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ABELIAN SCHUR GROUPS OF ODD ORDER

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Abstract. A finite group G is called a Schur group if any Schur ring over G is associated in a natural way with a subgroup of $\mathrm{Sym}(G)$ that contains all right translations. It is proved that the group $C_3 \times C_3 \times C_p$ is Schur for any prime p. Together with earlier results, this completes a classification of the abelian Schur groups of odd order.

Keywords: Schur rings, Schur groups, permutation groups.

1. Introduction

A *Schur ring* or *S-ring* over a finite group G can be defined as a subring of the group ring $\mathbb{Z}G$ that is a free $\mathbb{Z}$-module spanned by a partition of G closed under taking inverse and containing the identity element e of G as a class (see Section 2 for details). An important example of such a partition is given by the orbits of the point stabilizer K_e of a permutation group K such that

$$G_{right} \leq K \leq \mathrm{Sym}(G), \tag{1}$$

where G_{right} is the group induced by the right translations of G. The corresponding S-rings are said to be *schurian* in honor of I. Schur who studied the S-rings of this type.

In fact, there are a lot of non-shurian S-rings. An infinite family of them can be found in paper of R. Pöschel [10], where he introduced a concept of a *Schur group*:

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the group G is Schur if every S-ring over G is schurian. A motivation for being interested in the Schur groups comes from the problem of testing isomorphism of Cayley graphs, see [6, 9].

In [4], it was proved that every finite abelian Schur group belongs to one of several explicitly given families. Recent results [8, 11] show that two of them are indeed consist of Schur groups. The main result of the present paper is given in the theorem below concerning the next family.

Theorem 1.1. *For any prime p, all S-rings over a group $G = E_9 \times C_p$ are schurian. In particular, G is a Schur group.*

All cyclic Schur groups were classified in [3]. Therefore, as an immediate consequence of this theorem and the above mentioned results, we obtain a classification of all abelian Schur groups of odd order.

Theorem 1.2. *A noncyclic abelian group of odd order is Schur if and only if it is isomorphic to $C_3 \times C_{3^k}$ for an integer $k \geq 1$, or $E_9 \times C_p$ for a prime $p \geq 3$.*

In Section 6, we deduce Theorem 1.1 from Theorem 6.1 stating that any S-ring $\mathcal{A}$ over the group $G = E_9 \times C_p$ is either obtained from two S-rings over smaller groups (and then the schurity of $\mathcal{A}$ is under control) or is a cyclotomic S-ring (and then $\mathcal{A}$ is schurian by definition). The proof of Theorem 6.1 is mainly based on Theorem 5.1 giving a sufficient condition for $\mathcal{A}$ to be cyclotomic. This carried out in three steps. First, the S-rings over E_9 are completely described (Section 3). This enables us to prove that any class of the partition of G associated with $\mathcal{A}$ is an orbit of a suitable subgroup of $\mathrm{Aut}(G)$ (Section 4). At the last step, we show that this subgroup can be chosen the same for all classes (Section 5). For reader's convenience, we cite basic facts on S-rings in Section 2.

Notation.

As usual by $\mathbb{Z}$ we denote the ring of rational integers.

A finite field of order q is denoted by GF_q.

The projections of $X \subseteq A \times B$ to A and B are denoted by X_A and X_B, respectively.

The set of non-identity elements of a group G is denoted by $G^\#$.

The center of a group G is denoted by $Z(G)$.

Let $X \subseteq G$. The subgroup of G generated by X is denoted by $\langle X \rangle$; we also set $\mathrm{rad}(X) = \{g \in G : \ gX = Xg = X\}$.

Let $\sigma \in \mathrm{Aut}(G)$. The element $\sum_{x \in X} x^\sigma$ of the group ring $\mathbb{Z}G$ is denoted by $\underline{X}^\sigma$, and by $\underline{X}$ if σ is the identity.

The componentwise multiplication in the ring $\mathbb{Z}G$ is denoted by $\circ$.

The group of all permutations of G is denoted by $\mathrm{Sym}(G)$.

The induced action of $G \leq \mathrm{Sym}(\Omega)$ on an invariant set $\Delta \subseteq \Omega$ is denoted by G^Δ.

The cyclic group of order n is denoted by C_n.

The elementary abelian group of order p^k is denoted by E_{p^k}.

2. Preliminaries

2.1. **Definitions.** Let G be a finite group. A subring $\mathcal{A}$ of the group ring $\mathbb{Z}G$ is called a *Schur ring* (*S-ring*, for short) over G if there exists a partition $\mathcal{S} = \mathcal{S}(\mathcal{A})$ of G such that

(S1) $\{1_G\} \in \mathcal{S}$,

(S2) $X \in \mathcal{S} \ \Rightarrow\ X^{-1} \in \mathcal{S}$,
(S3) $\mathcal{A} = \mathrm{Span}\{\underline{X} :\ X \in \mathcal{S}\}$.

In particular, condition (S3) implies that $\mathcal{A}$ is closed with respect to the componentwise multiplication $\circ$. A group isomorphism $f : G \to G'$ is called a *Cayley isomorphism* from an S-ring $\mathcal{A}$ over G onto an S-ring $\mathcal{A}'$ over G' if $\mathcal{S}(\mathcal{A})^f = \mathcal{S}(\mathcal{A}')$. The set of Cayley isomorphisms from $\mathcal{A}$ to itself is denoted by $\mathrm{Iso}_{cay}(\mathcal{A})$. Up to notation, the following statement is known as the Schur theorem on multipliers (see [15, statement (a) of Theorem 23.9]).

Lemma 2.1. *Let $\mathcal{A}$ be an S-ring over an abelian group G. Then*

$$Z(\mathrm{Aut}(G)) \leq \mathrm{Iso}_{cay}(\mathcal{A}).$$

The elements of $\mathcal{S}$ and the number $\mathrm{rk}(\mathcal{A}) = |\mathcal{S}|$ are called, respectively, the *basic sets* and *rank* of the S-ring $\mathcal{A}$. Any union of basic sets is called an $\mathcal{A}$*-subset of* G or $\mathcal{A}$*-set*; the set of all of them is denoted by $\mathcal{S}(\mathcal{A})^{\cup}$. The latter set is closed with respect to taking inverse and product. Given $X \in \mathcal{S}(\mathcal{A})^{\cup}$ the submodule of $\mathcal{A}$ spanned by the set

$$\mathcal{S}(\mathcal{A})_X = \{Y \in \mathcal{S}(\mathcal{A}) :\ Y \subset X\}$$

is denoted by $\mathcal{A}_X$.

Any subgroup of G that is an $\mathcal{A}$-set is called an $\mathcal{A}$*-subgroup* of G or $\mathcal{A}$*-group*; the set of all of them is denoted by $\mathfrak{G}(\mathcal{A})$. With each $\mathcal{A}$-set X, one can naturally associate two $\mathcal{A}$-groups, namely $\langle X\rangle$ and $\mathrm{rad}(X)$ (see Propositions 23.6 and 23.5 in [15], respectively). The following useful lemma was proved in [7, Lemma 1.2] (see also [4, Lemma 2.1]).

Lemma 2.2. *Let $\mathcal{A}$ be an S-ring over a group G, $H \in \mathfrak{G}(\mathcal{A})$ and $X \in \mathcal{S}(\mathcal{A})$. Then the cardinality of the set $X \cap Hx$ does not depend on $x \in X$.*

Let $S = U/L$ be a section of G. It is called an $\mathcal{A}$*-section* if both U and L are $\mathcal{A}$-groups. Given $X \in \mathcal{S}(\mathcal{A})_U$, the module

$$\mathcal{A}_S = \mathrm{Span}\{\pi_S(X) :\ X \in \mathcal{S}(\mathcal{A})_U\}$$

is an S-ring over the group S, where $\pi_S : U \to S$ is the natural epimorphism. The basic sets of $\mathcal{A}_S$ are exactly the sets from the right-hand side of the formula.

2.2. **Wreath and tensor products.** Let $S = U/L$ be an $\mathcal{A}$-section. The S-ring $\mathcal{A}$ is called an S*-wreath product* if $L \trianglelefteq G$ and $L \leq \mathrm{rad}(X)$ for all basic sets X outside U; in this case, we write

$$\mathcal{A} = \mathcal{A}_U \wr_S \mathcal{A}_{G/L}, \tag{2}$$

and omit S when $U = L$. When the explicit indication of the section S is not important, we use the term *generalized wreath product* and omit S in the previous notation. The S-wreath product is *nontrivial* or *proper* if $1 \neq L$ and $U \neq G$.

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are S-rings over groups G_1 and G_2 respectively, then the subring $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ of the ring $\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 = \mathbb{Z}G$, where $G = G_1 \times G_2$, is an S-ring over the group G with

$$\mathcal{S}(\mathcal{A}) = \{X_1 \times X_2 : X_1 \in \mathcal{S}(\mathcal{A}_1),\ X_2 \in \mathcal{S}(\mathcal{A}_2)\}.$$

It is called the *tensor product* of $\mathcal{A}_1$ and $\mathcal{A}_2$. The following statement was proved in [4, Lemma 2.3]

Lemma 2.3. *Let $\mathcal{A}$ be an S-ring over an abelian group $G = C \times D$. Assume that C and D are $\mathcal{A}$-groups. Then*

(1) *X_C and X_D are basic sets of $\mathcal{A}$ for all $X \in \mathcal{S}(\mathcal{A})$,*
(2) *$\mathcal{A} \geq \mathcal{A}_C \otimes \mathcal{A}_D$, and the equality holds if $\mathcal{A}_C$ or $\mathcal{A}_D$ is the group ring.*

2.3. **Cyclotomic S-rings.** An S-ring $\mathcal{A}$ over a group G is said to be *cyclotomic* if there exists $M \leq \mathrm{Aut}(G)$ such that

$$\mathcal{S}(\mathcal{A}) = \mathrm{Orb}(M, G).$$

In this case, $\mathcal{A}$ is denoted by $\mathrm{Cyc}(M, G)$. Obviously, the group $K = G_{right}M$ satisfies condition (1). Thus, any cyclotomic S-ring is schurian. When M is a multiplicative subgroup of a finite field $\mathbb{F}$, we say that $\mathcal{A}$ is a cyclotomic S-ring over $\mathbb{F}$. For such a ring, the group $\mathrm{Iso}_{cay}(\mathcal{A})$ contains the Frobenius automorphism of the field $\mathbb{F}$. The following lemma has been reproved many times in various situations, see for example [10]. Below we formulate it in the form which is a special case of [1, Theorem 5.1].

Lemma 2.4. *For every prime p, each S-ring over a group C_p is cyclotomic.*

2.4. **Duality.** Let G be an abelian group. Denote by $\widehat{G}$ the group dual to G, i.e., the group of all irreducible complex characters of G. It is well known that there is a uniquely determined lattice antiisomorphism between the subgroups of G and $\widehat{G}$ [13]. The image of the group H with respect to this antiisomorphism is denoted by $H^{\perp}$.

The duality theory of S-rings over abelian groups goes back to [14]. Namely, for any S-ring $\mathcal{A}$ over the group G, the dual S-ring $\widehat{\mathcal{A}}$ over $\widehat{G}$ can be defined as follows: two irreducible characters of G belong to the same basic set of $\widehat{\mathcal{A}}$ if they have the same value on each basic set of $\mathcal{A}$ (for the exact definition, we refer to [1, 2]). One can prove that

$$\mathrm{rk}(\widehat{\mathcal{A}}) = \mathrm{rk}(\mathcal{A})$$

and the S-ring dual to $\widehat{\mathcal{A}}$ is equal to $\mathcal{A}$. The following statement collect some facts on the dual S-rings proved in [2, Sec. 2.3].

Lemma 2.5. *Let $\mathcal{A}$ be an S-ring over an abelian group G. Then*

(1) *the mapping $\mathfrak{G}(\mathcal{A}) \to \mathfrak{G}(\widehat{\mathcal{A}})$, $H \mapsto H^{\perp}$ is a lattice antiisomorphism,*
(2) *$\widehat{\mathcal{A}_H} = \widehat{\mathcal{A}}_{\widehat{G}/H^{\perp}}$ and $\widehat{\mathcal{A}_{G/H}} = \widehat{\mathcal{A}}_{H^{\perp}}$ for every $H \in \mathfrak{G}(\mathcal{A})$,*
(3) *$\mathcal{A} = \mathrm{Cyc}(K, G)$ for $K \leq \mathrm{Aut}(G)$ if and only if $\widehat{\mathcal{A}} = \mathrm{Cyc}(K, \widehat{G})$,*
(4) *$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ if and only if $\widehat{\mathcal{A}} = \widehat{\mathcal{A}_1} \otimes \widehat{\mathcal{A}_2}$,*
(5) *$\mathcal{A}$ is the U/L-wreath product if and only if $\widehat{\mathcal{A}}$ is the $L^{\perp}/U^{\perp}$-wreath product.*

2.5. **Subdirect product.** Let U and V be groups. Assume that φ and ψ be homomorphisms from U and from V onto isomorphic groups, i.e., there exists

$$f \in \mathrm{Iso}(\mathrm{im}(\varphi), \mathrm{im}(\psi)).$$

In this situation, one can define the subdirect product of the groups U and V with respect to the homomorphisms φ, ψ, and the isomorphism f as the following subgroup of $U \times V$:

$$U \prod_{f}^{\varphi,\psi} V = \{(u, v) \in U \times V : \ f(\varphi(u)) = \psi(v)\}.$$

It is easily seen that if the groups $\mathrm{im}(\varphi)$ and $\mathrm{im}(\psi)$ are trivial, then the subdirect product equals $U \times V$.

In what follows, we are interested in the orbits of subdirect products of permutation groups, in which the homomorphisms φ and ψ are induced by the actions of the groups on imprimitivity systems associated with normal subgroups. More exactly, let A and P be groups, and we are given the following data:

(P1) $U_0 \trianglelefteq U \le \mathrm{Aut}(A)$ and $V_0 \trianglelefteq V \le \mathrm{Aut}(P)$,
(P2) $X_A \in \mathrm{Orb}(U, A)$ and $X_P \in \mathrm{Orb}(V, P)$,
(P3) a bijection $f_0 : \mathrm{Orb}(U_0, X_A) \to \mathrm{Orb}(V_0, X_P)$.

Under these conditions, the sets $\Pi_A = \mathrm{Orb}(U_0, A)$ and $\Pi_P = \mathrm{Orb}(V_0, P)$ form imprimitivity systems of the (transitive) groups

$$U^{X_A} \le \mathrm{Sym}(X_A) \quad \text{and} \quad V^{X_P} \le \mathrm{Sym}(X_P).$$

Denote by φ and ψ the natural epimorphisms from U onto U^{Π_A} and from V onto V^{Π_P}, respectively. Note that the permutation groups U^{Π_A} and V^{Π_P} are regular (this follows from the normality of the groups U_0 and V_0). Therefore, each isomorphism from U^{Π_A} onto V^{Π_P} is induced by a certain bijection f_0 of the form given in condition (P3).

Lemma 2.6. *In the above notation, let $G = A \times P$. Assuming $f \in \mathrm{Iso}(U^{\Pi_A}, V^{\Pi_P})$, denote by K the subdirect product of the groups U and V with respect to the homomorphisms φ, ψ, and f. Then $K \le \mathrm{Aut}(A) \times \mathrm{Aut}(P)$ and*

$$\bigcup_{Y \in \Pi_A} Y \times Y^{f_0} \in \mathrm{Orb}(K, G).$$

3. S-rings over E_9

Up to Cayley isomorphism, there are exactly ten S-rings over E_9. This can be checked in a straightforward way or with the help of the GAP package COCO2 [5]. In this section, we cite relevant properties of these S-ring to be used in Sections 4 and 5. The first statement can be established by inspecting the above ten S-rings one after the other.

Theorem 3.1. *Every S-ring over a group E_9 is Cayley isomorphic to one of the S-rings listed below:*

(1) $\mathrm{Cyc}(M, \mathbb{F})$, *where* $\mathbb{F} = \mathrm{GF}_9$ *and* $1 < M \le \mathbb{F}^\times$,
(2) *the tensor product of two S-rings over* C_3,
(3) *the wreath product of two S-rings over* C_3.

In statement (1), M is a cyclic group of order 2, 4, or 8 (in the last case, the corresponding S-ring is of rank 2). In statement (2), there are three S-rings of ranks 4, 6, and 9 (the last one is $\mathbb{Z}E_9$). In statement (3), there are four S-rings of ranks 3, 4, 4, and 5. To establish some properties of this S-rings, we need an auxiliary notion.

Let G be an abelian group G and X an orbit of a subgroup of $\mathrm{Aut}(G)$; in particular, $X = X^{-1}$ or $X \cap X^{-1} = \varnothing$. A uniform partition[1] Π of X is said to be *regular* if the condition $X = X^{-1}$ implies that

(R1) $\Pi^{-1} = \Pi$,

[1] A partition of a set is said to be *uniform* if all its classes have the same cardinality

(R2) the permutation $Y \mapsto Y^{-1}$, $Y \in \Pi$, is either trivial or fixed point free,
(R3) $\underline{X} \circ \sum_{Y\in\Pi} \underline{Y}\,\underline{Y}^{-1} = \alpha\,\underline{X}$ for some integer $\alpha \geq 0$.

Thus, in the case $X \cap X^{-1} = \varnothing$, any uniform partition of X is regular. If $X = X^{-1}$, then the partition of X into one class is regular. A less trivial example is given by groups $L \trianglelefteq K \leq \mathrm{Aut}(G)$: in this case, one can take any $X \in \mathrm{Orb}(K,G)$ and $\Pi = \mathrm{Orb}(L,X)$.

Lemma 3.2. *Let $\mathcal{A}$ be an S-ring over a group $G = E_9$, and let $X \in \mathcal{S}(\mathcal{A})$. Then for any regular partition Π of X, there exist groups $L \leq M \leq \mathrm{Aut}(G)$ such that*

$$\Pi = \mathrm{Orb}(L,X) \quad \textit{and} \quad X \in \mathrm{Orb}(M,G).$$

Moreover, the group M is cyclic unless $\mathcal{A}$ is the tensor product of two S-rings of rank 2 and $|X| = 4$. In the latter case, $M = E_4$.

Proof. According to Theorem 3.1, we have the following cases:

(X1) X is an orbit of a Singer subgroup M of the group $\mathrm{Aut}(G) \cong \mathrm{GL}(2,p)$; in particular, $|X| = 2$, 4, or 8;
(X2) X is an orbit of a subgroup of $\mathrm{Aut}(C) \times \mathrm{Aut}(C') \leq \mathrm{Aut}(G)$, where $C \cong C_3 \cong C'$ are such that $G = C \times C'$; in particular, $|X| = 1$, 2, or 4;
(X3) X is an orbit of a subgroup $M \leq \mathrm{Aut}(G)$ of order 3 or 6 that stabilizes a group $C \cong C_3$; in particular, $|X| = 1$, 2, 3, or 6.

Let Π be a regular partition of X. Without loss of generality, we may assume that $1 < |\Pi| < |X|$. Since also $|\Pi|$ divides $|X|$,

$$(|X|,|\Pi|) = (4,2),\ (8,2),\ (8,4),\ (6,2),\ \text{or}\ (6,3), \tag{3}$$

where the first pair appears in cases (X1) and (X2), the second two appear in (X1), and the last two appear in (X3). In all these cases, the set X is symmetric. A simple counting argument using conditions (R2) and (R3) shows that the permutation defined in (R2) must be trivial unless $(|X|,|\Pi|) = (4,2)$ for the case (X2), and $(|X|,|\Pi|) = (6,2)$. Therefore, the number of possible partitions Π of the set X is 1 or 3, 3, 1, 1, and 1, respectively to cases listed in (3). Now a straightforward check in each case completes the proof. □

Lemma 3.2 shows that the group L equals the kernel of the homomorphism from M to $\mathrm{Sym}(\Pi)$ induced by the action of M on the $\mathrm{Orb}(L,X)$. In what follows, we say that (M,L) is a *standard pair* for the basic set X and regular partition Π; though the standard pair is not uniquely determined, the following statement holds true.

Lemma 3.3. *In the notation of Lemma 3.2, assume that the group M^{Π} is cyclic. Then for any regular cyclic group $C \leq \mathrm{Sym}(\Pi)$ centralizing the permutation in condition* (R2)*, there exist $\sigma \in \mathrm{Aut}(G)$ such that*

(1) *(M^{σ},L) is a standard pair for X and Π,*
(2) *$(M^{\sigma})^{\Pi} = C$.*

Proof. The statement is trivial if $|\Pi| \leq 3$, because $\mathrm{Sym}(\Pi)$ contains a unique cyclic subgroup of order $|\Pi|$. Since $|X| \leq 8$ and $|\Pi|$ divides $|X|$, we may assume that

$$(|X|,|\Pi|) = (4,4),\ (6,6),\ (8,8),\ \text{or}\ (8,4).$$

The condition on C implies that $C \leq \mathrm{Aut}(G)$ in the first three cases. This proves the required statement in these cases, because any two cyclic subgroups of $\mathrm{Aut}(G) \cong$

$\mathrm{GL}(2,3)$ of the same order at least 4 are conjugate. In case $(8,2)$, the condition on C implies that permutation in condition (R2) belongs to C. This leaves exactly three possibilities for C, and for each of them $C = M^{\Pi}$, where M is one of the three Singer subgroups of $\mathrm{Aut}(G)$. □

4. Dense S-rings over $E_9 \times C_p$: basic sets

Throughout this section, we assume that $p > 3$ and $G = A \times P$, where $A = E_9$ and $P = C_p$. In what follows, $\mathcal{A}$ is a *dense* S-ring over G, which means that A and P are $\mathcal{A}$-subgroups of G. By Lemma 2.1, we have

$$\langle\tau\rangle \times \mathrm{Aut}(P) = Z(\mathrm{Aut}(G)) \leq \mathrm{Iso}_{cay}(\mathcal{A}), \tag{4}$$

where $\tau \in \mathrm{Aut}(A)$ is the involution taking a to a^{-1}.

Let X be a basic set of the S-ring $\mathcal{A}$. In view of Lemma 2.3, the projections $X_A \subset A$ and $X_P \subset P$ are basic sets of the S-rings $\mathcal{A}_A$ and $\mathcal{A}_P$. Therefore

$$X_A \times X_P \in \mathcal{S}(\mathcal{A})^{\cup}.$$

By Lemmas 3.2 and 2.4, there exist groups $U(X) \leq \mathrm{Aut}(A)$ and $V(X) \leq \mathrm{Aut}(P)$ such that

$$X_A \in \mathrm{Orb}(U, A) \quad \text{and} \quad X_P \in \mathrm{Orb}(V, P),$$

where $U = U(X)$ and $V = V(X)$. For any element $a \in X_A$, each basic set inside $X_A \times X_P$ intersects $\{a\} \times X_P$. Since the group $V \leq \mathrm{Aut}(G)$ acts transitively on the latter set, formula (4) implies that V acts transitively on $\mathcal{S}(\mathcal{A})_{X_A \times X_P}$.

Lemma 4.1. *Let* $\Pi_P(X) = \{X(a)\}_{a \in X_A}$, *where*

$$X(a) = \{x \in X_P : \ (a, x) \in X\}. \tag{5}$$

Then there exists a group $V_0 \leq V$ *such that*

$$\Pi_P = \mathrm{Orb}(V_0, X_P).$$

In particular, Π_P *is an imprimitivity system for the group* V^{X_P}.

Proof. Denote by V_0 the subgroup of V leaving X fixed (as a set). From formula (4), it follows that each set X_a is contained in some $Y \in \mathrm{Orb}(V_0, X_P)$. On the other hand, $X(a)$ cannot be smaller than Y by the definition of V_0. Thus, $X_a = Y$. □

Let us define an equivalence relation $\sim$ on the set X_A by setting $a \sim b$ if and only if $X(a) = X(b)$. In particular, all the elements of X_A are $\sim$-equivalent if and only if $X = X_A \times X_P$. Denote by $\Pi_A = \Pi_A(X)$ the partition of X_A into the classes of the equivalence relation $\sim$. From our definitions and Lemma 4.1, it follows that the mapping

$$f_0 : \Pi_A \to \Pi_P, \ [a] \mapsto X(a), \tag{6}$$

is a well-defined bijection, where $[a]$ denotes the class of the equivalence relation $\sim$ that contains $a \in X_A$. Moreover,

$$X = \bigcup_{Y \in \Pi_A} Y \times Y^{f_0} \tag{7}$$

Lemma 4.2. Π_A *is a regular partition of* X *in the sense of Section 3.*

Proof. It is easily seen that $[a]y = X \cap Ay$ for all $a \in X_A$ and $y \in X(a)$. By Lemma 2.2, this implies that the partition Π_A is uniform. Without loss of generality, we may assume that X_A is symmetric. By formula (4) and since $\mathrm{Aut}(P)$ acts transitively on $\mathcal{A}_{X_A \times X_P}$, there exists $\sigma \in V$ such that $(X^\tau)^\sigma = X$. In view of equality (7) and Lemma 4.1, this implies that

$$([a] \times X(a))^{\tau\sigma} = [a]^{-1} \times X(a)^\sigma = [b] \times X(b)$$

for all $a \in X_A$, where the element $b \in X_P$ is defined by the condition $X(a)^\sigma = X(b)$. Thus, $[a]^{-1} = [b]$ and the partition Π_A satisfies condition (R1). Furthermore, if $[a] = [a]^{-1}$ for some $a \in X_A$, then $X^\tau = X$ by formula (7), and hence the permutation $[a] \to [a]^{-1}$ is trivial. This shows that Π_A satisfies condition (R2). Finally, again by formula (7) we have

$$\sum_{Y \in \Pi_A} \underline{Y}\,\underline{Y}^{-1} = \underline{A} \circ (\underline{X}\,\underline{X}^\tau) = \alpha \underline{X_A} + \xi$$

for some integer $\alpha \geq 0$ and $\xi \in \mathcal{A}$ such that $\underline{X_A} \circ \xi = 0$. This shows that Π_A satisfies condition (R3). □

From Lemmas 4.2 and 3.2, it follows that given $X \in \mathcal{S}(\mathcal{A})$ there exists a standard pair (U, U_0) for the set $X_A \in \mathcal{S}(\mathcal{A})$ and regular partition Π_A. The following statement is the main result of this section.

Theorem 4.3. *In the above notation, the set X is an orbit of the subdirect product $K = K(X)$ of the groups U and V with respect to the homomorphisms*

$$\varphi : U \to U^{\Pi_A} \quad \textit{and} \quad \psi : V \to V^{\Pi_P},$$

and the isomorphism $f : U^{\Pi_A} \to V^{\Pi_P}$ induced by bijection (6). Furthermore, $K \leq \mathrm{Aut}(G)$

Proof. According to our notation, we are in the situation described by the conditions (P1), (P2), and (P3). Moreover, the group V^{Π_P} is cyclic. First, assume that the group U^{Π_A} is also cyclic. Then by Lemma 3.3 for $(M, L) = (U, U_0)$, $\Pi = \Pi_A$, and $C = f^{-1} V_P^{\Pi_P} f$, the standard pair can be chosen so that

$$U^{\Pi_A} = f\, U^{\Pi_P} f^{-1},$$

i.e., $f \in \mathrm{Iso}(U^{\Pi_A}, V^{\Pi_P})$. Thus, from Lemma 2.6 and relation (7) it immediately follows that

$$K \leq \mathrm{Aut}(A) \times \mathrm{Aut}(P) = \mathrm{Aut}(G) \quad \text{and} \quad X \in \mathrm{Orb}(K, G),$$

as required.

To complete the proof, we show that the group U^{Π_A} must be cyclic. Assume on the contrary that this is not true. Then $|\Pi_A| = 4$. Moreover, by statement (1) of Lemma 3.2, the S-ring $\mathcal{A}_A$ is the tensor product of two trivial S-rings and $|X| = 4$. In particular, there are two distinct $\mathcal{A}_A$-groups C and D, each of order 3 and

$$X = C^\# \times D^\#.$$

Note that both C and D are also $\mathcal{A}$-groups, and $G = C \times (DP) = D \times (CP)$. By statement (1) of Lemma 2.3, this implies that $Y = X_{DP}$ and $Z = X_{CP}$ are basic sets of $\mathcal{A}$. It is easily seen that

$$Y_P = Z_P = X_P \quad \text{and} \quad |\Pi_P(Y)| = |\Pi_P(Z)| = 2 \quad \text{and} \quad \Pi_P(Y) \neq \Pi_P(Z).$$

By Lemma 4.1, this implies that the transitive cyclic group $V(X)^{X_P}$ has two distinct imprimitivity systems, each with exactly two blocks, a contradiction. □

For distinct basic sets X and Y of the S-ring $\mathcal{A}$, the groups $K(X)$ and $K(Y)$ are not necessarily equal: even if the standard pairs for X_A and Y_A are equal, the subdirect products $K(X)$ and $K(Y)$ may correspond to different isomorphisms f. The following statement provide a sufficient condition for Y to be an orbit of $K(X)$. In what follows, we set

$$\mathfrak{G}(\mathcal{A})' = \{H \in \mathfrak{G}(\mathcal{A}) :\ G = H \times H' \text{ for some } H' \in \mathfrak{G}(\mathcal{A})\}.$$

Lemma 4.4. *Let $\mathcal{A}$ be a dense S-ring over G, $X, Y \in \mathcal{S}(\mathcal{A})$, and $K = K(X)$. Then $Y \in \mathrm{Orb}(K, G)$ if at least one of the following conditions is satisfied:*

(1) *$Y = X_H$ for some $H \in \mathfrak{G}(\mathcal{A})'$,*
(2) *$Y = X^\sigma$ for some $\sigma \in Z(\mathrm{Aut}(G))$,*
(3) *$X_A = Y_A$.*

Proof. Under condition (1), the $\mathcal{A}$-groups H and H' are K-invariant. Therefore, it is easily seen that

$$X \in \mathrm{Orb}(K, G) \quad \Rightarrow \quad X_H, X_{H'} \in \mathrm{Orb}(K, G),$$

and we are done. Now assume that condition (2) is satisfied. Since the automorphism σ centralizes $K \leq \mathrm{Aut}(G)$, we conclude that

$$Y = X^\sigma \in \mathrm{Orb}(K^\sigma, G) = \mathrm{Orb}(K, G),$$

as required. To complete the proof, it suffices to note that condition (3) is a consequence of conditions (1) and (2). □

5. Dense S-rings over $E_9 \times C_p$ are cyclotomic

The main result of the present section is given in the following theorem. Along the proof, we freely use the notation introduced in Section 4

Theorem 5.1. *Every dense S-ring over $E_9 \times C_p$ is cyclotomic.*

Proof. Let $\mathcal{A}$ be a dense S-ring over the group $G = A \times P$ with $A = E_9$ and $P = C_p$ for a prime $p > 3$ (for $p = 2, 3$, the required statement can be verified by enumeration of the S-rings over small groups [16]). We divide the proof into three separate cases depending on which statement of Theorem 3.1 holds for the S-ring $\mathcal{A}_A$.

Case 1: $\mathcal{A}_A = \mathrm{Cyc}(M, \mathbb{F})$, where $\mathbb{F} = \mathrm{GF}_9$ and $1 < M \leq \mathbb{F}^\times$. In this case, M is a cyclic group of order $m \in \{2, 4, 8\}$. Fix a basic set

$$X \in \mathcal{S}(\mathcal{A})_{G \setminus (A \cup P)},\ |X_A| = m.$$

Denote by K the group $K(X)$ defined in Theorem 4.3. First assume that $|M| \neq 4$. Then $\mathrm{rk}(\mathcal{A}_A) = 2$ or $\mathfrak{G}(\mathcal{A}_A)$ contains all subgroups of A. It easily follows that

$$\mathcal{S}(\mathcal{A})^\# = \{(X_H)^\sigma :\ H \in \mathfrak{G}'(\mathcal{A}),\ \sigma \in Z(\mathrm{Aut}(G))\}. \tag{8}$$

By Lemma 4.4, this implies that $\mathcal{A} = \mathrm{Cyc}(K, G)$.

Now let $|M| = 4$. In this case, for any $Y \in \mathcal{S}(\mathcal{A})^\#$, the set Y_A is either trivial, or is equal to X_A or to $A^\# \setminus X_A$. By Lemma 4.4, it suffices to verify that in the last case, X and Y are orbits of a certain group $K' \le \mathrm{Aut}(G)$ (except for one case, K' will be equal to K). To this end, set

$$k_X = |\Pi_A(X)| \quad \text{and} \quad k_Y = |\Pi_A(Y)|. \tag{9}$$

Each of the numbers k_X and k_Y divides $|X| = |Y| = 4$, and hence is equal to 1, 2, or 4. Let us analyze all these possibilities. It is convenient to denote the eight nontrivial elements of the group A by $a_i^{\pm 1}$, $i = 1, 2, 3, 4$, so that the orbits X_A and Y_A of the group M are of the form:

$$X_A = \{a_1, a_3, a_1^{-1}, a_3^{-1}\} \quad \text{and} \quad Y_A = \{a_2, a_4, a_2^{-1}, a_4^{-1}\}.$$

Without loss of generality, we may assume that $a_2 = a_1 a_3$ and $a_4 = a_1 a_3^{-1}$, and also

$$f_X([a_1]) = f_Y([a_2]) \quad \text{and hence } f_X([a_1]) = f_Y([a_2]), \tag{10}$$

where f_X and f_Y are the bijections defined by formula (6) for X and Y, respectively.

Claim 1: $\{k_X, k_Y\} \ne \{4, 2\}$ and $\{k_X, k_Y\} \ne \{4, 1\}$. Assume, for instance, that $k_X = 4$. Then a straightforward calculation shows that

$$\begin{aligned}(X_A\, X) \cap (Y_A \times X_P) = {} & a_2 \; X(a_1 \;, a_3 \;) \cup a_4 \; X(a_1 \;, a_3^{-1}) \cup \\ & a_2^{-1} X(a_1^{-1}, a_3^{-1}) \cup a_4^{-1} X(a_1^{-1}, a_3 \;),\end{aligned} \tag{11}$$

where $X(a_i, a_j) = X(a_i) \cup X(a_j)$ for all i, j. Note that the left-hand side of (11) is an $\mathcal{A}$-set, because X_A, X, Y_A, and X_P are basic sets of $\mathcal{A}$. Furthermore, assumption (10) implies that it contains $a_2 X(a_1)$ and hence intersects Y nontrivially. Thus,

$$Y \subseteq (X_A\, X) \cap (Y_A \times X_P).$$

On the other hand, from the form of the right-hand side of (11), it follows that $k_Y \ne 1$, and if $k_Y = 2$, then the cyclic group $V(Y) = V(X)$ has two different imprimitivity systems, each consisting two blocks (Lemma 4.1). Since this is impossible, the claim is proved. □

Claim 2: if $\{k_X, k_Y\} = \{1, 2\}$, then $X, Y \in \mathrm{Orb}(K', G)$ for some group K' such that $K < K' \le \mathrm{Aut}(G)$. Without loss of generality, we may assume that $k_X = 1$ and $k_Y = 2$. Note that the Frobenius automorphism k' of the field $\mathbb{F}$ is an automorphism of the S-ring $\mathcal{A}_A$. It follows that $X_A, Y_A \in \mathrm{Orb}(U', A)$, where $U' = \langle U, k' \rangle$. Set

$$K' = U' \prod_{f}^{\varphi', \psi} V, \tag{12}$$

where $\varphi' : U' \to U'/U_0'$ and $U_0' = \langle M_0, k' \rangle$ with M_0 being the subgroup of M of order 2. Then by Lemma 2.6 applied to the set X and trivial bijection $f : \{X_A\} \to \{X_P\}$, and to the set Y with the bijection f_Y, we conclude that X and Y are orbits of the group $K' \le \mathrm{Aut}(G)$. □

By Claims 1 and 2, to complete the proof of the Case 1, we may assume that $k_X = k_Y := k$. If now $k = 1$ or 2, then the bijection (6) is unique and hence $Y \in \mathrm{Orb}(K, G)$. Assume that $k = 4$. In this case, the groups $K(X)$ and $K(Y)$ may correspond to subdirect products with different bijections f_X and f_Y. Namely, there are two possibilities:

$$f_X([a_3]) = f_Y([a_4]) \quad \text{or} \quad f_X([a_3]) = f_Y([a_4^{-1}]).$$

However, the first case is impossible, because the $\mathcal{A}$-set

$$(XY^{-1}) \cap A^{\#} = \{a_2^{\pm 1}, a_3^{\pm 1}\}$$

intersects each of the two different basic sets X_A and Y_A nontrivially. Since in the last case, $Y \in \mathrm{Orb}(K, G)$, we are done.

Case 2: $\mathcal{A}_A = \mathcal{A}_C \otimes \mathcal{A}_D$, where C and D are subgroups of A such that $A = C \times D$ and $|C| = |D| = 3$. First assume that one of the S-rings $\mathcal{A}_C$ or $\mathcal{A}_D$ is the group ring, say the first one. Then by statement (2) of Lemma 2.3 for $G_1 = C$ and $G_2 = DP$, we have

$$\mathcal{A} = \mathbb{Z}C \otimes \mathcal{A}_{DP}. \tag{13}$$

Since DP is a cyclic group and D, P are $\mathcal{A}_{DP}$-groups, the classification of S-rings over a cyclic group C_{pq} with primes p and q implies that the S-ring $\mathcal{A}_{DP}$ is cyclotomic (see [6]). Since also the S-ring $\mathbb{Z}C$ is cyclotomic, formula (13) shows that so is the S-ring $\mathcal{A}$.

To complete this case assume that both $\mathcal{A}_C$ and $\mathcal{A}_D$ are S-rings of rank 2. Take any $X \in \mathcal{S}(\mathcal{A})_{G\setminus A}$ such that $X_A = C^{\#} \times D^{\#}$. Then one can easily verify that formula (8) holds. By Lemma 4.4, this implies that $\mathcal{A} = \mathrm{Cyc}(K, G)$ with $K = K(X)$.

Case 3: $\mathcal{A}_A = \mathcal{A}_C \wr \mathcal{A}_{A/C}$, where $C \leq A$ is a group of order 3. Depending on whether $\mathcal{A}_{A/C}$ is of rank 2 or 3, the set $\mathcal{S}(\mathcal{A})_{A\setminus C}$ consists of one set of cardinality 6 or two sets of cardinalities 3, respectively. Fix a basic set

$$X \in \mathcal{S}(\mathcal{A})_{(A\setminus C)\times P^{\#}}.$$

First, assume that $\mathcal{A}_C$ is of rank 3. Since the number $|\Pi_A(X)|$ divides 6, there exists a standard pair (U, U_0) for X_A and $\Pi_A(X)$ such that the set $\mathrm{Orb}(U_0, C)$ consists of singletons. Assume that the group $K = K(X)$ is associated with this pair. Then obviously

$$Y \in \mathrm{Orb}(K, G) \quad \text{for all } Y \in \mathcal{S}(\mathcal{A})_{C\times P}.$$

Next, we observe that $X_A \times X_P$ is the union of X and X^{-1}, where depending on whether $\mathcal{A}_{A/C}$ is of rank 2 or 3, we have

$$X = X^{-1} \text{ and } |X_A| = 6 \quad \text{or} \quad X \cap X^{-1} = \varnothing \text{ and } |X_A| = |X_A^{-1}| = 3,$$

respectively. This easily implies that any $Y \in \mathcal{S}(\mathcal{A})$ contained in $(A \setminus C) \times P$ is of the form X^σ for some $\sigma \in Z(\mathrm{Aut}(G))$. Thus, again $Y \in \mathrm{Orb}(K, G)$ by Lemma 4.4 and hence $\mathcal{A} = \mathrm{Cyc}(K, G)$.

Let now $\mathcal{A}_C$ and $\mathcal{A}_{A/C}$ be of rank 2 and 3, respectively. Then

$$A^{\perp}, P^{\perp} \in \mathfrak{G}(\widehat{\mathcal{A}}) \tag{14}$$

by statement (1) of Lemma 2.5. It follows that $\widehat{\mathcal{A}}$ is a dense S-ring over the group $\widehat{G}$. The statements (2) and (3) of that lemma imply that the restriction of $\widehat{\mathcal{A}}$ to the group $A^{\perp}$ is the wreath product of the S-rings of rank 3 and rank 2. By the previous paragraph, we conclude that $\widehat{\mathcal{A}}$ is a cyclotomic S-ring over $\widehat{G}$. By statement (3) of Lemma 2.5, this proves that $\mathcal{A}$ is a cyclotomic S-ring over G.

In the remaining case, both $\mathcal{A}_C$ and $\mathcal{A}_{A/C}$ are of rank 2. It follows that $\mathcal{A}_A$ is of rank 3 and

$$\mathcal{S}^{\#}(\mathcal{A}) = \{C^{\#}, A \setminus C\}.$$

Fix arbitrary basic sets $X, Y \in \mathcal{S}(\mathcal{A})$ such that

$$X_A = A \setminus C, \quad Y_A = C^{\#}, \quad X_P = Y_P \neq 1_P.$$

By Lemma 4.4, it suffices to verify that X and Y are orbits of a certain group $K' \leq \mathrm{Aut}(G)$. To this end, we define the numbers k_X and k_Y by formula (9). Then from Lemmas 4.1 and 4.2, it follows that

$$k_X \in \{1,2,3,6\} \quad \text{and} \quad k_Y \in \{1,2\}. \tag{15}$$

As in Case 1, not each combination for the pair (k_X, k_Y) is possible.

Claim 3: $(k_X, k_Y) \neq (6,1)$ and $(k_X, k_Y) \neq (3,2)$. Let us consider the first case. Denote by r the cardinality of X_P. Then

$$|X| = r \quad \text{and} \quad |Y| = 2r.$$

The set $G \setminus (A \cup P)$ is partitioned into basic sets X^σ and Y^σ, where $\sigma \in Z(\mathrm{Aut}(G))$. Since $|X^\sigma| = |X|$, $|Y^\sigma| = |Y|$, and $|X_P| = |Y_P| = (p-1)/r$, we obtain

$$|\mathcal{S}(\mathcal{A})| = |\mathcal{S}(\mathcal{A}_A)| + |\mathcal{S}(\mathcal{A}_P)| - 1 + \frac{7(p-1)}{r}. \tag{16}$$

Next, let $\pi : G \to G/C$ be the natural epimorphism. Since $k_X = 6$, each of the sets $\pi(X^\sigma)$ is of cardinality r. It follows that

$$|\mathcal{S}(\mathcal{A}_{G/C})| = |\mathcal{S}(\mathcal{A}_{A/C})| + |\mathcal{S}(\mathcal{A}_{CP/C})| - 1 + \frac{2(p-1)}{r}. \tag{17}$$

For the S-ring $\widehat{\mathcal{A}}$ dual to $\mathcal{A}$, relation (14) holds. Since the S-rings $\mathcal{A}_A$ and $\mathcal{A}_{G/A}$ as well as $\mathcal{A}_P$ and $\mathcal{A}_{G/P}$ are isomorphic, statement (2) of Lemma 2.5 and equality (16) yield

$$|\mathcal{S}(\widehat{\mathcal{A}})_{\widehat{G} \setminus (A^\perp \cup P^\perp)}| = \frac{7(p-1)}{r}. \tag{18}$$

Furthermore, $C^\perp \cong C_{3p}$ is an $\widehat{\mathcal{A}}$-group and the restriction of $\widehat{\mathcal{A}}$ to this group is isomorphic to the S-ring $\mathcal{A}_{G/C}$. Therefore from equality (17), it follows that

$$|\mathcal{S}(\widehat{\mathcal{A}})_{C^\perp \setminus (A^\perp \cup P^\perp)}| = \frac{2(p-1)}{r}. \tag{19}$$

Now using equalities (18) and (19), we conclude that there are exactly $5(p-1)/r$ basic sets of $\widehat{\mathcal{A}}$ outside $A^\perp$, $P^\perp$, and $C^\perp$. All of these basic sets are obtained from any one of them by applying an automorphism from $Z(\mathrm{Aut}(\widehat{G}))$. Therefore they have the same size, say k. This implies that

$$k \frac{5(p-1)}{r} = |\widehat{G} \setminus (A^\perp \cup P^\perp \cup C^\perp)| = 6(p-1). \tag{20}$$

On the other hand, the S-rings $\mathcal{A}_P = \mathrm{Cyc}(V, P)$ and $\widehat{\mathcal{A}}_{P^\perp}$ are isomorphic, where $V = V(X) = V(Y)$ is a subgroup of $\mathrm{Aut}(P)$ of order r. Therefore the nonidentity basic sets of the latter S-ring are of cardinality r. It follows that r divides k, which contradicts equality (20).

The proof of the Claim 3 for the case $(k_X, k_Y) = (3,2)$ differs from the previous argument only in the values of the parameters. Namely, here $|X| = 2r$ and $|Y| = r$. Therefore, the last summands on the right-hand sides in formulas (16) and (18) are equal to $5(p-1)/r$, and those in formulas (17) and (19) are $(p-1)/r$. Thus, equality (20) leads to the equality $4k/r = 6$, which is also impossible, because r divides k. The claim is proved. □

Let us return to the remaining part of Case 3. By Claim 3 and formula (15), there are six possibilities for the pair (k_X, k_Y). For each of them, we define a group $K' \leq \mathrm{Aut}(G)$ by formula (12), where the standard pair (U', U'_0) is given in the second and third columns of Table 1 below; in the fourth column contains the sizes of the U'_0-orbits.

(k_X, k_Y)	U'	U'_0	$\mathrm{Orb}(U'_0, A)$
$(1,1)$	C_6	C_6	$[1,2,6]$
$(1,2)$	D_{12}	$\mathrm{Sym}(3)$	$[1,1,1,6]$
$(2,1)$	D_{12}	$\mathrm{Sym}(3)$	$[1,2,3,3]$
$(2,2)$	C_6	C_3	$[1,3,3,1,1]$
$(3,1)$	C_6	C_2	$[1,2,2,2,2]$
$(6,2)$	C_6	1	$[1,\ldots,1]$

TABLE 1. Standard groups for Case 3

A straightforward check shows that in each case, $Y \in \mathrm{Orb}(K', G)$, as required. □

6. THE PROOF OF THEOREM 1.1

We deduce Theorem 1.1 in the end of the section from the theorem below giving a complete description of all S-rings over the group $E_9 \times C_p$.

Theorem 6.1. *Let $\mathcal{A}$ be an S-ring over a group $G = E_9 \times C_p$, where $p > 3$ is a prime. Then one of the following statements holds:*

(1) *$\mathcal{A}$ is trivial or cyclotomic,*
(2) *$\mathcal{A}$ is the tensor product of a trivial S-ring and an S-ring over C_3,*
(3) *$\mathcal{A}$ is a proper S-wreath product with $|S| \leq 3$.*

Proof. If the S-ring $\mathcal{A}$ is dense, then we are done with statement (1) by Theorem 5.1. Assume that $\mathcal{A}$ is not dense. Then A or P is not an $\mathcal{A}$-subgroup of G. By the duality (see Lemma 2.5), we may assume that $A \notin \mathfrak{G}(\mathcal{A})$. Denote by C the maximal $\mathcal{A}$-subgroup of the group A. Clearly, this group is trivial or of order 3. The lemma below is a special case of [4, Lemma 6.2].

Lemma 6.2. *In the above notation, one of the following statements holds:*

(1) *$\mathcal{A} = \mathcal{A}_C \wr \mathcal{A}_{G/C}$ and also $\mathrm{rk}(A_{G/C}) = 2$,*
(2) *$\mathcal{A}$ is the U/L-wreath product, where $P \leq L < G$ and $U = CL$.*

Without loss of generality, we may assume that $\mathcal{A}$ is as in statement (2) of Lemma 6.2: otherwise this lemma implies that either $\mathcal{A}$ is trivial (if $C = 1$) and statement (1) of Theorem 6.1 holds, or statement (3) of this theorem holds with $S = C/C$. Furthermore, if $U < G$ then $|U/L| \leq 3$ and statement (3) of Theorem 6.1 holds. Thus, we may also assume that $U = G$. Then $C \not\leq L$, for otherwise $G = L$, a contradiction. Since $|C| = 3$, it follows that $C \cap L = e$. Thus,

$$G = C \times L \quad \text{and} \quad |L| = 3p.$$

Note that the S-ring $\mathcal{A}_L$ is circulant. Moreover, since A is not an $\mathcal{A}$-group, the subgroup of L of order 3 is not an $\mathcal{A}_L$-group. According to [6], this implies that

$$\text{(21)} \qquad \mathrm{rk}(\mathcal{A}_L) = 2 \quad \text{or} \quad \mathcal{A}_L = \mathcal{A}_P \wr \mathcal{A}_{L/P}.$$

Assume that $\mathcal{A}_C = \mathbb{Z}C$. Then $\mathcal{A} = \mathcal{A}_C \otimes \mathcal{A}_L$ by statement 2 of Lemma 2.3. In particular, statement (2) of Theorem 6.1 holds, whenever $\mathrm{rk}(\mathcal{A}_L) = 2$. On the other hand, if $\mathcal{A}_L$ is not trivial, then $\mathcal{A}$ is obviously the CP/P-wreath product and statement (3) of Theorem 6.1 holds. Thus, we may assume that

$$\mathrm{rk}(\mathcal{A}_C) = 2.$$

Denote by L_0 the trivial subgroup of L if $\mathrm{rk}(\mathcal{A}_L) = 2$, and the group P otherwise. In view of (21), we have $L_0 \in \mathfrak{G}(\mathcal{A})$. In particular, $L \setminus L_0$ is an $\mathcal{A}$-set.

Lemma 6.3. *If $X \in \mathcal{S}(\mathcal{A})$ is contained in $C^\# \times (L \setminus L_0)$, then $X = X_C \times X_L$.*

Proof. For all $X \in \mathcal{S}(\mathcal{A})$ contained in $C^\# \times L^\#$, we have

$$|X_L| \le |X| \le |C^\#|\,|X_L| = 2|X_L|.$$

It follows that $X_C \times X_L$ is the union of two basic sets X and X' of the same cardinality. Now if $X = X'$, then $X = X_C \times X_L$ and we are done. In the remaining case,

$$\text{(22)} \qquad |X| = \frac{|X_C| \cdot |X_L|}{2} = \frac{2|X_L|}{2} = |X_L|.$$

Assume that $X_L \subseteq L \setminus L_0$. From the definition of the group L_0, it follows that $|L \setminus L_0| \le 3p - 1$. Therefore, equality (22) yields

$$\text{(23)} \qquad |X| = |X_L| \le |L \setminus L_0| \le 3p - 1.$$

On the other hand, if $L_0 = 1$, then $\mathrm{rk}(\mathcal{A}_L) = 2$ and hence $X_L = L \setminus L_0$. Furthermore, if $L_0 = P$, then $\mathcal{A}_L = \mathcal{A}_P \wr \mathcal{A}_{L/P}$ and hence $X_L = P$ or $X = L \setminus P$. However, the first case is impossible, because by Lemma 2.2 for $H = P$ the number $|X|$ must be even. Thus, in any case, $X_L = L \setminus L_0$ and hence

$$X \cup X' = C^\# \times (L \setminus L_0).$$

The the right-hand side includes the set $C_0 := A \setminus C$ of cardinality 6. Therefore, at least one of X or X', say X, contains three elements from C_0. According to [4, Lemma 6.1], this implies that

$$|X| \ge |(X \cap C_0)P| \ge 3p,$$

which contradicts inequality (23). □

From Lemma 6.3, it follows that if $\mathrm{rk}(\mathcal{A}_L) = 2$, then $\mathcal{A} = \mathcal{A}_C \otimes \mathcal{A}_L$ and we are done with statement (2) of Theorem 6.1. To complete the proof, in view of (21) we may assume that $\mathcal{A}_L = \mathcal{A}_P \wr \mathcal{A}_{L/P}$. In this case, statement (1) of Lemma 2.3 and Lemma 6.3 imply that $P \le \mathrm{rad}(X)$ for all $X \in \mathcal{S}(\mathcal{A})_{G \setminus CP}$. It follows that $\mathcal{A}$ is the CP/P-product and we and we are done with statement (3) of Theorem 6.1. □

Proof of the Theorem 1.1. For $p \le 3$, the statement follows from the computational results obtained in [16, p. 498]. Let $p > 3$ and $\mathcal{A}$ an S-ring over G. Then by Theorem 6.1, this ring is obviously schurian if statement (1) of this theorem holds. In case of statement (2), the S-ring $\mathcal{A}$ being the tensor product of two schurian S-rings is also schurian. To complete the proof, we may assume that $\mathcal{A}$ is a proper S-wreath product with $|S| \le 3$. Then the S-ring $\mathcal{A}_S$ is either trivial or a group ring.

Thus, the group $\mathrm{Aut}(\mathcal{A}_S)$ is permutation isomorphic to S_{right}, $\mathrm{Sym}(3)$, or $\mathrm{Alt}(3)$. In any case, according to a criterion of schurity of a generalized wreath product [8, Corollary 10.3], the S-ring $\mathcal{A}$ is schurian. □

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