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SUBEXTENSIONS FOR CO-INDUCED MODULES

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Abstract. Using cohomological methods, we find a criterion for the embedding of a group extension with abelian kernel into the split extension of a co-induced module. This generalises some earlier similar results. We also prove an assertion about the conjugacy of complements in split extensions of co-induced modules. Both results follow from a relation between homomorphisms of certain cohomology groups.

Keywords: subextension, co-induced module, group cohomology.

To Prof. Victor D. Mazurov on his 75th birthday

1. Introduction

The natural action of $G = \mathrm{PSL}_n(q)$ on the projective space $\mathbb{P}^{n-1}$ over $\mathbb{F}_q$ gives rise to the permutation wreath product of $L = \mathbb{Z}/r\mathbb{Z}$ and G, where r is a prime divisor of $(n, q-1)$. The criterion of when this product contains a subgroup isomorphic to the nonsplit central extension of L by G was obtained in [9]. Namely, it was proved that the containments holds iff r does not divide $(q-1)/(n, q-1)$. In the present paper, using some cohomology theory, we generalise this fact by finding a criterion for embedding extensions with an abelian kernel into a split extension. To state the results more precisely, we introduce some terminology. In what follows, we use right modules and right composition of maps.

Let R be a commutative ring, G a group (possibly infinite), and let L and M be RG-modules. Assume that

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$$0 \longrightarrow L \xrightarrow{\varepsilon} M \tag{1}$$

$$0 \longrightarrow L \xrightarrow{\iota} S \xrightarrow{\pi} G \longrightarrow 1, \tag{2}$$

$$0 \longrightarrow M \xrightarrow{\lambda} E \xrightarrow{\rho} G \longrightarrow 1 \tag{3}$$

are exact sequences of modules and groups, where the conjugation action of S on $L\iota$ agrees with the module structure of L, i. e. $(l\iota)^s = l(s\pi)\iota$ for all $l \in L$, $s \in S$, and similarly for M and E. We say that S is a *subextension* of E with respect to the embedding ε if there exists a group homomorphism β that makes the following diagram commutative:

$$\begin{array}{ccccccccc} & & 0 & & 1 & & & & \\ & & \downarrow & & \vdots & & & & \\ 0 & \to & L & \to & S & \to & G & \to & 1 \\ & & \downarrow \varepsilon & & \vdots \beta & & \| & & \\ 0 & \to & M & \to & E & \to & G & \to & 1 \end{array} \tag{4}$$

Should β exist, it must be a monomorphism, which follows from diagram chase.

The map ε induces a homomorphism of the second cohomology groups

$$\varepsilon^{(2)} : H^2(G, L) \longrightarrow H^2(G, M). \tag{5}$$

Let $\overline{\delta} \in H^2(G, L)$ and $\overline{\gamma} \in H^2(G, M)$ be the elements that define, respectively, the extensions S and E up to equivalence. The following fact holds.

Lemma 1. [8, Lemma 2] *In the above notation, S is a subextension of E with respect to ε if and only if $\overline{\delta}\varepsilon^{(2)} = \overline{\gamma}$.*

This general criterion sometimes can be made more explicit. For example, in the situation where $G = \mathrm{PSL}_n(q)$ described earlier, we clearly have a central extension of $R = \mathbb{Z}/r\mathbb{Z}$ by G as a subextension of the wreath product with respect to the diagonal embedding of the principal RG-module into the permutation module, and the above criterion for the existence of this subextension is purely number-theoretic. Since permutation modules are co-induced, we can generalise this as follows.

We say that a subgroup $H \leqslant G$ is *liftable* to S, where S is as in (2), if $H\pi^{-1}$ splits over $L\iota$. Given an RH-module N, we recall that

$$\mathrm{Coind}_H^G(N) = \mathrm{Hom}_{RH}(RG, N)$$

is an RG-module with the action of $g \in G$ on $\mu \in \mathrm{Coind}_H^G(N)$ given by

$$(\mu g)(x) = \mu(gx)$$

for all $x \in G$.

Our main result is as follows.

Theorem 2. *Let G be a group, $H \leqslant G$, and let L be an RG-module. Denote $M = \mathrm{Coind}_H^G(L_H)$ and let ε be the canonical embedding*

$$0 \longrightarrow L \xrightarrow{\varepsilon} M. \tag{6}$$

Then an extension

$$0 \longrightarrow L \longrightarrow S \longrightarrow G \longrightarrow 1 \tag{7}$$

is a subextension of the natural semidirect product

$$0 \longrightarrow M \longrightarrow M \leftthreetimes G \longrightarrow G \longrightarrow 1$$

with respect to ε *if and only if* H *is liftable to* S.

We recall that the embedding ε in (6) is the image of the identity map of L_H under the natural isomorphism

$$\mathrm{Hom}_{RH}(L_H, L_H) \cong \mathrm{Hom}_{RG}(L, \mathrm{Coind}_H^G(L_H)).$$

Explicitly, we have

$$(l\varepsilon)(g) = lg \tag{8}$$

for all $l \in L$, $g \in G$, see [1, Corollary 2.8.3(ii)].

A few remarks are due about Theorem 2. Suppose a group S has an abelian normal subgroup L and quotient $G = S/L$. Then conjugation defines on L the structure of a $\mathbb{Z}G$-module. If we take H to be the trivial subgroup of G then it is liftable to S, and so Theorem 2 ensures existence of the embedding $S \to M \leftthreetimes G$, where $M = \mathrm{Coind}_H^G(L_H)$. It is readily seen that in this case $M \leftthreetimes G$ is isomorphic to the unrestricted regular wreath product $L \,\mathrm{wr}\, G$, and hence the same embedding also follows from

Theorem 3 (Kaloujnine–Krasner, [3]). *Every group* S *with a normal subgroup* L *can be embedded into the unrestricted regular wreath product* $L \,\mathrm{wr}\, S/L$.

Therefore, we give and alternative cohomological proof of this result in the case of abelian L and specify a necessary and sufficient condition for the embedding.

Now, let L be the principal RG-module and suppose that the index $|G : H|$ is finite. Then M is just the transitive permutation module corresponding to the action of G on the cosets of H and $L\varepsilon$ is its diagonal submodule. In [10], we have considered this situation restricted to the case where R has prime characteristic but generalised to not necessarily transitive action and shown without using cohomology that the liftability of H to S is necessary for the existence of the required subextension which must be a central extension in this case. Conversely, the sufficiency of liftability in the general case can also be deduced without applying cohomological methods using a generalisation of the Kaloujnine–Krasner theorem [6, Theorem 2.10.9] which is originally due to B. H. Neumann and is related to the so-called twisted wreath products.

As we show below, Theorem 2 follows from a group-theoretic interpretation in dimension 2 of the equality of kernels of homomorphisms between certain cohomology groups (see Corollary 7) which holds in arbitrary dimension. Since cohomology in dimension 1 is usually also meaningful for groups, we prove the corresponding corollary as well which is as follows.

Theorem 4. *Let* G *be a group,* $H \leqslant G$, *and let* L *be an* RG-*module. Denote* $M = \mathrm{Coind}_H^G(L_H)$ *and let* ε *be the canonical embedding* (6). *Then a complement to* L *in* $L \leftthreetimes G$ *is* M-*conjugate to* G *if and only if its intersection with* $L \leftthreetimes H$ *is* L-*conjugate to* H.

In the statement of Theorem 4, we assume that $L \leftthreetimes G$ is embedded in $M \leftthreetimes G$ via $(g, l) \mapsto (g, l\varepsilon)$ for $g \in G$, $l \in L$, and by X-conjugacy we mean the conjugacy by elements of X, where $X \in \{M, L\}$.

2. H^n AS A FUNCTOR

We recall that H^n, $n \geqslant 0$, can be viewed as a functor from the category $\mathscr{D}$ of pairs (G, M) to abelian groups, where M is a G-module, see [2, §III.8]. A morphism in $\mathscr{D}$ is a map

$$(\alpha, \varphi) : (H, N) \to (G, M)$$

with $\alpha : H \to G$ a group homomorphism and $\varphi : M \to N$ a homomorphism of H-modules, where M is considered as an H-module via α, i. e.

$$(m(h\alpha))\varphi = (m\varphi)h \tag{9}$$

for all $m \in M$, $h \in H$. The image of (α, φ) under H^n is a homomorphism of abelian groups

$$(\alpha, \varphi)^{(n)} : H^n(G, M) \to H^n(H, N).$$

Three particular cases are of interest to us.

(*i*) Suppose $H = G$ and $\alpha = \mathrm{id}_H$. Then $(\mathrm{id}_H, \varphi)^{(n)}$ is just the standard induced homomorphism $H^n(G, \varphi)$.

(*ii*) Suppose $\alpha : H \hookrightarrow G$ is an embedding and $N = M_H$. If $\varphi = \mathrm{id}_M$ then the compatibility condition (9) holds and we have a homomorphism $(\alpha, \mathrm{id}_M)^{(n)}$.

(*iii*) Suppose $\alpha : H \hookrightarrow G$ is an embedding and $M = \mathrm{Coind}_H^G(N)$. If φ is the canonical epimorphism $\psi : M \to N$ given by

$$\mu\psi = \mu(1) \tag{10}$$

for $\mu \in M$, the compatibility condition (9) holds. In this case, the induced map $(\alpha, \psi)^{(n)} : H^n(G, M) \to H^n(H, N)$ is known to be an isomorphism due to the following result.

Lemma 5 (Shapiro's lemma, [7, §6.3])**.** *If $H \leqslant G$ and N is an H-module then $H^n(G, \mathrm{Coind}_H^G(N)) \cong H^n(H, N)$.*

The fact that the isomorphism in Shaprio's lemma coincides with the map $(\alpha, \psi)^{(n)}$ is well known, see [2, Proposition (III.6.2) and §8, Exercise 2].

3. CO-INDUCED MODULES

Let $\alpha : H \hookrightarrow G$ be an embedding of groups and let L be a G-module. Denote $M = \mathrm{Coind}_H^G(L_H)$. The canonical embedding $\varepsilon : L \to M$ gives rise to a homomorphism $(\mathrm{id}_G, \varepsilon)^{(n)} : H^n(G, L) \to H^n(G, M)$ as in (*i*) above. By the above discussion in (*ii*), (*iii*), we also have the homomorphisms $(\alpha, \mathrm{id}_L)^{(n)}$ and $(\alpha, \psi)^{(n)}$ which fit into the diagram

$$\begin{array}{ccc} H^n(G,L) & \xrightarrow{(\mathrm{id}_G,\varepsilon)^{(n)}} & H^n(G,M) \\ {\scriptstyle (\alpha,\mathrm{id}_L)^{(n)}}\big\downarrow & \swarrow{\scriptstyle (\alpha,\psi)^{(n)}} & \\ H^n(H,L_H) & & \end{array} \tag{11}$$

where the map $\psi : M \to L_H$ is as in (10).

Lemma 6. *Diagram* (11) *is commutative.*

Proof. Since H^n is a functor, we only have to prove the commutativity of the following diagram in the category $\mathscr{D}$:

$$\begin{array}{ccc} (G,L) & \xrightarrow{(\mathrm{id}_G,\varepsilon)} & (G,M) \\ {\scriptstyle (\alpha,\mathrm{id}_L)}\downarrow & \swarrow{\scriptstyle (\alpha,\psi)} & \\ (H,L_H) & & \end{array}$$

We have

$$(\mathrm{id}_G,\varepsilon)(\alpha,\psi) = (\alpha,\varepsilon\psi) = (\alpha,\mathrm{id}_L),$$

since $\varepsilon\psi = \mathrm{id}_L$, because from (8) and (10) we have

$$l(\varepsilon\psi) = (l\varepsilon)\psi = (l\varepsilon)(1) = l1 = l$$

for every $l \in L$. The claim follows. □

For simplicity, we will denote $\varepsilon^{(n)} = (\mathrm{id}_G,\varepsilon)^{(n)}$ and $\alpha^{(n)} = (\alpha,\mathrm{id}_L)^{(n)}$. The map $(\alpha,\psi)^{(n)}$ is an isomorphism by Lemma 5. Therefore, Lemma 6 implies

Corollary 7. $\ker\varepsilon^{(n)} = \ker\alpha^{(n)}$.

We note that henceforth instead of G-modules we may as well consider arbitrary RG-modules. This follows from the next result which essentially says that co-induced modules and cohomology groups are independent of the ground ring.

Lemma 8. *For $H \leqslant G$, let M be an RG-module and N an RH-module. Then the following isomorphisms of abelian groups hold:*

(*i*) $\mathrm{Hom}_{RH}(RG,N) \cong \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G,N)$;
(*ii*) $\mathrm{Ext}^n_{RG}(R,M) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},M)$.

Proof. (*i*) Both abelian groups equal

$$\{f : G \to N \mid (gh)f = (gf)h \quad \forall g \in G, h \in H\}$$

with the natural additive structure.

(*ii*) See [4, Lemma 9.4.13]. □

4. Proof of main results

We now prove Theorem 2.

Proof. Since the split extension $M \leftthreetimes G$ is defined by the zero element of $H^2(G,M)$, Lemma 1 implies that S is a subextension of $M \leftthreetimes G$ with respect to ε if and only if $\overline{\delta} \in \ker\varepsilon^{(2)}$, where $\overline{\delta} \in H^2(G,L)$ defines S. By Corollary 7 specialised to dimension 2, we have $\ker\varepsilon^{(2)} = \ker\alpha^{(2)}$, where $\alpha^{(2)} : H^2(G,L) \to H^2(H,L_H)$ and α is the embedding $H \hookrightarrow G$. However, $\overline{\delta}$ lies in $\ker\alpha^{(2)}$ if and only if it is mapped to the zero element of $H^2(H,L_H)$ which defines the split extension $L \rightthreetimes H$, i. e. this is possible if and only if H is liftable to S, as is required. □

In a similar fashion, Theorem 4 can be proved as follows.

Proof. The L-conjugacy classes of complements to L in $L \leftthreetimes G$ are in a one-to-one correspondence with the elements of $H^1(G,L)$ with the class of G corresponding to the zero of $H^1(G,L)$, see [5, 11.1.3]. Therefore, by considering the action on 1-cocycles, one sees that the elements of the kernel of $\varepsilon^{(1)} : H^1(G,L) \to H^1(G,M)$ correspond to the L-conjugacy classes of complements in $L \leftthreetimes G$ that merge to the

M-conjugacy class of G. On the other hand, Corollary 7 specialised to dimension 1 implies that $\ker \varepsilon^{(1)} = \ker \alpha^{(1)}$. Again, by considering the action on 1-cocycles, we see that the elements of the kernel of $\alpha^{(1)} : H^1(G, L) \to H^1(H, L_H)$ correspond to the L-conjugacy classes of complements in $L \leftthreetimes G$ that intersect $L \leftthreetimes H$ in an L-conjugate of H. The claim follows from these remarks. $\square$

5. Defining subgroups

Given an RG-module L and a subgroup $H \leqslant G$, we say that an extension

$$0 \longrightarrow L \xrightarrow{\iota} S \xrightarrow{\pi} G \longrightarrow 1 \tag{12}$$

is *defined* by H if L is a subextension of $M \leftthreetimes G$, where $M = \mathrm{Coind}_H^G(L_H)$, with respect to the natural embedding $\varepsilon : L \to M$ given in (8).

Lemma 9. *Let $H \leqslant G$, let L be an RG-module, and let S be the extension* (12) *that is defined by H. Then*

(*i*) *S is defined by K for every $K \leqslant H$;*
(*ii*) *S is defined by H^g for every $g \in G$.*

Proof. By Theorem 2, the fact that S is defined by H is equivalent to the liftability of H to S which clearly implies the liftability of both K and H^g, hence the claim.

Observe that we can also prove this lemma without using Theorem 2. Indeed, let $M = \mathrm{Coind}_H^G(L_H)$ and let $\beta : S \to M \leftthreetimes G$ be the subextension embedding.

First, suppose $K \leqslant H$ and denote $N = \mathrm{Coind}_K^G(L_K)$. There is a canonical RG-embedding $\varphi : M \to N$ which acts identically on every element of M viewed as a map $G \to L$. In particular, $\delta = \varepsilon\varphi$ is the natural embedding $L \to N$. Also, φ uniquely extends to a map $\alpha : M \leftthreetimes G \to N \leftthreetimes G$ so that $\beta\alpha$ gives the required subextension embedding $S \to N \leftthreetimes G$ with respect to δ.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & S & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \beta} & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & M \leftthreetimes G & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \alpha} & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & N \leftthreetimes G & \longrightarrow & G & \longrightarrow & 1
\end{array}$$

Second, suppose $g \in G$ and denote $U = \mathrm{Coind}_{H^g}^G(L_{H^g})$. Since the RH- and RH^g-modules L_H and L_{H^g} are conjugate by g, there is an RG-isomorphism $\psi : M \to U$ given by $(\mu\psi)(x) = \mu(xg^{-1})g$ for all $\mu \in M$, $x \in G$. We see that $\varepsilon\psi$ is the natural embedding $L \to U$, because $(l\varepsilon\psi)(x) = (l\varepsilon)(xg^{-1})g = lxg^{-1}g = lx$. Hence, as above we have the required subextension embedding $S \to U \leftthreetimes G$. $\square$

By Lemma 9, the study of defining subgroups for a given extension (12) reduces to the the study up to conjugacy of maximal liftable to S subgroups of G. The set of such subgroups is nonempty as the identity subgroup is always liftable.

For example, consider the particular case of alternating groups and their central double covers.

Problem 1. *Let $G = A_n$ be the alternating group of degree $n \geqslant 4$ and let $S = 2.A_n$ be its nonsplit double cover.*

(*i*) *Describe the maximal liftable to S subgroups of G.*

(*ii*) *Find a function* $f : \mathbb{N} \to \mathbb{N}$ *such that* $f(n)$ *is the minimal number with the property that* S *is embedded to a semidirect product* $M \rtimes G$*, where* M *is an elementary abelian group of order* $2^{f(n)}$.

(*iii*) *Describe the maximal subgroups of* G *that lift to* S.

It follows from Theorem 2 that the value $f(n)$ in item (*ii*) is bounded above by the minimal index of liftable subgroups. The case (*iii*), where a maximal subgroup of G is liftable to S, is of special interest because we then obtain the most 'economic' subextension embedding in view of Lemma 9(*i*). This need not always happen, however, as we saw, for example, in the case $G = \mathrm{PSL}_n(q)$ above. For $G = A_n$, it can be shown that no maximal subgroup is liftable to $2.A_n$ for $n = 5, 6, 7, 8$, but there are three conjugacy classes of maximal subgroup of A_9 that lift to $2.A_9$. These subgroups have indices 120 (two classes) and 840 (one class) and are isomorphic to $\mathrm{L}_2(8){:}3$ and $\mathrm{ASL}_2(3)$, respectively.

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