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# TEMPORAL MULTI-VALUED LOGIC WITH LOST WORLDS IN THE PAST

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Abstract. We study many–valued temporal multi–agent logics based on non-transitive models. The semantical basis, i. e., relational models, are used for modelling of computational processes and analysis of databases with incomplete information, for instance, with information forgotten in the past. In the situation we consider, the agents' accessibility relations may have lacunas: agents may have no access to some potentially known and stored information. Yet innovative point is that in the relational models we consider various valuations $V_i$ for agents' knowledge and a global valuation based on these valuations. Besides, agents' logical operations inside formulas may be nested, as a consequence they may interfere; that is, we consider not autonomous but cooperating agents. Satisfiability and decidability issues are discussed. We find algorithms solving satisfiability problem and hence we obtain the decidability of the decidability problem. Open problems are discussed.

**Keywords:** many–valued logic, multi–agent logic, temporal logic, computability, information, satisfiability, decidability, deciding algorithms, non–transitive time.

## 1. Introduction

Looking at information sciences and non-classical logic, one can find various fruitful approaches to checking the correctness of models and to the development of mathematical models of human logic which use multi–valued logics. Initially, multi–valued logics aimed at the representing of the truth relations for the boolean logic, which is the basic logical language. That may be dated to Łukasiewicz (1917) and his three–valued and many–valued propositional calculi, as well as to Godel (1932),

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who refuted the finite–validness of intuitionistic logic. In their pioneering works, A. Tarski (1951) and S. Kripke (1960th) suggested semantical models for the studies of modal and temporal logics such as topological boolean algebras and relational models (Kripke–Hintikka models); these models are multi–valued by nature.

Applications of logic in the AI and Information Sciences use the logical approach to the automated logical inference which has a great potential for solving problems and deriving conclusions from facts. The logics used in this approach (such as modal, temporal, and description logics) are essentially non-classical; all these logics are used in the semantic web. Temporal logic has various applications in CS. A particular modal logic–based system of temporal logic was introduced by Arthur Prior in the late 1950s. Nowadays temporal logic is very popular, highly technical, and fruitful area (cf. e. g. Gabbay and Hodkinson [8, 9, 10]) with various particular areas of applications in CS and in the AI.

The approach via symbolic logic works for the verification of correct behavior of computational processes, the verification of correct representation of information and knowledge, etc. (cf. for example Wooldridge et al [24, 25, 26], Lomuscio et al [11, 3], Balbiani and Vakarelov [4], Vakarelov [23]). This framework is usually based on various non–classical logics, in particular on those close to modal logics.

Concerning the multi–agency, the technique of mathematical logic translated for description logics is useful for the study of ontologies, e. g., F. Baader et al [1], F. Wolter, [27], F. Walter et al [12]. Earlier we also studied the multi–agent logic with distances, the satisfiability problem for it (Rybakov et al [18]), and the models for the conception of Chance Discovery in multi–agent environment (Rybakov [19, 21]). A logic modelling the uncertainty via agents' views was also investigated (cf. McLean et al [13]); the study of the conception of knowledge from the viewpoint of multi agency based on temporal logic is contained in the works by Rybakov [14, 16, 17]. From the technical point of view, perhaps the very first approach to multi–valued modal logics (when different valuations are taken on algebraic lattices) may be found in the works by M. Fitting [6, 7]; the multi–valued approaches were also used in such a popular area as the model checking (cf. e. g. G. Bruns, P. Godefroid [5]).

In this paper we study many–valued temporal multi–agent logics based on non–transitive time models. The first innovative idea here is to consider relational models (for the representation of knowledge) not with a single valuation (which was typical earlier) but with many valuations — each agent has its own separate valuation. Besides, to involve agents' interaction, we introduce a global valuation based on the agents' knowledge.

The second innovative point is to consider knowledge models based on special relational frames with partial accessibility agents' relations so that to capture the changing environment with incomplete information, e. g., with elements of the forgettable past. An individual agent accessibility relation may have lacunas, e. g., an agent may have no access to some potentially known and already stored information which might be accessible to the other agents. We use the logical language where agents' logical operations inside formulas may be nested, and so they may interfere; that is, we consider not autonomous but cooperating agents. The usage of non–transitive time is also unusual (as a rule, temporal logics in CS are transitive and linear, the same one can say about the linear temporal logic). But here we get rid of this restriction and show that in some situations the assumption of the linear transitive time is too strong and does not look realistic. Satisfiability and decidability issues are discussed. We find algorithms solving the satisfiability problem and hence the decidability problem. We also provide some examples for the chosen logical language.

The paper is organized as follows. In Section 2 (Syntax and Semantics) we recall some general definitions, introduce logical language, and define the semantics — the relational multi-valued logic based on it, comment on the relevance of this approach and give some illustrative examples. In section 3 (Truncated Models) we develop preliminary techniques for the satisfiability by reducing the case to the small models. Section 4 (Solving Satisfiability) introduces the kernel mathematical techniques and all the preliminary results before we approach the main problems resolved in this paper: satisfiability problem and decidability problem. We find an algorithm deciding whether a formula (statement) is satisfiable using the invented technique of special small models. By a similar technique we address the validness of the rules or their refutation. In Section 5 we describe the remaining open problems.

## 2. Syntax and Semantics

We start with the description of a logical language we will use. This language contains boolean logics, so we have a potentially infinite set of propositional letters $P$ and boolean logical operations $\wedge, \vee, \rightarrow, \neg$. It also has binary temporal operations $U_l$ ("until" for each agent $l$, where $l \in Ag$ and $Ag$ is a finite set of all agents) and unary operations "next": $N_l$, for each $l \in Ag$. The formation rules for formulas are standard. More precisely: for any $p \in P$, $p$ is a formula; if $\varphi$ and $\psi$ are formulas then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\neg\varphi$, $\varphi U_l \psi$ and $N_l\varphi$ are formulas, for all $l \in Ag$. Thus, everything looks the same as for temporal logics with UNTIL and NEXT, but the difference is that here we consider temporal logical operations referred to any individual agent. There is a big difference with the mere gluing together the copies of logical languages because the temporal operations are allowed to be nested which creates a very interesting opportunity to explore agents interaction for collecting knowledge. We will demonstrate it a bit later with examples but first we introduce a semantics for our logical language. Recall some notations. For any two natural numbers $n$ and $m$ where $n < m$, $[n, m]$ is the interval of all natural numbers situated between $n$ and $m$ (including $n$ and $m$ themselves); $(n, m) := [n, m] \setminus \{n, m\}$; $[n, m) := [n, m] \setminus \{m\}$. Let the set of agents $Ag$ from the definition of formulas have $k$ different agents', i. e., $Ag := [1, k]$.

**Definition 1.** *A multi–agent non–transitive frame is a tuple*

$$\mathcal{F}^{nt}_{ma} = \left\langle W^{nt}_{ma}, \left(\bigcup_{\xi \in In, j \in [1,k]} \langle R_{\xi,j} \rangle\right), Nxt \right\rangle$$

*such that*

- $W^{nt}_{ma} := \bigcup_{\xi \in In \subset N} It[\xi] = N$*, where $N$ is the set of all natural numbers and the following holds: for any number $\xi \in In$, a number $d(\xi) \in In$ is chosen and fixed, where $d(\xi) > \xi$; $It[\xi]$ is the closed interval of all natural numbers situated between $\xi$ and $d(\xi)$: $It[\xi] := [\xi, d(\xi)]$; the intervals may intersect only at the boundary: $[\xi, d(\xi)] \cap [d(\xi), d(d(\xi))] = \{d(\xi)\}$; so,*
- $\forall \xi_1, \xi_2 \in In,\ \xi_1 \neq \xi_2 \Rightarrow (\xi_1, d(\xi_1)) \cap (\xi_2, d(\xi_2)) = \emptyset$;
- *any $R_{\xi,j}$ is the restriction of the standard linear order $(\leq)$ on the interval $It[\xi]$ to a subset $Dom_{\xi,j} \subseteq It(\xi)$;*

- *Nxt is the standard "next" relation on* $N$: $[n\ Nxt\ m]$ *iff* $m = n+1$; *we will write* $Nxt(a) = b$ *to denote that* $b = a+1$ .

For any $\mathcal{F}^{nt}_{ma}$, we denote its base set $W^{nt}_{ma}$ by $|\mathcal{F}^{nt}_{ma}|$. For brevity, we write $a \in \mathcal{F}^{nt}_{ma}$ instead of $a \in |\mathcal{F}^{nt}_{ma}|$. The function $d$ from this definition is called the *distance function*. The number $d(\xi) - \xi$ for any $\xi$ is the measure of intransitivity; we interpret $[\xi, d(\xi)]$ as the maximal interval of time the agents may remember at time point $\xi$. The set $In$ is interpreted as the set of all checkpoints where agents reason and exchange knowledge. Notice that we also consider $\left(\bigcup_{\xi \in In, j \in [1,k]} \langle R_{\xi,j} \rangle\right)$ not as a binary relation, but as an infinite countable set of finite binary relations $R_{\xi,j}$; they are individual accessibility relations of agents within intervals bounded by the checkpoints.

To illustrate this definition we may consider the frame $\mathcal{F}^{nt}_{ma}$ in which $In$ consists of all numbers of kind $10^j$, $j \in N$; all the natural numbers are chopped into intervals $[10^i, 10^{i+1}]$.

To explain why do we consider time to be intransitive, note that evidently in the models of computational processes the transition of information from one computational run to another one may be non–transitive. The information is passed only in the new computational run and we cannot predict outputs of the next computations, and not the whole information collected so far may be completely and correctly transferred to all the future computational runs. If we model human knowledge, the agents may remember not all the past, and their ancestors may remember what these current time agents do not know.

The interesting feature of our definition above is that these accessibility relations may have lacunas — time points or intervals of time points which individual agents do not remember (cannot reach) within the time intervals which they remember (may be interpreted as forgettable time in the remembered past).

Overall, these frames model the agents' knowledge for some computational (or upcoming) processes analyzed for the past or for the future. Therefore, later on in the comments and examples we will tell something about past or future intervals of time. The information or knowledge of the agents will be represented by truth valuations in such frames i. e., by constituting models.

**Definition 2.** *A multi–valued, multi–agent non–transitive model* $\mathcal{MA}^{nt}_{ma}$ *is a pair* $\langle \mathcal{F}^{nt}_{ma}, \{V_l \mid l \in Ag\}\rangle$ *where*

(i) $\mathcal{F}^{nt}_{ma}$ *is a multi–agent non–transitive frame;*
(ii) *any* $V_l$ *is an agent's valuation for the agent* $l$ *of a set* $P$ *of propositional letters in this frame, which is fixed for* $\mathcal{MA}^{nt}_{ma}$, *that is, for any letter* $p$ *holds* $V_l(p) \subseteq |\mathcal{F}^{nt}_{ma}|$.

*We will use the notation* $\left(\mathcal{MA}^{nt}_{ma}, a\right) \Vdash_{V_l} p$ *iff* $a \in V_l(p)$.

For brevity, if it does not lead to a confuse, we may write $a \in \mathcal{MA}^{nt}_{ma}$ instead of $a \in |\mathcal{F}^{nt}_{ma}|$. For any $\mathcal{MA}^{nt}_{ma}$, $\left(\mathcal{MA}^{nt}_{ma}, a\right) \Vdash_{V_l} p$ is the truth relation in this model; we say then that $p$ is true in the world (state) $a$ w. r. t. $V_l$. This encodes the fact that the agent $l$ knows (thinks?) that the statement $p$ is true at state $a$.

Thus, we have $k$ agents, $Ag := [1, \ldots, k]$ and in such models any agent $l \in Ag$ has an individual opinion about the truth of some statements. We model the cooperation between agents by introducing a summarizing (global) valuation $V_0$ which is defined as follows:

$$(\mathcal{M}A^{mv}_{nt}, a) \Vdash_{V_0} p \;\Leftrightarrow\; \big\|\{l \mid (\mathcal{M}A^{nt}_{ma}, a) \Vdash_{V_l} p\}\big\| > TH,$$

where $TH$ (a threshold) is a fixed number bigger than $k/2 + 1$.

The choice of a threshold can differ from the pointed one and it can include, for example, the weights of agents' competences etc. Here we have many ways to define $TH$. For instance, it could be the same for all models or individual for each model. But after we decide which way to use and to fix it, the rest of the paper will not be affected by our choice. Thus, we fix an agreement about $TH$ and thus we obtain the set of all such models with all possible valuations.

To work with a global valuation, in any model we set a new additional accessibility relation: each $R_{\xi,0}$ is just the linear order on any $[\xi, d(\xi)]$. To compute truth values of compound formulas, we use the following rules.

**Definition 3.** *For any $a \in \mathcal{M}A^{nt}_{ma}$ and any $j \in Ag := [1, k]$:*

$$\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} \neg\varphi \quad\Leftrightarrow\quad \big(\mathcal{M}A^{nt}_{ma}, a\big) \nVdash_{V_j} \varphi;$$

$$\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} (\varphi \,\wedge\, \psi) \quad\Leftrightarrow\quad \big(\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} \varphi\big) \wedge \big(\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} \psi\big);$$

$$\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} (\varphi \,\vee\, \psi) \quad\Leftrightarrow\quad \big(\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} \varphi\big) \vee \big(\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} \psi\big);$$

$$\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} (\varphi \,\to\, \psi) \quad\Leftrightarrow\quad \big(\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} \psi\big) \vee \big(\big(\mathcal{M}A^{nt}_{ma}, a\big) \nVdash_{V_j} \varphi\big).$$

*For all formulas $\varphi U_l \psi$ and any $V_j$ we define the truth values as follows (note that $a \in It(\xi)$, for any $a$):*

$$\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} (\varphi\ U_l\ \psi) \quad\Leftrightarrow\quad (a \in It\,(\xi)) \;\wedge$$
$$\exists b\big[\,(aR_{\xi,j}b) \wedge \big(\big(\mathcal{M}A^{nt}_{ma}, b\big) \Vdash_{V_l} \psi\big) \wedge$$
$$\forall c\big[\,(c \in Dom_{\xi,j}\ \&\ (aR_{\xi,j}c)\ \&\ c < b) \Rightarrow \big(\mathcal{M}A^{nt}_{ma}, c\big) \Vdash_{V_l} \varphi\big]\big];$$

$$\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} N_l\varphi \quad\Leftrightarrow\quad \big[(a\ Nxt\ b) \Rightarrow \big(\mathcal{M}A^{nt}_{ma}, b\big) \Vdash_{V_l} \varphi\big].$$

It is important to note here that the possible presence of lacunas — time points or intervals of time the individual agents do not remember — is formalized here by $c \in Dom_{\xi,j}$, and both $j$ and $l$ (agents' notations) do interfere in this definition. Illustrating examples will be given shortly below. Notice again the switching of agents in definition of $\big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_j} N_l\varphi$.

The modal logical operations 'possible' and 'necessary' may be introduced via the operation UNTIL in a usual way: $\Diamond_l p := [(q \to q)\, U_l p]$; $\Box_l := \neg\Diamond_l\neg$. If we compute the truth values of such operations w. r. t. the valuation $V_l$ then these modal operations work as the standard 'possible' and 'necessary'.

But if, for example, we compute the truth value of $\Diamond_l p$ w. r. t. a different valuation $V_j$ then the meaning changes: that would mean that it is possible for the agent $j$ to access a state $s$ where the statement $p$ is true from the viewpoint of the agent $l$ (by the way this $s$ might be impossible to reach for the agent $l$ itself). We feel that such an opportunity of switching agents may give us very useful and unusual instruments for the analysis of information.

If for a formula $\varphi$ and for some $a$ holds $(\mathcal{M}A^{nt}_{ma}, a) \Vdash_{V_l} \varphi$ then we say that *the formula $\varphi$ is true (valid)* at the state (world) $a$ w. r. t. the valuation $V_l$. Now we pause briefly to illustrate the usage of our logical language and to provide some examples. Evidently, our language includes all the features of the usual linear temporal logic and the interval logic, and so it can formalize everything that might be formalized in these logics. But in addition to this, we may express some interesting fine things about the agents' knowledge.

**EXAMPLES**

(1) The formula $\Diamond_1 p \wedge \neg\Diamond_2 p$ being true w. r. t. $V_1$ says that the accessibility relation for the agent 2 has a hole (lacuna) which is nonetheless accessible for the agent 1.

(2) Total opposition for the whole initial interval of time:
$\varphi_{op} := [\Box_1 p \to \Box_2\neg p] \wedge [\Box_2 p \to \Box_1\neg p]$. This formula being evaluated via a global valuation $V_0$ says that both these agents are totally opposite in their opinion about stable facts at all states accessible for them.

(3) Agree at all visible time but after this the agents are in a complete opposition:
$\Box_0[((\Box_1 p \to \Box_2 p) \wedge (\Box_2 p \to \Box_1 p)] \wedge \Diamond_0 N_0\varphi_{op}$. Being evaluated via a global valuation $V_0$ this formula says that both the agents always agree in the whole time visible for them but after this they always disagree.

(4) Revolt: $\Diamond_1\top \wedge \Box_1[p \to \Box_2 p] \wedge \Box_1[\neg p \to \Box_2\neg p] \wedge \Diamond_1 N_1[p \wedge \Diamond_2\neg p]$ (being evaluated either by $V_1$ or $V_0$). Currently agent 1 totally dominates the other one and enforces 2 to think that if the thing $p$ is true now then for agent 2 it is always true (total domination), but in the next time interval this is not the case: there is a state where 1 thinks $p$ is true but agent 2 sees a state were $p$ fails to be true.

(5) Total recall: $\Diamond_1 p \wedge \Box_1(p \to \Diamond_1[\neg p \wedge \neg\Box_1\neg p]) \wedge \Diamond_1\Diamond_1\Box_1 p$. This formula (w. r. t. $V_1$) says that agent 1 always swaps its opinion about the truth of $p$ from true to false and vice versa or loses $p$ during the whole initial interval of time, but after some time it decides $p$ to be always true.

Define *the logic* $MA^{Int}_{Lin}(TH)$ as the set of all formulas which are valid in any model $\mathcal{M}A^{nt}_{ma}$ for all states and valuations.

We are mostly interested in the satisfiability of formulas: how can we check that a formula is satisfiable. Recall that for any logic $L$, the satisfiability problem is to determine by any given formula $\varphi$ whether it is satisfiable in $L$, more exactly whether there is a model and a state of this model for which this formula is true. If there is an algorithm answering this question for any given formula $\varphi$ then the satisfiability problem is said to be decidable.

For a logic $L$ which is semantically based on some relational models with a single valuation, $L$ is said to be decidable if there is an algorithm answering for any formula $\varphi$ whether $\varphi \in L$ holds (if $\varphi$ is a theorem of $L$), that is whether the formula is true at all states of all models w. r. t. all valuations. For all such logics, these two mentioned problems are mutually connected: $\varphi$ is satisfiable in $L$ iff $\neg\varphi \notin L$; $\varphi \in L$ iff $\neg\varphi$ is not satisfiable. By this the decidability itself implies that the satisfiability problem is decidable and vice versa. If we use models with multiple valuations $V_l$, the situation fortunately is similar, which will be shown below.

## 3. Truncated Models

To work with the satisfiability, we will need special small finite models. We recall that for our current framework, the definition of satisfiability looks as follows.

**Definition 4.** *We say that a formula $\varphi$ is satisfiable in a model $\mathcal{M}A^{nt}_{ma}$ by a valuation $V_l$ from this model iff there exists a world $a$ from the model $\mathcal{M}A^{nt}_{ma}$ such* *that* $(\mathcal{M}A^{nt}_{ma}, a) \Vdash_{V_l} \varphi$. *A formula* $\varphi$ *is said to be* refuted *in a world* $a \in \mathcal{M}A^{nt}_{ma}$ *by a valuation* $V_l$ *iff* $(\mathcal{M}A^{nt}_{ma}, a) \nVdash_{V_l} \varphi$.

Assume that a model $\mathcal{M}A^{nt}_{ma}$ is based on its frame

$$\mathcal{F}^{nt}_{ma} = \left\langle W^{nt}_{ma}, \left(\bigcup_{\xi \in In, j \in [1,k]} \langle R_{\xi,j} \rangle\right), Nxt \right\rangle$$

and let $m \geq 0$ be a natural number.

**Definition 5.** *For any model* $\mathcal{M}A^{nt}_{ma}$ *and any* $m \in N$, *a truncated model* $\mathcal{M}A^{nt}_{ma}(m)$ *is the model based on the frame with the base set* $|\mathcal{M}A^{nt}_{ma}| \setminus \{x \mid x \in N, x > d^{m+2}(0)\}$ *and having the following relations and valuations.*

(i) *Relations* $Nxt$ *are the same as before but we redefine* $Nxt$ *at* $d^{m+2}(0)$ *as* $Nxt(d^{m+2}(0)) = d^{m+2}(0)$, *where* $d$ *is the distance function of the frame of* $\mathcal{M}A^{nt}_{ma}$.

(ii) *We transfer all the agents' accessibility relations to this base set but we assume that* $d^{m+2}(0)$ *does not belong to the union of the domains of all agents' accessibility relations.*

We may transfer the rules for computation of the truth values of formulas to truncated models without any amendments. For the formulas with bounded temporal degree, these models will give us a useful tool for the satisfiability problem.

**Definition 6.** *For a formula* $\varphi$, *its temporal degree* $td(\varphi)$ *is defined inductively as follows. If* $\varphi$ *is a propositional letter then* $td(\varphi) := 0$. *If* $\varphi = \varphi_1 \circ \varphi_2$ *where* $\circ$ *is a binary Boolean logical operation, then* $td(\varphi) := max\{td(\varphi_1), td(\varphi_2)\}$; *if* $\varphi = \neg\varphi_1$ *then* $td(\varphi) := td(\varphi_1)$. *If* $\varphi = N_l\varphi_1$ *then* $td(\varphi) := td(\varphi_1) + 1$. *If* $\varphi = \varphi_1 U_l \varphi_2$ *then* $td(\varphi) := max\{td(\varphi_1), td(\varphi_2)\} + 1$.

Recall that for natural numbers $n$ and $m$ such that $n < m$, $[n, m)$ is the interval of all numbers strictly smaller than $m$ and bigger or equal than $n$.

It is clear that to study the satisfiability, we can restrict ourselves with the consideration of states from $[0, d(0))$ which may satisfy formulas (since we can always consider the upper part of the frame starting at a state satisfying a formula).

**Lemma 1.** *Assume that a model* $\mathcal{M}A^{nt}_{ma}$ *based on a frame* $\mathcal{F}^{nt}_{ma}$ *is given and a formula* $\alpha$ *with temporal degree* $n$ *is satisfied in this model at a state* $a$ *from* $[0, d(0))$ *w. r. t a valuation* $V_l$. *Then* $\alpha$ *is satisfied at* $0$ *by the valuation* $V_l$ *at the truncated model* $\mathcal{M}A^{nt}_{ma}(n+1)$.

*Proof.* To obtain this result we need the following more general statement:

**Lemma 2.** *For any valuation* $V_l$, *formula* $\beta$, *and any* $m, n \in N$, *if* $td(\beta) \leq n$ *then* $\forall a \in \left[d^m(0), d^{m+1}(0)\right) \subseteq |\mathcal{M}A^{nt}_{ma}|$

$$(1) \qquad \left(\mathcal{M}A^{nt}_{ma}, a\right) \Vdash_{V_l} \beta \Leftrightarrow \left(\mathcal{M}A^{nt}_{ma}(m+n+1), a\right) \Vdash_{V_l} \beta.$$

*Proof.* We proceed by induction on $n$. The case $n = 0$ is obvious. Assume that (1) is true for all $n \leq n_1$ and we have a formula $\beta$ of temporal degree $n_1 + 1$.

In this case the formula $\beta$ is constructed from some formulas $\beta_i$ with temporal degree at most $n_1$ and some formulas $\gamma_i$ with temporal degree $n_1 + 1$ by means of boolean logical operations. For all formulas $\alpha$ with temporal degree at most $n_1$ we apply (1) to the interval $[d^m(0), d^{m+1}(0))$ itself and obtain:

$$\forall m \forall V_l \forall a \in \left[d^m(0), d^{m+1}(0)\right) \; \left(\mathcal{M}A^{nt}_{ma}, a\right) \Vdash_{V_l} \alpha \Leftrightarrow$$

$$\big(\mathcal{M}A^{nt}_{ma}(m+n_1+1), a\big) \Vdash_{V_l} \alpha. \tag{2}$$

For any formula $\gamma_i$, it suffices to consider only the cases when

(i) $\gamma_i = \delta_{i,1}\, U_j \delta_{i,2}$ or

(ii) $\gamma_i = N_j \alpha_i$,

where some formulas from $\delta_{i,1}$ or $\delta_{i,2}$, or $\alpha_i$ respectively, have temporal degree $n_1$.

In case (i), $\gamma_i = \delta_{i,1}\, U_j \delta_{i,2}$. By (2) we obtain:

$$\forall m \forall V_l \forall a \in \big[d^m(0) d^{m+1}(0)\big) \;\; \big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_l} \delta_{i,k} \Leftrightarrow$$

$$\big(\mathcal{M}A^{nt}_{ma}(m+n_1+1), a\big) \Vdash_{V_l} \delta_{i,k}. \tag{3}$$

Therefore

$$\forall m \forall a \in [d^m(0), d^{m+1}(0))\big[\, \big(\mathcal{M}A^{nt}_{ma}, a\big) \Vdash_{V_l} \delta_{i,1}\, U_j \delta_{i,2} \Leftrightarrow$$

$$\Leftrightarrow \; \big(\mathcal{M}A^{nt}_{ma}(m+n_1+1+1), a\big) \Vdash_{V_l} \delta_{i,1}\, U_j \delta_{i,2}\big].$$

(Notice that we do not consider here the case $a = d^{m+1}(0)$ since it is not required in the view of $a \in [d^m(0), d^{m+1}(0))$ ). So, we are done.

Consider now the case (ii), $\gamma_i = N_j \alpha_i$. If $a < d^{m+1}(0) - 1$ then we apply the same reasoning as for $\gamma_i = \delta_{i,1}\, U_j \delta_{i,2}$ above. If $a = d^{m+1}(0) - 1$ then we first apply (2) to the interval $[d^{m+1}(0), d^{m+2}(0))$ and obtain

$$\forall m \forall V_l \forall b \in [d^{m+1}(0), d^{m+2}(0))\Big[\, \big(\mathcal{M}A^{nt}_{ma}, b\big) \Vdash_{V_l} \alpha_i \Leftrightarrow$$

$$\Leftrightarrow \; \big(\mathcal{M}A^{nt}_{ma}(m+1+n_1+1), b\big) \Vdash_{V_l} \alpha_i\Big].$$

Consequently,

$$\big(\mathcal{M}A^{nt}_{ma}, d^{m+1}(0)\big) \Vdash_{V_l} \alpha_i \Leftrightarrow \big(\mathcal{M}A^{nt}_{ma}(m+n_1+2), d^{m+1}(0)\big) \Vdash_{V_l} \alpha_i$$

and hence

$$\big(\mathcal{M}A^{nt}_{ma}, d^{m+1}(0)-1\big) \Vdash_{V_l} N_j\alpha_i \Leftrightarrow \big(\mathcal{M}A^{nt}_{ma}(m+n_1+2), d^{m+1}(0)-1\big) \Vdash_{V_l} N_j\alpha_i.$$

Thus, we have shown that (1) holds for $n = n_1 + 1$, and by induction it holds for all $n$. Lemma is complete.

Lemma 1 immediately follows from this lemma for $m = 0$.

**Lemma 3.** *If a formula $\alpha$ is refuted in a model $\mathcal{M}A^{nt}_{ma}(m)$ at a a state $a$ from the interval $[0, d(0))$ by a valuation $V_l$, then $\alpha$ may be refuted in a model based on a standard frame $\mathcal{F}^{nt}_{ma}$.*

*Proof* is very easy: it suffices to blow out the frame $\mathcal{M}A^{nt}_{ma}(m)$ up to an infinite one. Indeed, consider the final interval $[d^{m+1}(0)), d^{m+2}(0)]$ of the frame $\mathcal{M}A^{nt}_{ma}(m)$ and, starting from $[d^{m+1}(0), d^{m+2}(0)]$, adjoin to $\mathcal{M}A^{nt}_{ma}(m)$ the infinite sequence of intervals $[a_i, b_i], i \in N$ with $Nxt(b_i) = a_{i+1}$. Define the valuation of the letters on these new worlds to be the same as it was at the state $d^{m+2}(0)$. It is easy to see that this modification will not effect the truth values of formulas on the initial interval $[0, d^{m+2}(0))$. Lemma is complete.

Therefore, due to Lemmas 1 and 3, if we want to find an algorithm solving the satisfiability problem, we can restrict ourselves with models based on frames of kind $\mathcal{M}A^{nt}_{ma}(m)$ only.

## 4. Solving the Satisfiability

Working with satisfiability problem, we are not in a position to immediately use the known techniques for decidability (as e. g. was proposed in [17, 16]) or some conventional techniques such as the filtration. Here we face the following problems:

(i) We have several valuations for truth values of formulas working simultaneously;
(ii) The frames in consideration have lacunas for the accessibility relations (we have some forgettable states in the past);
(iii) The frames in consideration are non-transitive.

Going to comment the technique, it is easy to see that, e. g., the non–transitivity hampers to convert formulas into more suitable and simple forms, to some canonical or similar ones. Therefore here we will use the technique of reduction of formulas to rules (which we have already used earlier many times for different purposes, cf. e. g. [17, 20, 22]).

This approach efficiently simplifies all the proofs because it allows to consider very simple and uniform formulas without nested temporal operations. We briefly recall this technique.

A *rule* is an expression $\mathbf{r} := \varphi_1(x_1,\ldots,x_n),\ldots,\ \varphi_s(x_1,\ldots,x_n)\ /\ \psi(x_1,\ldots,x_n)$, where all $\varphi_k(x_1,\ldots,x_n)$ and $\psi(x_1,\ldots,x_n)$ are formulas constructed out of letters (variables) $x_1,\ldots,x_n$.

Formulas $\varphi_k(x_1,\ldots,x_n)$ are called *premises* and $\psi(x_1,\ldots,x_n)$ is called the *conclusion*. The rule $\mathbf{r}$ means that $\psi(x_1,\ldots,x_n)$ (conclusion) follows (logically follows) from the assumptions $\varphi_1(x_1,\ldots,x_n)$, $\ldots$, $\varphi_s(x_1,\ldots,x_n)$. The definition of validity of a rule is the same for any relational model. However, since we deal with models with multi–valuations, we need some modifications.

Assume that a truncated model $\mathcal{M}A^{nt}_{ma}(m)$ and a rule $\mathbf{r}$ are given.

**Definition 7.** *The rule* $\mathbf{r}$ *is true (or valid) on* $\mathcal{M}A^{nt}_{ma}(m)$ *iff*

$$\left[\forall V_l \forall a\left(\left(\mathcal{M}A^{nt}_{ma}(m),a\right)\Vdash_{V_l}\bigwedge_{1\leq i\leq s}\varphi_i\right)\right]\ \ \Rightarrow\ \ \left[\forall V_l\forall a\left(\left(\mathcal{M}A^{nt}_{ma}(m),a\right)\Vdash_{V_l}\psi\right)\right].$$

*If* $\forall V_l\forall a\left(\left(\mathcal{M}A^{nt}_{ma}(m),a\right)\Vdash_{V_l}\bigwedge_{1\leq i\leq s}\varphi_i\right)$ *but* $\exists V_l\exists a\left(\left(\mathcal{M}A^{nt}_{ma}(m),a\right)\nVdash_{V_l}\psi\right)$*, then we say that* $\mathbf{r}$ *is refuted in* $\mathcal{M}A^{nt}_{ma}(m)$ *by* $V_l$*; and we denote this fact as* $\mathcal{M}A^{nt}_{ma}\nVdash_{V_l}\mathbf{r}$.

If we need to verify the satisfiability of a formula $\varphi$ in a model $\mathcal{M}A^{nt}_{ma}(m)$, we may do it using the rules. Indeed, $\varphi$ is satisfiable in $\mathcal{M}A^{nt}_{ma}(m)$ w. r. t. some valuation iff the rule $x\to x/\neg\varphi$ is refuted in $\mathcal{M}A^{nt}_{ma}(m)$ w. r. t. this valuation. And vise versa, if $x\to x/\varphi$ is true in $\mathcal{M}A^{nt}_{ma}(m)$ then the formula $\varphi$ is true in $\mathcal{M}A^{nt}_{ma}(m)$ for all agents' valuations of this model. Thus we have

**Lemma 4.** *If there is an algorithm verifying for any given rule and any given model* $\mathcal{M}A^{nt}_{ma}(m)$ *whether this rule may be refuted in this model then there exists an algorithm verifying whether any given formula is satisfiable.*

Now we need to consider rules in some uniform simple form, in particular — without nested temporal operations.

**Definition 8.** *A rule* $\mathbf{r}$ *is said to be in* reduced normal form *if* $\mathbf{r}=\varepsilon/x_1$ *where*

$$\varepsilon = \bigvee_{1\le j\le m} \Big[ \bigwedge_{1\le i\le n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\le i\le n; 1\le l\le k} (N_l x_i)^{t(j,i,l,1)} \wedge$$
$$\wedge \bigwedge_{l\in[1,k], 1\le i,k_1\le n} (x_i\, U_l x_{k_1})^{t(j,i,k_1,l,2)} \Big],$$

*$t(j,i,0)$, $t(j,i,l,1)$, $t(j,i,k_1,l,2) \in \{0,1\}$ and, for any formula $\alpha$ above $\alpha^0 := \alpha$, $\alpha^1 := \neg\alpha$.*

**Definition 9.** *For any given rule* $\mathbf{r}$*, a rule* $\mathbf{r_{nf}}$ *in the reduced normal form is said to be a* reduced normal form of $\mathbf{r}$ *iff*

(i) $\mathbf{r_{nf}}$ *contains all variable–letters from* $\mathbf{r}$ *and maybe some extra ones;*
(ii) *For any truncated model $\mathcal{M}A^{nt}_{ma}(m)$, the rule* $\mathbf{r}$ *may be refuted in $\mathcal{M}A^{nt}_{ma}(m)$ if and only if the rule* $\mathbf{r_{nf}}$ *may be refuted in this model;*

**Theorem 1.** *There exists an algorithm running in (single) exponential time which given any rule* $\mathbf{r}$ *constructs some its reduced form* $\mathbf{r_{nf}}$*. The variables of* $\mathbf{r_{nf}}$ *are all variables of* $\mathbf{r}$ *together with the set of new variables denoting all subformulas of* $\mathbf{r}$*.*

The proofs of the similar statement for various relative relational models and rules for the case of models with single valuation was suggested by us quite a while ago (e. g. cf. Lemma 5 in [2], or the proofs of similar statements in [15]). In our case the models are multi–valued but it does not affect the proof because we modified (correctly) the definition for truth rules in the multi–valued models of this paper. Theorem is complete.

Thus, if we are interested to investigate the problem of refutation for rules, we may restrict ourselves with considerations of rules in the reduced form only. Recall that now we consider truncated models.

**Lemma 5.** *If a rule in a reduced normal form* $\mathbf{r_{nf}}$ *is refuted in a truncated model $\mathcal{M}A^{nt}_{ma}(g)$ for some $g$ then* $\mathbf{r_{nf}}$ *can be refuted in some such model where $\forall \xi \in In$, $d(\xi) - \xi \le dis(r) \times v + 2$, where $dis(r)$ is the number of disjuncts in* $\mathbf{r_{nf}}$ *and $v$ is the number of valuations in $\mathcal{M}A^{nt}_{ma}(g)$.*

*Proof.* Let $\mathbf{r_{nf}} = \varepsilon/x_1$, where $\varepsilon = \bigvee_{1\le j\le m} \theta_j$,

$$\theta_j = \Big[ \bigwedge_{1\le i\le n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\le i\le n; 1\le l\le k} (N_l x_i)^{t(j,i,l,1)} \wedge$$
$$\wedge \bigwedge_{l\in[1,k], 1\le i,k_1\le n} (x_i\, U_l x_k)^{t(j,i,k_1,l,2)} \Big],$$

and $\mathbf{r_{nf}}$ is refuted in a given model $\mathcal{M}A^{nt}_{ma}(g)$. Then for all valuations the premise of the rule is true at any state but for some valuation the conclusion of the rule is refuted.

Hence, for all states $a$ of $\mathcal{M}A^{nt}_{ma}(g)$ and all valuations $V_l$ there exists certain unique (for $V_l$) disjunct $\theta_l(a)$ from the premise of the rule $\mathbf{r_{nf}}$ such that

(4) $$\big(\mathcal{M}A^{nt}_{ma}(g), a\big) \Vdash_{V_l} \theta_l(a);$$

but at the same time, for some $b$ and some $V_l$ holds $\big(\mathcal{M}A^{nt}_{ma}(g), b\big) \nVdash_{V_l} x_1$.

To make a proper rarefication, first fix some interval $[\xi, d(\xi)]$ for $\xi \in In$. For all remaining intervals we will execute the similar procedure. At the very beginning, if $d(\xi) = \xi + 1$ we do nothing. If this is not a case then we consider $Nxt(\xi)$ and the greatest number $Nx_1(\xi)$ from $[\xi, d(\xi))$ strictly bigger than the state $Nxt(\xi)$ (if exists) such that

$$\forall V_l[\theta_l(Nxt(\xi)) = \theta_l(Nx_1(\xi))]. \tag{5}$$

(If there is no such numbers strictly bigger than $Nxt(\xi)$, we consider $Nxt(\xi)$ as $\xi$ and follow all the described rarefication steps). At the first step, we delete from $[\xi, d(\xi)]$ some states: we remove all the states situated strictly between $\xi$ and $Nx_1(\xi)$. We set now that $Nxt(\xi) = Nx_1(\xi)$. Now we transfer all the valuations $V_s$ from the original model to the resulted model being intact; denote the so obtained model by $\mathcal{M}A(r)$. Let $Base(\mathcal{M}A(r))$ be the set of all states from the model $\mathcal{M}A(r)$.

**Lemma 6.** $\forall a \in [\xi, d(\xi)] \cap Base(\mathcal{M}A(r))$

$$(\mathcal{M}A(r), a) \Vdash_{V_l} \theta_l(a). \tag{6}$$

*Proof.* First, in case $a \geq Nx_1(\xi)$ the statement of this lemma is evident because we did not change anything above the state $Nx_1(\xi)$. In case $a = \xi$, the part of the statement of this lemma for all subformulas of the formula $\theta_l(a)$ except the ones containing operations $U_s$ are evident. Consider the remaining parts.

Let $\left(\mathcal{M}A^{nt}_{ma}(g)\right), \xi) \Vdash_{V_l} x_{i_1} U_s x_{i_2}$. In this case there exists a $b \in Dom_{\xi,l}$ such that

$$(\xi R_{\xi,l} b) \wedge \left(\left(\mathcal{M}A^{nt}_{ma}(g), b\right) \Vdash_{V_s} x_{i_2}\right) \wedge$$
$$\forall c \left[(c \in Dom_{\xi,l} \ \& \ \xi R_{\xi,l} c \ \& \ c < b) \Rightarrow \left(\mathcal{M}A^{nt}_{ma}(g), c\right) \Vdash_{V_s} x_{i_1}\right].$$

Take minimal $b$ with these properties. If $b = \xi$ then everything is clear. If $b > \xi$ then $b \geq Nxt(\xi)$. If $b \geq Nx_1(\xi)$ then we have

$$\left(\mathcal{M}A^{nt}_{ma}(g), Nx_1(\xi)\right) \Vdash_{V_l} x_{i_1} U_s x_{i_2},$$

and consequently $(\mathcal{M}A(r), \xi) \Vdash_{V_l} x_{i_1} U_s x_{i_2}$. It remains to consider the case when $b < Nx_1(\xi)$ and $b > \xi$, that is $Nxt(\xi) \leq b < Nx_1(\xi)$. From our assumption (5) that $\forall V_{s_t} [\theta_{s_t} (Nxt(\xi)) = \theta_{s_t} (Nx_1(\xi))]$ we conclude that

$$\left(\mathcal{M}A^{nt}_{ma}(g), Nx_1(\xi)\right) \Vdash_{V_l} x_{i_1} U_s x_{i_2} \text{ and } (\mathcal{M}A(r), Nx_1(\xi)) \Vdash_{V_l} x_{i_1} U_s x_{i_2}.$$

Then, since $\left(\mathcal{M}A^{nt}_{ma}(g), \xi\right) \Vdash_{V_l} x_{i_1} U_s x_{i_2}$, we conclude that if $\xi \notin Dom_{\xi,l}$ all is done, and otherwise we have $\left(\mathcal{M}A^{nt}_{ma}(g), \xi\right) \Vdash_{V_s} x_{i_1}$, and consequently

$$(\mathcal{M}A(r), \xi) \Vdash_{V_l} x_{i_1} U_s x_{i_2}.$$

For the opposite direction, let $(\mathcal{M}A(r), \xi) \Vdash_{V_l} x_{i_1} U_s x_{i_2}$. Then

$$\exists b \in Base(\mathcal{M}A(r)) \Big[\xi R_{\xi,l} b \wedge (\mathcal{M}A(r), b) \Vdash_{V_s} x_{i_2}) \wedge$$
$$\forall c \big[ (c \in \mathcal{M}A(r) \cap Dom_{\xi,l} \wedge \xi R_{\xi,l} c \wedge c < b) \Rightarrow (\mathcal{M}A(r), c) \Vdash_{V_s} x_{i_1}\big]\Big].$$

Take minimal $b$ with this property. If $b = \xi$ then all is done. Otherwise $b \geq Nx_1(\xi)$ and consequently

$$(\mathcal{M}A(r), Nx_1(\xi)) \Vdash_{V_l} x_{i_1} U_s x_{i,2} \text{ and } \left(\mathcal{M}A^{nt}_{ma}(g), Nx_1(\xi)\right) \Vdash_{V_l} x_{i_1} U_s x_{i,2}.$$

Since $\forall V_{s_t} [\theta_{s_t}(Nxt(\xi)) = \theta_{s_t}(Nx_1(\xi))]$ (cf. (5)), we conclude that

$$\left(\mathcal{M}A^{nt}_{ma}(g), Nxt(\xi)\right) \Vdash_{V_l} x_{i_1} U_s x_{i,2}.$$

If $\xi \notin Dom_{\xi,l}$ then we obtain $\left(\mathcal{M}A^{nt}_{ma}(g), \xi\right) \Vdash_{V_l} x_{i_1} U_s x_{i_2}$.

If $\xi \in Dom_{\xi,l}$ then $(\mathcal{M}A(r),\xi) \Vdash_{V_l} x_{i_1} U_s x_{i_2}$ implies $(\mathcal{M}A(r),\xi) \Vdash_{V_s} x_{i_1}$ and $\big(\mathcal{M}A^{nt}_{ma}(g),\xi\big) \Vdash_{V} x_{i_1}$, which together with $\big(\mathcal{M}A^{nt}_{ma}(g),Nxt(\xi)\big) \Vdash_{V_l} x_{i_1} U_s x_{i_2}$ allows us to conclude that $(\mathcal{M}A(r),\xi) \Vdash_{V_l} x_{i_1} U_s x_{i,2}$. Lemma 6 is complete.

Now, instead of $[\xi, d(\xi)]$ we consider the interval $[Nx_1(\xi),(d(\xi))]$ and apply to $[Nx_1(\xi),d(\xi))]$ the same rarefication procedure as for $[\xi, d(\xi)]$ above. An analog of Lemma 6 will hold and it will not affect the truth of all formulas $\theta_s(\xi)$ on $\xi$. We then continue this procedure. Executing this transformation subsequently for all $\xi \in In$, this procedure stops in at most $g+2$ steps (cf. $g$ is specified in the definition of $\mathcal{M}A^{nt}_{ma}(g)$). As a result, we obtain that $d(\xi)-\xi \leq dis(r)\times v+2$, which accomplishes the proof of Lemma 5. Lemma is complete.

Combining Lemmas 1, 3, 4, 5 and Theorem 1 we obtain

**Theorem 2.** *The satisfiability problem for the logic $MA^{Int}_{Lin}(TH)$ is decidable: there exists an algorithm described in the series of cited lemmas which verifies the satifiability.*

As we have demonstrated, rules provide a more general formalization and they do cover the case of simple formulas. Rules may be more flexible for checking properties of the knowledge inside models. Refutation of a rule, in particular, says that a formula — which is the premise of the rule — is true at all states of a model. If we consider transitive temporal models only (as e. g., for LTL itself), then the refutation or the validness (truth) of a rule may be easily represented by a formula. For example, a rule $\varphi/\psi$ is valid iff $\Box\varphi \wedge \varphi \to \psi$ is a theorem. But for intransitive temporal or modal logics this does not work because any operation $\Box$ and even any finite composition of it $\Box^n$ are always local. Therefore, the computation of truth of formulas does refer to some initial finite fragment of a model only. For instance, for any number $n$, the rule $\Box^n_0 x/x$ is valid for any model of $MA^{Int}_{Lin}$, but there is no way to describe it by a formula. Therefore, to check refutation or validness of rules, we need some extra work to be done to extract it from our previous results. The definition of refutation is already given above. Here is the definition of validness.

**Definition 10.** *A rule $\mathbf{r}$ is valid in logic $MA^{Int}_{Lin}(TH)$ if it is valid in any model $\mathcal{M}A^{nt}_{ma}$.*

So, a rule is invalid in a logic if there is a model refuting it.

**Theorem 3.** *There is an algorithm verifying if a rule may be refuted in some model for $MA^{Int}_{Lin}(TH)$. Logic $MA^{Int}_{Lin}(TH)$ is decidable w. r. t. valid inference rules.*

*Proof.* We give a sketch proof only because it is very similar to the proof for satisfiability given above. Assume that a rule $\mathbf{r}$ is refuted in a usual model for $\mathcal{M}A^{nt}_{ma}$. As in Theorem 1 (but using reduced normal forms for not truncated but original models) we can construct a reduced normal form $\mathbf{r}_{nf}$ for $\mathbf{r}$. Then $\mathbf{r}_{nf}$ is also refuted in the model and following the proof of Lemma 5 in details we can make the intervals of intransitivity $[\xi, d(\xi)]$ in this model to be small and computable from the length of $\mathbf{r}_{nf}$.

Next, when in the future the first intransitivity interval occurs which repeats some earlier interval (w. r. t. the frame order, size, and valuations) we build the circling model by directing the predecessor state of the last state of the repeating interval to the last sate of its original earlier copy. The resulting model will refute $\mathbf{r_{nf}}$ again. It will be finite and of size computable from the length of $\mathbf{r_{nf}}$, and it will refute **r** as well. Vice versa, if **r** is refutable in such a model then it can be refuted in a usual original model for our logic obtained by the usual unraveling of the circle to the future. Theorem is complete.

## 5. Open problems

In the study of intransitive multi–valued models, a set of problems which are actual for any logic remain unsolved, such as axiomatization, unifiability, and decidability with respect to admissible inference rules. The next interesting open question is the extension of our results to the branching time logics. We did not consider yet the extended versions of our logic for the case with the future and the past. Also the case when intervals of time are not discrete but continuous is not investigated.

The next open avenue for research is the embedding fuzzy logics into this framework — the case when truth values of formulas at any state are not binary but again multi–valued. Here some tools borrowed from Lukasiewicz logic or modern fuzzy–logic with continuous intervals of truth values may be used. In this case it is very interesting to formalize, how different agents interact and, in particular, when, as in this paper, each agent has its own valuation of the basic propositions.

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