathbb{F}^\times$,
2. the tensor product of two S-rings over $C_3$,
3. the wreath product of two S-rings over $C_3$.

In statement (1), $M$ is a cyclic group of order 2, 4, or 8 (in the last case, the corresponding S-ring is of rank 2). In statement (2), there are three S-rings of ranks 4, 6, and 9 (the last one is $ZE_9$). In statement (3), there are four S-rings of ranks 3, 4, 4, and 5. To establish some properties of this S-rings, we need an auxiliary notion.

Let $G$ be an abelian group $G$ and $X$ an orbit of a subgroup of $\text{Aut}(G)$; in particular, $X = X^{-1}$ or $X \cap X^{-1} = \varnothing$. A uniform partition of $X$ is said to be regular if the condition $X = X^{-1}$ implies that

(R1) $\Pi^{-1} = \Pi$.

$^1$A partition of a set is said to be uniform if all its classes have the same cardinality.
(R2) the permutation $Y \mapsto Y^{-1}$, $Y \in \Pi$, is either trivial or fixed point free,
(R3) $X \circ \sum_{Y \in \Pi} Y^{-1} = \alpha X$ for some integer $\alpha \geq 0$.
Thus, in the case $X \cap X^{-1} = \emptyset$, any uniform partition of $X$ is regular. If $X = X^{-1}$, then the partition of $X$ into one class is regular. A less trivial example is given by groups $L \leq K \leq \text{Aut}(G)$: in this case, one can take any $X \in \text{Orb}(K, G)$ and $\Pi = \text{Orb}(L, X)$.

Lemma 3.2. Let $\mathcal{A}$ be an $S$-ring over a group $G = E_3$, and let $X \in \mathcal{S}(\mathcal{A})$. Then for any regular partition $\Pi$ of $X$, there exist groups $L \leq M \leq \text{Aut}(G)$ such that
$$\Pi = \text{Orb}(L, X) \quad \text{and} \quad X \in \text{Orb}(M, G).$$
Moreover, the group $M$ is cyclic unless $\mathcal{A}$ is the tensor product of two $S$-rings of rank 2 and $|X| = 4$. In the latter case, $M = E_4$.

Proof. According to Theorem 3.1, we have the following cases:

(X1) $X$ is an orbit of a Singer subgroup $M$ of the group $\text{Aut}(G) \cong \text{GL}(2, p)$; in particular, $|X| = 2, 4, 8, 8$;
(X2) $X$ is an orbit of a subgroup of $\text{Aut}(C) \times \text{Aut}(C') \leq \text{Aut}(G)$, where $C \cong C' \cong C''$ are such that $G = C \times C''$; in particular, $|X| = 1, 2, 4, 4$;
(X3) $X$ is an orbit of a subgroup $M \leq \text{Aut}(G)$ of order 3 or 6 that stabilizes a group $C \cong C_3$; in particular, $|X| = 1, 2, 3, 6$.

Let $\Pi$ be a regular partition of $X$. Without loss of generality, we may assume that $1 < |\Pi| < |X|$. Since also $|\Pi|$ divides $|X|$, (3)
$$(|X|, |\Pi|) = (4, 2), (8, 2), (8, 4), (6, 2), \text{ or } (6, 3),$$
where the first pair appears in cases (X1) and (X2), the second two appear in (X1), and the last two appear in (X3). In all these cases, the set $X$ is symmetric. A simple counting argument using conditions (R2) and (R3) shows that the permutation defined in (R2) must be trivial unless $(|X|, |\Pi|) = (4, 2)$ for the case (X2), and $(|X|, |\Pi|) = (6, 2)$. Therefore, the number of possible partitions $\Pi$ of the set $X$ is 1 or 3, 3, 1, 1, and 1, respectively to cases listed in (3). Now a straightforward check in each case completes the proof.

Lemma 3.2 shows that the group $L$ equals the kernel of the homomorphism from $M$ to $\text{Sym}(\Pi)$ induced by the action of $M$ on the $\text{Orb}(L, X)$. In what follows, we say that $(M, L)$ is a standard pair for the basic set $X$ and regular partition $\Pi$; though the standard pair is not uniquely determined, the following statement holds true.

Lemma 3.3. In the notation of Lemma 3.2, assume that the group $M^\Pi$ is cyclic. Then for any regular cyclic group $C \leq \text{Sym}(\Pi)$ centralizing the permutation in condition (R2), there exist $\sigma \in \text{Aut}(G)$ such that

1. $(M^\sigma, L)$ is a standard pair for $X$ and $\Pi$,
2. $(M^\sigma)^\Pi = C$.

Proof. The statement is trivial if $|\Pi| \leq 3$, because $\text{Sym}(\Pi)$ contains a unique cyclic subgroup of order $|\Pi|$. Since $|X| \leq 8$ and $|\Pi|$ divides $|X|$, we may assume that $(|X|, |\Pi|) = (4, 4), (6, 6), (8, 8), \text{ or } (8, 4)$.

The condition on $C$ implies that $C \leq \text{Aut}(G)$ in the first three cases. This proves the required statement in these cases, because any two cyclic subgroups of $\text{Aut}(G) \cong \text{GL}(2, p)$
GL(2,3) of the same order at least 4 are conjugate. In case (8,2), the condition on \( C \) implies that permutation in condition (R2) belongs to \( C \). This leaves exactly three possibilities for \( C \), and for each of them \( C = M^{\Pi} \), where \( M \) is one of the three Singer subgroups of \( \text{Aut}(G) \). \( \square \)

4. Dense S-rings over \( E_9 \times C_p \): basic sets

Throughout this section, we assume that \( p > 3 \) and \( G = A \times P \), where \( A = E_9 \) and \( P = C_p \). In what follows, \( A \) is a dense S-ring over \( G \), which means that \( A \) and \( P \) are \( A \)-subgroups of \( G \). By Lemma 2.1, we have

\[ (\tau) \times \text{Aut}(P) = Z(\text{Aut}(G)) \leq \text{Iso}_{cay}(A), \]

where \( \tau \in \text{Aut}(A) \) is the involution taking \( a \) to \( a^{-1} \).

Let \( X \) be a basic set of the S-ring \( A \). In view of Lemma 2.3, the projections \( X_A \subset A \) and \( X_P \subset P \) are basic sets of the S-rings \( A_A \) and \( A_P \). Therefore

\[ X_A \times X_P \in S(A)^{\tau}. \]

By Lemmas 3.2 and 2.4, there exist groups \( U(X) \leq \text{Aut}(A) \) and \( V(X) \leq \text{Aut}(P) \) such that

\[ X_A \in \text{Orb}(U, A) \quad \text{and} \quad X_P \in \text{Orb}(V, P), \]

where \( U = U(X) \) and \( V = V(X) \). For any element \( a \in X_A \), each basic set inside \( X_A \times X_P \) intersects \( \{a\} \times X_P \). Since the group \( V \leq \text{Aut}(G) \) acts transitively on the latter set, formula (4) implies that \( V \) acts transitively on \( S(A)^{X_A \times X_P} \).

**Lemma 4.1.** Let \( \Pi_P(X) = \{ X(a) \}_{a \in X_A} \), where

\[ X(a) = \{ x \in X_P : (a, x) \in X \}. \]

Then there exists a group \( V_0 \leq V \) such that

\[ \Pi_P = \text{Orb}(V_0, X_P). \]

In particular, \( \Pi_P \) is an imprimitivity system for the group \( V^{X_P} \).

**Proof.** Denote by \( V_0 \) the subgroup of \( V \) leaving \( X \) fixed (as a set). From formula (4), it follows that each set \( X_a \) is contained in some \( Y \in \text{Orb}(V_0, X_P) \). On the other hand, \( X(a) \) cannot be smaller than \( Y \) by the definition of \( V_0 \). Thus, \( X_a = Y \). \( \square \)

Let us define an equivalence relation \( \sim \) on the set \( X_A \) by setting \( a \sim b \) if and only if \( X(a) = X(b) \). In particular, all the elements of \( X_A \) are \( \sim \)-equivalent if and only if \( X = X_A \times X_P \). Denote by \( \Pi_A = \Pi_A(X) \) the partition of \( X_A \) into the classes of the equivalence relation \( \sim \). From our definitions and Lemma 4.1, it follows that the mapping

\[ f_0 : \Pi_A \rightarrow \Pi_P, \quad [a] \mapsto X(a), \]

is a well-defined bijection, where \([a]\) denotes the class of the equivalence relation \( \sim \) that contains \( a \in X_A \). Moreover,

\[ X = \bigcup_{Y \in \Pi_A} Y \times Y^{f_0} \]

**Lemma 4.2.** \( \Pi_A \) is a regular partition of \( X \) in the sense of Section 3.
In the above notation, the set $\circ$ and the group $U$ satisfies condition (R3). The contrary that this is not true. Then note that both $f$, i.e., $f$ and the isomorphism $f$ for some integer $\leq K$ and the group $V$. The proof follows that $A$. In particular, there are two distinct $A$ and $A$. Thus, $A$.

Thus, $A$.

Theorem 4.3. In the above notation, the set $X$ is an orbit of the subdirect product $K = K(X)$ of the groups $U$ and $V$ with respect to the homomorphisms

$$\varphi : U \to U^{\Pi_A} \quad \text{and} \quad \psi : V \to V^{\Pi_P},$$

and the isomorphism $f : U^{\Pi_A} \to V^{\Pi_P}$ induced by bijection (6). Furthermore, $K \leq \text{Aut}(G)$

Proof. According to our notation, we are in the situation described by the conditions (P1), (P2), and (P3). Moreover, the group $V^{\Pi_P}$ is cyclic. First, assume that the group $U^{\Pi_A}$ is also cyclic. Then by Lemma 3.3 for $(M, L) = (U, U_0)$, $\Pi = \Pi_A$, and $0 = f^{-1}V^{\Pi_P}$, the standard pair can be chosen so that

$$U^{\Pi_A} = f U^{\Pi_P} f^{-1},$$

i.e., $f \in \text{Iso}(U^{\Pi_A}, V^{\Pi_P})$. Thus, from Lemma 2.6 and relation (7) it immediately follows that

$$K \leq \text{Aut}(A) \times \text{Aut}(P) = \text{Aut}(G) \quad \text{and} \quad X \in \text{Orb}(K, G),$$

as required.

To complete the proof, we show that the group $U^{\Pi_A}$ must be cyclic. Assume on the contrary that this is not true. Then $|\Pi_A| = 4$. Moreover, by statement (1) of Lemma 3.2, the S-ring $A_A$ is the tensor product of two trivial S-rings and $|X| = 4$. In particular, there are two distinct $A_A$-groups $C$ and $D$, each of order 3 and $X = C^# \times D^#$. Note that both $C$ and $D$ are also $A$-groups, and $G = C \times (DP) = D \times (CP)$. By statement (1) of Lemma 2.3, this implies that $Y = X_{DP}$ and $Z = X_{CP}$ are basic sets of $A$. It is easily seen that

$$Y_P = Z_P = X_P \quad \text{and} \quad |\Pi_P(Y)| = |\Pi_P(Z)| = 2 \quad \text{and} \quad \Pi_P(Y) \neq \Pi_P(Z).$$
By Lemma 4.1, this implies that the transitive cyclic group \( V(X)^X \) has two distinct imprimitivity systems, each with exactly two blocks, a contradiction.

For distinct basic sets \( X \) and \( Y \) of the S-ring \( A \), the groups \( K(X) \) and \( K(Y) \) are not necessarily equal: even if the standard pairs for \( X_A \) and \( Y_A \) are equal, the subdirect products \( K(X) \) and \( K(Y) \) may correspond to different isomorphisms \( f \).

The following statement provide a sufficient condition for \( Y \) to be an orbit of \( K \).

In what follows, we set
\[
G'(A) = \{ H \in G(A) : G = H \times H' \text{ for some } H' \in G(A) \}.
\]

Lemma 4.4. Let \( A \) be a dense S-ring over \( G \), \( X, Y \in S(A) \), and \( K = K(X) \). Then \( Y \in \text{Orb}(K,G) \) if at least one of the following conditions is satisfied:

1. \( Y = X_H \) for some \( H \in G'(A) \),
2. \( Y = X^\sigma \) for some \( \sigma \in Z(\text{Aut}(G)) \),
3. \( X_A = Y_A \).

Proof. Under condition (1), the \( A \)-groups \( H \) and \( H' \) are \( K \)-invariant. Therefore, it is easily seen that
\[
X \in \text{Orb}(K,G) \implies X_H, X_{H'} \in \text{Orb}(K,G),
\]
and we are done. Now assume that condition (2) is satisfied. Since the automorphism \( \sigma \) centralizes \( K \leq \text{Aut}(G) \), we conclude that
\[
Y = X^\sigma \in \text{Orb}(K^\sigma, G) = \text{Orb}(K, G),
\]
as required. To complete the proof, it suffices to note that condition (3) is a consequence of conditions (1) and (2).

5. Dense S-rings over \( E_9 \times C_p \) are cyclotomic

The main result of the present section is given in the following theorem. Along the proof, we freely use the notation introduced in Section 4

Theorem 5.1. Every dense S-ring over \( E_9 \times C_p \) is cyclotomic.

Proof. Let \( A \) be a dense S-ring over the group \( G = A \times P \) with \( A = E_9 \) and \( P = C_p \) for a prime \( p > 3 \) (for \( p = 2, 3 \), the required statement can be verified by enumeration of the S-rings over small groups [16]). We divide the proof into three separate cases depending on which statement of Theorem 3.1 holds for the S-ring \( A_A \).

Case 1: \( A_A = \text{Cyc}(M, F) \), where \( F = GF_9 \) and \( 1 < M \leq F^\times \). In this case, \( M \) is a cyclic group of order \( m \in \{2, 4, 8\} \). Fix a basic set
\[
X \in S(A)_{G'(A \cup P)}, |X_A| = m.
\]
Denote by \( K \) the group \( K(X) \) defined in Theorem 4.3. First assume that \( |M| \neq 4 \). Then \( r_k(A_A) = 2 \) or \( S(A_A) \) contains all subgroups of \( A \). It easily follows that
\[
S(A)^\# = \{(X_H)^\sigma : H \in G'(A), \sigma \in Z(\text{Aut}(G))\}.
\]
By Lemma 4.4, this implies that \( A = \text{Cyc}(K, G) \).

Now let \( |M| = 4 \). In this case, for any \( Y \in S(A)^\# \), the set \( Y_A \) is either trivial, or is equal to \( X_A \) or to \( A^\# \setminus X_A \). By Lemma 4.4, it suffices to verify that in the
last case, $X$ and $Y$ are orbits of a certain group $K' \leq \text{Aut}(G)$ (except for one case, $K'$ will be equal to $K$). To this end, set
\begin{equation}
(9) \quad k_X = |\Pi_A(X)| \quad \text{and} \quad k_Y = |\Pi_A(Y)|.
\end{equation}
Each of the numbers $k_X$ and $k_Y$ divides $|X| = |Y| = 4$, and hence is equal to 1, 2, or 4. Let us analyze all these possibilities. It is convenient to denote the eight nontrivial elements of the group $A$ by $a_i^{\pm 1}$, $i = 1, 2, 3, 4$, so that the orbits $X_A$ and $Y_A$ of the group $M$ are of the form:
\begin{equation}
X_A = \{a_1, a_3, a_1^{-1}, a_3^{-1}\} \quad \text{and} \quad Y_A = \{a_2, a_4, a_2^{-1}, a_4^{-1}\}.
\end{equation}
Without loss of generality, we may assume that $a_2 = a_1a_3$ and $a_4 = a_1a_3^{-1}$, and also
\begin{equation}
(10) \quad f_X([a_1]) = f_Y([a_2]) \quad \text{and hence} \quad f_X([a_1]) = f_Y([a_2]),
\end{equation}
where $f_X$ and $f_Y$ are the bijections defined by formula (6) for $X$ and $Y$, respectively.

**Claim 1**: $\{k_X, k_Y\} \neq \{4, 2\}$ and $\{k_X, k_Y\} \neq \{4, 1\}$. Assume, for instance, that $k_X = 4$. Then a straightforward calculation shows that
\begin{equation}
(11) \quad (X_A X) \cap (Y_A \times X_P) = a_2' \cdot X(a_1, a_2) \cup a_4 \cdot X(a_1, a_4^{-1}) \cup a_2^{-1} \cdot X(a_1^{-1}, a_3^{-1}) \cup a_4^{-1} \cdot X(a_1^{-1}, a_3),
\end{equation}
where $X(a_i, a_j) = X(a_i) \cup X(a_j)$ for all $i, j$. Note that the left-hand side of (11) is an $A$-set, because $X_A$, $X$, $Y_A$, and $X_P$ are basic sets of $A$. Furthermore, assumption (10) implies that it contains $a_2X(a_1)$ and hence intersects $Y$ nontrivially. Thus,
\begin{equation}
Y \subseteq (X_A X) \cap (Y_A \times X_P).
\end{equation}
On the other hand, from the form of the right-hand side of (11), it follows that $k_Y \neq 1$, and if $k_Y = 2$, then the cyclic group $V(Y) = V(X)$ has two different imprimitivity systems, each consisting two blocks (Lemma 4.1). Since this is impossible, the claim is proved.

**Claim 2**: if $\{k_X, k_Y\} = \{1, 2\}$, then $X, Y \in \text{Orb}(K', G)$ for some group $K'$ such that $K < K' \leq \text{Aut}(G)$. Without loss of generality, we may assume that $k_X = 1$ and $k_Y = 2$. Note that the Frobenius automorphism $k'$ of the field $F$ is an automorphism of the S-ring $A_A$. It follows that $X_A, Y_A \in \text{Orb}(U', A)$, where $U' = \langle U, k' \rangle$. Set
\begin{equation}
(12) \quad K' = U' \prod_{f} V_f,
\end{equation}
where $\varphi' : U' \rightarrow U'/U_0'$ and $U_0' = \langle M_0, k' \rangle$ with $M_0$ being the subgroup of $M$ of order 2. Then by Lemma 2.6 applied to the set $X$ and trivial bijection $f : \{X_A\} \rightarrow \{X_P\}$, and to the set $Y$ with the bijection $f_Y$, we conclude that $X$ and $Y$ are orbits of the group $K' \leq \text{Aut}(G)$.

By Claims 1 and 2, to complete the proof of the Case 1, we may assume that $k_X = k_Y := k$. If now $k = 1$ or 2, then the bijection (6) is unique and hence $Y \in \text{Orb}(K, G)$. Assume that $k = 4$. In this case, the groups $K(X)$ and $K(Y)$ may correspond to subdirect products with different bijections $f_X$ and $f_Y$. Namely, there are two possibilities:
\begin{equation}
f_X([a_1]) = f_Y([a_1]) \quad \text{or} \quad f_X([a_1]) = f_Y([a_1^{-1}]).
\end{equation}
However, the first case is impossible, because the $A$-set

$$(XY^{-1}) \cap A^# = \{a_2^{\pm 1}, a_3^{\pm 1}\}$$

intersects each of the two different basic sets $X_A$ and $Y_A$ nontrivially. Since in the last case, $Y \in \text{Orb}(K, G)$, we are done.

**Case 2:** $A_A = A_C \otimes A_D$, where $C$ and $D$ are subgroups of $A$ such that $A = C \times D$ and $|C| = |D| = 3$. First assume that one of the S-rings $A_C$ or $A_D$ is the group ring, say the first one. Then by statement (2) of Lemma 2.3 for $G_1 = C$ and $G_2 = DP$, we have

$$X \in S(A)_{C \times P}.$$ 

Then obviously $Y \in \text{Orb}(K, G)$ for all $Y \in S(A)_{C \times P}$.

Next, we observe that $X_A \times X_P$ is the union of $X$ and $X^{-1}$, where depending on whether $A_A/C$ is of rank 2 or 3, we have $X = X^{-1}$ and $|X_A| = 6$ or $X \cap X^{-1} = \emptyset$ and $|X_A| = |X_A^\perp| = 3$, respectively. This easily implies that any $Y \in S(A)$ contained in $(A \setminus C) \times P$ is of the form $X^\sigma$ for some $\sigma \in \hat{Z}(\text{Aut}(G))$. Thus, again $Y \in \text{Orb}(K, G)$ by Lemma 4.4 and hence $A = \text{Cyc}(K, G)$.

Let now $A_C$ and $A_A/C$ be of rank 2 and 3, respectively. Then

$$A^\perp, P^\perp \in S(A)$$

by statement (1) of Lemma 2.5. It follows that $\tilde{A}$ is a dense S-ring over the group $\tilde{G}$. The statements (2) and (3) of that lemma imply that the restriction of $\tilde{A}$ to the group $A^\perp$ is the wreath product of the S-rings of rank 3 and rank 2. By the previous paragraph, we conclude that $\tilde{A}$ is a cyclotomic S-ring over $\tilde{G}$. By statement (3) of Lemma 2.5, this proves that $A$ is a cyclotomic S-ring over $G$.

In the remaining case, both $A_C$ and $A_A/C$ are of rank 2. It follows that $A_A$ is of rank 3 and

$$S^#(A) = \{C^#, A \setminus C\}.$$
Fix arbitrary basic sets $X, Y \in S(A)$ such that
\[ X_A = A \setminus C, \quad Y_A = C^\#, \quad X_P = Y_P \neq 1_P. \]
By Lemma 4.4, it suffices to verify that $X$ and $Y$ are orbits of a certain group $K' \leq \text{Aut}(G)$. To this end, we define the numbers $k_X$ and $k_Y$ by formula (9). Then from Lemmas 4.1 and 4.2, it follows that
\[ k_X \in \{1, 2, 3, 6\} \quad \text{and} \quad k_Y \in \{1, 2\}. \]
As in Case 1, not each combination for the pair $(k_X, k_Y)$ is possible.

**Claim 3**: $(k_X, k_Y) \neq (6, 1)$ and $(k_X, k_Y) \neq (3, 2)$. Let us consider the first case.

Denote by $r$ the cardinality of $X_P$. Then
\[ |X| = r \quad \text{and} \quad |Y| = 2r. \]
The set $G \setminus (A \cup P)$ is partitioned into basic sets $X^\sigma$ and $Y^\sigma$, where $\sigma \in Z(\text{Aut}(G))$. Since $|X^\sigma| = |X|, \ |Y^\sigma| = |Y|$, and $|X_P| = |Y_P| = (p - 1)/r$, we obtain
\[ |S(A)| = |S(A_A)| + |S(A_P)| - 1 + \frac{7(p - 1)}{r}. \]
Next, let $\pi : G \to G/C$ be the natural epimorphism. Since $k_X = 6$, each of the sets $\pi(X^\sigma)$ is of cardinality $r$. It follows that
\[ |S(\hat{A}_{G/C})| = |S(\hat{A}_{A/C})| + |S(\hat{A}_{C/P/C})| - 1 + \frac{2(p - 1)}{r}. \]
For the S-ring $\hat{A}$ dual to $A$, relation (14) holds. Since the S-rings $A_A$ and $A_{G/A}$ as well as $A_P$ and $A_{G/P}$ are isomorphic, statement (2) of Lemma 2.5 and equality (16) yield
\[ |S(\hat{A})| = \frac{7(p - 1)}{r}. \]
Furthermore, $C^\perp \cong C_{3p}$ is an $\hat{A}$-group and the restriction of $\hat{A}$ to this group is isomorphic to the S-ring $A_{G/C}$. Therefore from equality (17), it follows that
\[ |S(\hat{A})|_{C^\perp(A_A \cup P)} = \frac{2(p - 1)}{r}. \]
Now using equalities (18) and (19), we conclude that there are exactly $5(p - 1)/r$ basic sets of $\hat{A}$ outside $A^\perp, P^\perp$, and $C^\perp$. All of these basic sets are obtained from any one of them by applying an automorphism from $Z(\text{Aut}(\hat{G}))$. Therefore they have the same size, say $k$. This implies that
\[ k \frac{5(p - 1)}{r} = |\hat{G} \setminus (A^\perp \cup P^\perp \cup C^\perp)| = 6(p - 1). \]
On the other hand, the S-rings $\hat{A}_P = \text{Cyc}(V, P)$ and $\hat{A}_{P\perp}$ are isomorphic, where $V = V(X) = V(Y)$ is a subgroup of $\text{Aut}(P)$ of order $r$. Therefore the nonidentity basic sets of the latter S-ring are of cardinality $r$. It follows that $r$ divides $k$, which contradicts equality (20).

The proof of the Claim 3 for the case $(k_X, k_Y) = (3, 2)$ differs from the previous argument only in the values of the parameters. Namely, here $|X| = 2r$ and $|Y| = r$. Therefore, the last summands on the right-hand sides in formulas (16) and (18) are equal to $5(p - 1)/r$, and those in formulas (17) and (19) are $(p - 1)/r$. Thus,
equality (20) leads to the equality $4k/r = 6$, which is also impossible, because $r$ divides $k$. The claim is proved.

Let us return to the remaining part of Case 3. By Claim 3 and formula (15), there are six possibilities for the pair $(k_X, k_Y)$. For each of them, we define a group $K' \leq \text{Aut}(G)$ by formula (12), where the standard pair $(U', U'_0)$ is given in the second and third columns of Table 1 below; in the fourth column contains the sizes of the $U'_0$-orbits.

<table>
<thead>
<tr>
<th>$(k_X, k_Y)$</th>
<th>$U'$</th>
<th>$U'_0$</th>
<th>$\text{Orb}(U'_0, A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 1)</td>
<td>$C_6$</td>
<td>$C_6$</td>
<td>[1, 2, 6]</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>$D_{12}$</td>
<td>$\text{Sym}(3)$</td>
<td>[1, 1, 6]</td>
</tr>
<tr>
<td>(2, 1)</td>
<td>$D_{12}$</td>
<td>$\text{Sym}(3)$</td>
<td>[1, 2, 3, 3]</td>
</tr>
<tr>
<td>(2, 2)</td>
<td>$C_6$</td>
<td>$C_3$</td>
<td>[1, 3, 1, 1]</td>
</tr>
<tr>
<td>(3, 1)</td>
<td>$C_6$</td>
<td>$C_2$</td>
<td>[1, 2, 2, 2]</td>
</tr>
<tr>
<td>(6, 2)</td>
<td>$C_6$</td>
<td>1</td>
<td>[1, \ldots, 1]</td>
</tr>
</tbody>
</table>

Table 1. Standard groups for Case 3

A straightforward check shows that in each case, $Y \in \text{Orb}(K', G)$, as required.

6. The proof of Theorem 1.1

We deduce Theorem 1.1 in the end of the section from the theorem below giving a complete description of all $S$-rings over the group $E_9 \times C_p$.

**Theorem 6.1.** Let $A$ be an $S$-ring over a group $G = E_9 \times C_p$, where $p > 3$ is a prime. Then one of the following statements holds:

1. $A$ is trivial or cyclotomic,
2. $A$ is the tensor product of a trivial $S$-ring and an $S$-ring over $C_3$,
3. $A$ is a proper $S$-wreath product with $|S| \leq 3$.

**Proof.** If the $S$-ring $A$ is dense, then we are done with statement (1) by Theorem 5.1. Assume that $A$ is not dense. Then $A$ or $P$ is not an $A$-subgroup of $G$. By the duality (see Lemma 2.5), we may assume that $A \not\in \mathcal{G}(A)$. Denote by $C$ the maximal $A$-subgroup of the group $A$. Clearly, this group is trivial or of order 3. The lemma below is a special case of [4, Lemma 6.2].

**Lemma 6.2.** In the above notation, one of the following statements holds:

1. $A = AC \wr A_{G/C}$ and also $\text{rk}(A_{G/C}) = 2$,
2. $A$ is the $U/L$-wreath product, where $P \leq L < G$ and $U = CL$.

Without loss of generality, we may assume that $A$ is as in statement (2) of Lemma 6.2: otherwise this lemma implies that either $A$ is trivial (if $C = 1$) and statement (1) of Theorem 6.1 holds, or statement (3) of this theorem holds with $S = C/C$. Furthermore, if $U < G$ then $|U/L| \leq 3$ and statement (3) of Theorem 6.1 holds. Thus, we may also assume that $U = G$. Then $C \not\leq L$, for otherwise $G = L$, a contradiction. Since $|C| = 3$, it follows that $C \cap L = e$. Thus,

$$G = C \times L \quad \text{and} \quad |L| = 3p.$$
Note that the S-ring $A_L$ is circulant. Moreover, since $A$ is not an $A$-group, the subgroup of $L$ of order 3 is not an $A_L$-group. According to [6], this implies that
\begin{equation}
\text{rk}(A_L) = 2 \quad \text{or} \quad A_L = A_P \wr A_{L/P}.
\end{equation}
Assume that $A_C = ZC$. Then $A = A_C \otimes A_L$ by statement 2 of Lemma 2.3. In particular, statement (2) of Theorem 6.1 holds, whenever $\text{rk}(A_L) = 2$. On the other hand, if $A_L$ is not trivial, then $A$ is obviously the $CP/P$-wreath product and statement (3) of Theorem 6.1 holds. Thus, we may assume that
\[ \text{rk}(A_C) = 2. \]
Denote by $L_0$ the trivial subgroup of $L$ if $\text{rk}(A_L) = 2$, and the group $P$ otherwise.

In view of (21), we have $L_0 \in \mathfrak{G}(A)$. In particular, $L \setminus L_0$ is an $A$-set.

**Lemma 6.3.** If $X \in S(A)$ is contained in $C^* \times (L \setminus L_0)$, then $X = X_C \times X_L$.

**Proof.** For all $X \in S(A)$ contained in $C^* \times L^*$, we have
\[ |X_L| \leq |X| \leq |C^*| |X_L| = 2|X_L|. \]
It follows that $X_C \times X_L$ is the union of two basic sets $X$ and $X'$ of the same cardinality. Now if $X = X'$, then $X = X_C \times X_L$ and we are done. In the remaining case,
\begin{equation}
|X| = \frac{|X_C| \cdot |X_L|}{2} = \frac{2|X_L|}{2} = |X_L|.
\end{equation}
Assume that $X_L \subseteq L \setminus L_0$. From the definition of the group $L_0$, it follows that $|L \setminus L_0| \leq 3p - 1$. Therefore, equality (22) yields
\begin{equation}
|X| = |X_L| \leq |L \setminus L_0| \leq 3p - 1.
\end{equation}
On the other hand, if $L_0 = 1$, then $\text{rk}(A_L) = 2$ and hence $X_L = L \setminus L_0$. Furthermore, if $L_0 = P$, then $A_L = A_P \wr A_{L/P}$ and hence $X_L = P$ or $X = L \setminus P$. However, the first case is impossible, because by Lemma 2.2 for $H = P$ the number $|X|$ must be even. Thus, in any case, $X_L = L \setminus L_0$ and hence
\[ X \cup X' = C^* \times (L \setminus L_0). \]
The right-hand side includes the set $C_0 := A \setminus C$ of cardinality 6. Therefore, at least one of $X$ or $X'$, say $X$, contains three elements from $C_0$. According to [4, Lemma 6.1], this implies that
\[ |X| \geq |(X \cap C_0)P| \geq 3p, \]
which contradicts inequality (23). \qed

From Lemma 6.3, it follows that if $\text{rk}(A_L) = 2$, then $A = A_C \otimes A_L$ and we are done with statement (2) of Theorem 6.1. To complete the proof, in view of (21) we may assume that $A_L = A_P \wr A_{L/P}$. In this case, statement (1) of Lemma 2.3 and Lemma 6.3 imply that $P \leq \text{rad}(X)$ for all $X \in S(A)_{G \setminus CP}$. It follows that $A$ is the $CP/P$-product and we are done with statement (3) of Theorem 6.1. \qedsymbol

**Proof of the Theorem 1.1.** For $p \leq 3$, the statement follows from the computational results obtained in [16, p. 498]. Let $p > 3$ and $A$ an $S$-ring over $G$. Then by Theorem 6.1, this ring is obviously schurian if statement (1) of this theorem holds. In case of statement (2), the $S$-ring $A$ being the tensor product of two schurian $S$-rings is also schurian. To complete the proof, we may assume that $A$ is a proper $S$-wreath product with $|S| \leq 3$. Then the $S$-ring $A_S$ is either trivial or a group ring.
Thus, the group $\text{Aut}(\mathcal{A}_S)$ is permutation isomorphic to $S_{\text{right}}$, $\text{Sym}(3)$, or $\text{Alt}(3)$. In any case, according to a criterion of schurity of a generalized wreath product [8, Corollary 10.3], the $S$-ring $\mathcal{A}$ is schurian.

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