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ON FINITE GROUPS ISOSPECTRAL TO THE SIMPLE GROUPS $S_4(q)$

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Abstract. The *spectrum* of a finite group is the set of its element orders. A finite group G is *critical* with respect to a subset ω of the natural numbers if ω coincides with the spectrum of G and does not coincide with the spectra of proper sections of G. We study the structure of groups with spectra equal to the spectra of the simple symplectic groups $PSp(4,q)$, where $q>3$ and $q\neq 5$. In particular, we describe the structure of the groups critical with respect to the spectra of $PSp(4,q)$.

Keywords: finite group, spectrum, critical group, nonabelian simple group.

1. Introduction

All groups in this article are assumed to be finite. Let G be a group. We denote by $\omega(G)$ the *spectrum* of G, i. e. the set of element orders of G. Groups with the same spectrum are referred to as *isospectral*. Since $\omega(G)$ is closed under division, it is uniquely defined by its subset $\mu(G)$, that is the set of division-maximal elements of $\omega(G)$.

We say that G is *recognizable* (more precisely, *recognizable by spectrum* in the class of finite groups) if every finite group isospectral to G is isomorphic to G. A group G is *almost recognizable* if there exist only finitely many pairwise nonisomorphic groups isospectral to G. Otherwise G is called *unrecognizable*.

In [1] it is proved that a group is unrecognizable if and only if it is isospectral to a group containing a nontrivial solvable normal subgroup. The definition of a critical group is given in the same work. Let ω be a subset of the natural numbers. A group G is called *critical with respect to* ω (or ω*-critical*) if ω coincides with the

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spectrum of G and does not coincide with the spectrum of any proper section of G (i. e. section not equal to G). By *sections* of G we mean homomorphic images of its subgroups.

As a result of almost thirty years of research it was established that, as a general rule, finite simple groups are almost recognizable. More precisely, all nonabelian simple alternating groups of degree $n \neq 6, 10$ are recognizable [2] and so are all simple sporadic groups apart from J_2 [3]. All exceptional groups of Lie type except for ${}^3D_4(2)$ are almost recognizable [4]. All simple groups of Lie type of a sufficiently large dimension are almost recognizable [5].

Thus in order to solve the problem of recognizability by spectrum for simple groups it is relevant to study the groups isospectral to unrecognizable simple groups, and this is the purpose of this work. All known unrecognizable nonabelian simple groups are listed in the table below. In this table H denotes a group isospectral to G that contains a nontrivial solvable normal subgroup (which always exists due to [1]).

TABLE 1. The known unrecognizable simple groups

G	Conditions on G	H	Bibliography
A_6		$2^4 : A_5$	[6]
A_{10}		$(7^4 \times 3^{12}) : (2.L_2(5).2)$	[7]
J_2		$2^6 : A_8$	[8]
${}^3D_4(2)$		$2^{24} : {}^3D_4(2)$	[9]
$L_3(3)$		$13^4 : (2.S_4)$	[10]
$L_4(13^{24})$		$13^{96} : L_4(13^{24})$	[11]
$U_3(q)$	$q = 5$	$2^{18} : L_3(4)$	[7]
	q a Mersenne prime, $q^2 - q + 1$ prime	$2^i : U_3(q)$, i natural	[12]
$U_5(2)$		$3^5 : M_{11}$	[7]
$S_4(q)$	$q = 3$	$3^4 : S_5$	[7]
	$q = 2^m$, $m > 1$	$2^{8m} : L_2(q^2)$	[13]
	$q = 3^{2m}$	$3^{28m} : L_2(q^2)$	[10]
	$q = p^m$, $p > 3$, p prime	$p^{8m} : (L_2(q^2).2)$	[10]
$S_8(q)$	$q = p^m$, $p \neq 2, 7$, p prime	$p^{8m} : (O_8^-(q).2)$	[14]
$O_9(q)$	$q = p^m$, p prime	$p^{8m} : O_8^-(q)$	[15]

A complete description of groups critical with respect to the spectra of A_6, A_{10}, and J_2 is given in [16, 17, 18] (in each of these cases the number of such pairwise nonisomorphic groups is at most 3). Furthermore, in [9] it is proved that every group isospectral to ${}^3D_4(2)$ has a section isomorphic to ${}^3D_4(2)$ and thus ${}^3D_4(2)$ is up to isomorphism the only critical group with such spectrum. Modulo the already known results, this implies that all groups critical with respect to the spectra of nonabelian simple alternating, sporadic groups and exceptional groups of Lie type are known. Therefore in what follows we will only consider classical simple groups. All critical groups isospectral to $L_3(3)$ are listed in [16] (there are two such groups up to isomorphism). Also in [18] there is a partial description of critical groups isospectral to $U_3(3)$ (up to isomorphism there are at least 7 such groups).

In this article we study the structure of groups isospectral to the simple symplectic groups $S_4(q) = PSp(4, q)$, where $q > 3$ and $q \neq 5$. We note that the following theorem does not hold for $q = 2, 3$, and the case $q = 5$ yields an exceptional case that we wish to omit here (see [19] and remark 1 in §4 for further explanation).

Theorem. *Let G be a finite group isospectral to $S_4(q)$, where q is a power of a prime p, $q > 3$, and $q \neq 5$. Then there exists a nilpotent normal subgroup K of G such that $P \leq G/K \leq \operatorname{Aut} P$, where P is isomorphic to either $S_4(q)$ or $L_2(q^2)$. The subgroup K is trivial if $P \simeq S_4(q)$ and a p-group if $P \simeq L_2(q^2)$. In the latter case K is nontrivial if $p > 2$. The group G/K is an extension of P by a field automorphism τ of order 2^m, $m \geq 0$.*

2. Preliminary results

The spectrum $\omega(G)$ of a group G defines the *Gruenberg-Kegel graph* (or the *prime graph*) $GK(G)$ of G. The vertices of this graph are the elements of $\pi(G)$, i. e. the set of all prime divisors of the order of G. Distinct vertices p and q are adjacent if $pq \in \omega(G)$. We denote by $s(G)$ the number of connected components of $GK(G)$, $\pi_i(G)$ denotes the i-th connected component, $i = 1, \ldots, s$. If the order of G is even, we suppose that $2 \in \pi_1(G)$. Let $\omega_i(G)$ (respectively, $\mu_i(G)$) be the set of all $n \in \omega(G)$ ($n \in \mu(G)$) such that every prime divisor of n is contained in $\pi_i(G)$.

Lemma 1 (Gruenberg, Kegel, [21])**.** *If G is a finite group with a disconnected graph $GK(G)$ then one of the following holds.*

1. *$s(G) = 2$ and G is a Frobenius group, i. e. G has a nontrivial normal nilpotent Hall subgroup A and $C_G(a) \leq A$ for every nonidentity $a \in A$.*
2. *$s(G) = 2$ and G is a 2-Frobenius group, i. e. G has a normal Frobenius subgroup B with kernel A such that G/A is a Frobenius group with kernel B/A.*
3. *There exists a nonabelian simple group P such that*

$$P \leq \overline{G} = G/K \leq \operatorname{Aut} P$$

for some nilpotent normal $\pi_1(G)$-subgroup K of G and $\overline{G}/P$ is a $\pi_1(G)$-group. Moreover, the graph $GK(P)$ is disconnected, $s(P) \geq s(G)$, and for every integer i, $2 \leq i \leq s(G)$, there exists j, $2 \leq j \leq s(P)$ such that $\omega_i(G) = \omega_j(P)$.

In particular, G possesses at most one unsolvable composition factor.

In [22] it is proved that if P is a finite simple group with a disconnected prime graph then $|\mu_j(P)| = 1$ for $2 \leq j \leq s(P)$. In what follows we will denote by $n_j = n_j(P)$ the sole element of $\mu_j(P)$, $j \geq 2$.

Lemma 2 ([23])**.** *Let G be a finite group, N a normal subgroup of G and let G/N be a Frobenius group with a kernel F and a cyclic complement H. If $(|F|, |N|) = 1$ and F is not contained in $NC_G(N)/N$ then G contains an element of order $p|H|$ for some $p \in \pi(N)$.*

Proposition 1. *Let G be a finite group with a disconnected graph $GK(G)$ and suppose that for some nontrivial nilpotent normal $\pi_1(G)$-subgroup K of G the quotient group G/K is simple. Then for every subgroup H of G the group $KC_H(K)/K$ is trivial.*

Proof. Denote $G/K = P$. Let $KC_H(K)/K > 1$ for some $H \leq G$. Then $KC_G(K)/K$ is a nontrivial normal subgroup of P and so $KC_G(K)/K = P$ and $KC_G(K) = G$. Let $g \in G$ be an element of order $n_2(G)$. Since K is a $\pi_1(G)$-group, we can assume that $g \in C_G(K)$. It follows that the components $\pi_1(G)$ and $\pi_2(G)$ are connected in $GK(G)$; a contradiction. □

3. Proof of the theorem

Throughout the proof of the theorem L will denote the group $S_4(q)$, where $q = p^n$, p prime, $n \geq 1$.

Lemma 3 ([13, 24] or [25])**.** *The following holds.*

1. *If $p = 2$ then the spectrum of L consists of all divisors of the following numbers*:

$$4,\ p(q-1),\ p(q+1),\ q^2-1,\ q^2+1.$$

In particular, $s(L) = 2$ and $n_2(L) = q^2 + 1$.

2. *If $p > 3$ then the spectrum of L consists of all divisors of the following numbers*:

$$p(q-1),\ p(q+1),\ (q^2-1)/2,\ (q^2+1)/2.$$

If $p = 3$, then, additionally, the spectrum of L contains 9.

In particular, $s(L) = 2$ and $n_2(L) = (q^2+1)/2$.

Let G be a finite group isospectral to L. By Lemma 3 the graph $GK(G)$ is disconnected, so it follows from Lemma 1 and [19] that there exists a nonabelian simple group P, such that $P \leq G/K \leq \operatorname{Aut} P$ for some nilpotent normal $\pi_1(G)$-subgroup K of G, $s(P) \geq 2$, and $n_j(P) = n_2(L)$ for some $2 \leq j \leq s(P)$. Denote $n_2(L)$ by m and fix this notation until indicated otherwise.

First we prove that P is isomorphic to one of the groups $S_4(q)$, $L_2(q^2)$. Due to Theorems 1 and 2 from [26] the group P cannot be an alternating or a sporadic group or the Tits group ${}^2F_4(2)'$. Also if P is a group of Lie type in characteristic p, this result follows from Theorem 3 from [26]. Therefore in what follows we can assume that P is a group of Lie type in characteristic different from p.

Consider the case $p = 2$. The spectrum of L does not contain elements of order $4t$, $t > 1$, so it follows from [13, Lemma 8] together with what we already know that if P is not $S_4(q)$ or $L_2(q^2)$ then P is ${}^2G_2(3^{2l+1})$, $l \geq 1$. In this case $n_2(P) = 3^{2l+1} + 3^{l+1} + 1$, $n_3(P) = 3^{2l+1} - 3^{l+1} + 1$. It is clear that none of these numbers can be presented as $2^{2n} + 1$. So P can only be $S_4(q)$ or $L_2(q^2)$.

If $p = 3$, the theorem follows from [10, Lemma 14]. Therefore, in what follows we are going to assume that $p \geq 5$. The following simple lemma will be useful.

Lemma 4. *The following holds.*

1. *The least common multiple of the numbers from $\omega_1(L)$ equals $p(m-1)$.*
2. *Let N be the least common multiple of the numbers from $\omega(P)\backslash\omega_j(P)$, where $n_j(P) = m$. Then $N/n_j(P) < p$.*

Proof of Lemma 4. The first assertion is obvious. Since $\omega(P)\backslash\omega_j(P) \subseteq \omega_1(L)$, it follows that

$$N/n_j(P) = N/m \leq p(m-1)/m < p.$$

□

All finite simple groups with disconnected prime graphs are listed in [10, Tbl. 1a–1c]. For every group P in these tables there are also given the numbers $n_j(P)$, $j \geq 2$. We are going to consider every group from these tables apart from the aforementioned alternating and sporadic groups and ${}^2F_4(2)'$. We start with the finite simple groups P such that $s(P) = 2$.

Suppose that $P = L_r(u)$, where r is an odd prime and $(r, u) \neq (3, 2), (3, 4)$. We have $m = \dfrac{u^r - 1}{(u-1)(r, u-1)}$. The spectrum of P contains numbers $\dfrac{u^{r-1} - 1}{(r, u-1)}$ and

$u^{r-2}-1$ (see [27]). Their least common multiple is divisible by $\dfrac{(u^{r-1}-1)(u^{r-2}-1)}{(u-1)(r,u-1)}$, so by Lemma 4 we have $\dfrac{(u^{r-1}-1)(u^{r-2}-1)}{u^r-1} < p$, wherefrom $u^{r-3} \leq p$. Using this in the equality $n_2(P) = m = (p^{2n}+1)/2$ we get $r \leq 3 + \dfrac{4}{2n-1}$, and the inequality is strict for $u > 2$. Thus, $r \leq 7$ if $u = 2$ and $r \leq 5$ in the other cases. Note that the case $u = 2$ is impossible since none of the numbers $2^r - 1$ for $r \leq 7$ can be presented as $(q^2+1)/2$. So $u > 2$ and $r \leq 5$.

Consider the case $r = 5$. In the previous paragraph we concluded that $u^2 \leq p$, wherefrom $u^2 - 1 < p$. Suppose that $(5, u-1) = 1$. Then $m = u^4 + u^3 + u^2 + u + 1$. By Lemma 4 every element of $\omega_1(P)$ must divide $p(m-1)$. We have

$$u^2 - 1 \mid p(m-1) = p(u^4 + u^3 + u^2 + u).$$

If $u > 3$, we get $p \mid u^2 - 1$, which contradicts the inequality obtained earlier. In turn, the case $u = 3$ is impossible since $q^2 \neq 241$.

Assume now that $(5, u-1) = 5$. Then $m = (u^4 + u^3 + u^2 + u + 1)/5$, and so

$$u^2 - 1 \mid p(u^4 + u^3 + u^2 + u - 4).$$

Again we have $p \mid u^2 - 1$. Therefore, the case $r = 5$ is impossible.

We are left with the case $r = 3$. Suppose that $(3, u-1) = 1$. Then $m = u^2+u+1$. Every element from $\omega_1(P)$ must divide $p(m-1)$, which in turn divides $q^2(m-1)$. We have

$$u^2 - 1 \mid q^2(m-1) = (2u^2 + 2u + 1)(u^2 + u),$$

therefore, $u^2 - 1 \mid 5(u+1)$ and $u - 1 \mid 5$. So $u = 3$, wherefrom $m = 13$ and $q = 5$. But the spectrum of $L_3(3)$ contains the number 8, which is not in the spectrum of $S_4(5)$, thus this case is impossible.

Suppose that $(3, u-1) = 3$. Then $m = (u^2 + u + 1)/3$. Similarly we get

$$u^2 - 1 \mid (2u^2 + 2u - 1)(u^2 + u - 2),$$

and so $u^2 - 1 \mid u - 1$, which is impossible. Therefore, P is not $L_r(u)$, where r is an odd prime.

Suppose that $P = L_{r+1}(u)$, where r is an odd prime and $u - 1$ divides $r + 1$. Then $m = \dfrac{u^r - 1}{u - 1}$. In $\omega_1(P)$ there are numbers $\dfrac{u^{r+1}-1}{(u-1)^2}$ and $u^{r-1} - 1$. Their least common multiple is divisible by $\dfrac{(u^{r+1}-1)(u^{r-1}-1)}{(u-1)^2(u^2-1)}$, so by Lemma 4 we have $u^{r-3} \leq p$, wherefrom $r \leq 7$. If $r = 7$, then $u \in \{2,3,5,9\}$. If $r = 5$, then $u \in \{2,3,4,7\}$. Lastly, if $r = 3$, then $u \in \{2,3,5\}$. If follows from the equality $m = n_2(P)$ that $(r,u) = (3,3)$. In this case $P = L_4(3)$ and $L = S_4(5)$. But the spectrum of $L_4(3)$ contains 9, which is not in the spectrum of $S_4(5)$, so this case is impossible.

Let $P = U_r(u)$, where r is an odd prime. Then $m = \dfrac{u^r + 1}{(u+1)(r, u+1)}$. The spectrum $\omega_1(P)$ contains numbers $\dfrac{u^{r-1}-1}{(r,u+1)}$ and $u^{r-2}+1$. Arguing as before, using Lemma 4 we get $u^{r-3} < p$ and hence $r \leq 5$. Note that in this case $u > 2$ since none of the numbers $(2^r + 1)/3$, $r \leq 5$, can be presented as $(q^2+1)/2$.

Consider the case $r = 5$. In the previous paragraph we obtained the inequality $u^2 < p$, wherefrom $u^2 - 1 < p$. Suppose that $(5, u+1) = 1$. Then $m = u^4 - u^3 + u^2 - u + 1$. By Lemma 4 we have

$$u^2 - 1 \mid p(m-1) = p(u^4 - u^3 + u^2 - u).$$

Since $u > 2$, we get $p \mid u^2 - 1$, which contradicts the inequality obtained earlier. If $(5, u+1) = 5$, then $m = (u^4 - u^3 + u^2 - u + 1)/5$, and consequently

$$u^2 - 1 \mid p(u^4 - u^3 + u^2 - u - 4).$$

Again we get $p \mid u^2 - 1$. Ergo the case $r = 5$ is impossible.

Let $r = 3$ and assume that $(3, u+1) = 1$. Then $m = u^2 - u + 1$. Using Lemma 4 we get

$$u^2 - 1 \mid q^2(m-1) = (2u^2 - 2u + 1)(u^2 - u),$$

and hence $u^2 - 1 \mid 5(u-1)$ and $u + 1 \mid 5$, which is possible only when $u = 4$. In this case $P = U_3(4)$ and $L = S_4(5)$.

Prove that this case is impossible. Note that here $\mu(L) = \{12, 13, 20, 30\}$ and $\mu(P) = \{4, 10, 13, 15\}$. Let G be a finite group isospectral to $S_4(5)$ and $P \leq G/K \leq \operatorname{Aut} P$ for some nilpotent normal $\{2,3,5\}$-subgroup K of G. It follows from [28] that if $P < H \leq \operatorname{Aut} P$, then H contains an element of order 8. Therefore $P = G/K$.

Suppose that $3 \in \pi(K)$. Let $K = P \times Q$, where P is a Sylow 3-subgroup of K. Agree on the notation: $\overline{X}$ is the image of X under factorization by Q. Then $\overline{K}$ is a 3-group and $\overline{G}/\overline{K} \simeq P$. Let a be an element of order 13 from P. Then $N = N_P(a)$ is a Frobenius group with kernel of order 13 and complement of order 3. Let F be the preimage of N in $\overline{G}$. Lemma 2 implies that F contains an element of order 9, which contradicts the groups' G and L being isospectral.

Thus, the order of K is not divisible by 3. Since $12 \in \omega(G)$, it follows that $2 \in \pi(K)$ and there exists an element $x \in G$ of order 3 that centralizes an element $y \in K$ of order 4. Suppose that some element of order 3 from G centralizes an element of order 5 from K. Since in P all elements of order 3 are conjugate (see [28]), it follows that x also centralizes some element $z \in P$ of order 5. Since K is nilpotent, the element xyz is of order 60; a contradiction.

Therefore, an element of order 30 from G is a preimage of an element of order 15 from P. It follows that there is a G-chief factor W of the Sylow 2-subgroup of K, such that P fixes a nontrivial point in W. We can regard W as an absolutely irreducible P-module. Since $\omega(W \leftthreetimes P) \subseteq \omega(G)$, elements of order 13 from P do not fix nontrivial points in W. A complete description of absolutely irreducible representations of P in characteristic 2 can be found in [29], and it can be checked directly (for example, using GAP [30]) that there are no absolutely irreducible P-modules W in characteristic 2 such that an element of order 15 from P fixes a nontrivial point in W and elements of order 13 from P do not fix any nontrivial points in W. Ergo this case is impossible, i. e. $P \neq U_3(4)$.

So, in the case $P = U_r(u)$, where r is an odd prime, we are left with the case $r = 3$ and $(3, u+1) = 3$. In this case $m = (u^2 - u + 1)/3$. Arguing as before, we get

$$u^2 - 1 \mid (2u^2 - 2u - 1)(u^2 - u - 2),$$

and hence $u^2 - 1 \mid u + 1$, which is impossible. Therefore P is not $U_r(u)$, where r is an odd prime.

Assume that $P = U_{r+1}(u)$, where r is an odd prime, $u+1$ divides $r+1$, and $(r,u) \neq (3,3),(5,2)$. In this case $m = \dfrac{u^r+1}{u+1}$. The set $\omega_1(P)$ contains numbers $\dfrac{u^{r+1}-1}{(u+1)^2}$ and $u^{r-1}-1$. As before, it follows from Lemma 4 that $u^{r-4} < p$, and consequently $r \leq 7$. Thus $(r,u) \in \{(5,5),(7,3),(7,7)\}$. It is easy to check that in none of these cases the number $\dfrac{u^r+1}{u+1}$ can be presented as $(q^2+1)/2$.

The case $P = U_4(2)$ is impossible since $n_2(U_4(2)) = 5$ and $m \geq 13$.

The case $P = O_{2k+1}(u)$, where $k = 2^l \geq 4$ and u odd, is impossible, since in this case $n_2(P) = (u^k+1)/2$, so the equality $n_2(P) = m$ is only possible if P is a group of Lie type in characteristic p, which contradicts our initial assumption.

Let $P = O_{2r+1}(3)$, where r is an odd prime. Then $m = (3^r-1)/2$. In $\omega_1(P)$ there are coprime numbers $(3^r+1)/4$ and $3^{r-1}-1$. By Lemma 4 we have $3^{r-2} < p$, and hence $r = 3$. In this case $P = O_7(3)$ and $L = S_4(5)$. But the spectrum of P contains the number 8 which is not in the spectrum of L, thus this case is impossible.

Suppose that $P = S_{2k}(u)$, where $k = 2^l \geq 2$. Then $n_2(P) = \dfrac{u^k+1}{(2,u-1)}$. We need to show that this case is possible only when $k = 2$ and $u = q$. If u is even, then we have $(q^2+1)/2 = 2^l+1$ for some l. But then $(q^2-1)/2 = 2^l$, which is impossible since the number $(q^2-1)/2$ is divisible by 3. If u is odd then the desired conditions follow from the equality $m = n_2(P)$.

Let $P = S_{2r}(u)$, where r is an odd prime and $u \in \{2,3\}$. First assume that $u = 2$. In this case $m = 2^r-1$. The spectrum of P contains coprime numbers 2^r+1 and $2(2^{r-1}+1)$. Using Lemma 4 we get $2^r < p$, which is impossible since $m = 2^r-1$. The case $u = 3$ is similar to the case $P = O_{2r+1}(3)$: it is also only possible if $r = 3$. In this case $P = S_6(3)$, $L = S_4(5)$, which cannot be since $\omega(P)$ contains 8.

Assume that $P = O^+_{2r}(u)$, where $r \geq 5$ is prime and $u \in \{2,3,5\}$. In this case $m = \dfrac{u^r-1}{u-1}$. Suppose that $u = 2$. Then $\omega_1(P)$ contains coprime numbers $2^{r-1}+1$ and $2^{r-2}+1$. It follows from Lemma 4 that $2^{r-3} < p$, which is only possible if $r = 5$. But the number 31 cannot be presented as $(q^2+1)/2$, therefore this case is impossible. Let $u = 3$. Then $\omega_1(P)$ contains coprime numbers $(3^{r-1}-1)/2$ and $(3^{r-2}-1)/2$. By Lemma 4 we get $3^{r-4} < p$, wherefrom $r \leq 7$. But the equality $m = n_2(P)$ does not hold for $r \leq 7$, so this case is also impossible. Lastly, suppose that $u = 5$. Then in $\omega_1(P)$ there are numbers $(5^{r-1}-1)/2$ and $(5^{r-2}-1)/2$, whose greatest common divisor equals 2, so Lemma 4 implies that $5^{r-4} < p$, wherefrom again we have $r \leq 7$. But if $r \leq 7$, the equality $m = n_2(P)$ does not hold. Ergo, P is not $O^+_{2r}(u)$, where $r \geq 5$ is prime and $u \in \{2,3,5\}$.

Let $P = O^+_{2r+2}(u)$, where r is an odd prime and $u \in \{2,3\}$. Then $m = \dfrac{u^r-1}{(2,u-1)}$. Suppose that $u = 2$. Then $\omega_1(P)$ contains numbers $2^{r+1}-1$ and $2^{r-1}-1$, whose greatest common divisor equals 3, so by Lemma 4 we have $2^r-1 < 3p+1$. It follows from the equality $m = n_2(P)$ that $q^2 < 6p+1$, which is only possible if $q = 5$. But $2^r-1 \neq 13$ so the case $u = 2$ is impossible. Assume that $u = 3$. Then $\omega_1(P)$ contains numbers $(3^{r+1}-1)/2$ and $(3^{r-1}-1)/2$, whose greatest common divisor equals 4, therefore Lemma 4 implies that $3^r-1 < 8p+2$. From the equality $m = n_2(P)$ we conclude that $q^2 < 8p+1$, which is only possible if $q \in \{5,7\}$. If $q = 5$, then $r = 3$. This case is impossible since the group $O^+_8(3)$ contains an element of order 8 and there are no elements of such order in $S_4(5)$. Lastly, the case $q = 7$ is impossible since $(3^r - 1)/2 \neq 25$.

Suppose that $P = O^-_{2k}(u)$, where $k = 2^l \geq 4$. Then $n_2(P) = \dfrac{u^k + 1}{(2, u+1)}$. The number u cannot be odd since in this case P is a group of Lie type in characteristic p. If u is even then the equality $m = n_2(P)$ implies that $q^2 - 1 = 2^t$ for some t, which is impossible since the number $q^2 - 1$ is divisible by 3.

The case $P = O^-_{2k}(2)$, where $k = 2^l + 1 \geq 5$, is ruled out similarly since in this case $n_2(P) = 2^{k-1} + 1$.

Assume that $P = O^-_{2r}(3)$, where r is a prime such that $r \geq 7$ and $r \neq 2^l + 1$. Then $n_2(P) = (3^r + 1)/4$. The spectrum of P contains coprime numbers $(3^{r-1} + 1)/2$ and $(3^{r-2} + 1)/2$. Using Lemma 4 we get $3^{r-3} < p$, which is impossible since $r \geq 7$ and $m = n_2(P)$.

If $P = O^-_{2k}(3)$, where k is a composite number of form $2^l + 1$, then $n_2(P) = (3^{k-1} + 1)/2$, thus this case is impossible for $p \geq 5$.

Let $P = G_2(u)$, where u is not a power of 3 and $u > 2$. Suppose that $u \equiv 1 \pmod 3$. Then $n_2(P) = u^2 - u + 1$. In $\omega_1(P)$ there are numbers $u^2 + u + 1$ and $u^2 - 1$ (see [31] or [32]), whose greatest common divisor equals 3. By Lemma 4 we have $u^2 - u + 1 < 3p$. This is only possible if $q = 5$. In this case $u = 4$. But the group $G_2(4)$ contains elements of order $7 \notin \omega(S_4(5))$, so this case is impossible.

Assume now that $u \equiv -1 \pmod 3$. Then $n_2(P) = u^2 + u + 1$. Again, $\omega_1(P)$ contains numbers $u^2 - u + 1$ and $u^2 - 1$, and their greatest common divisor is 3, so it follows from Lemma 4 that $(u-1)^2 < 3p$. Using this in the equality $m = n_2(P)$, we get $(q^2 + 1)/2 < 3(p + \sqrt{3p} + 1)$, which is only possible if q equals 5 or 7. If $q = 5$, then $n_2(P) = 13$ and $u = 3$, which contradicts our choice of u. If $q = 7$, then the equality $m = n_2(P)$ does not hold for any u.

Suppose that $P = {}^3D_4(u)$. Then $n_2(P) = u^4 - u^2 + 1$. Since $m = n_2(P)$, by Lemma 4 every element from $\omega_1(P)$ divides the number

$$q^2(u^4 - u^2) = (2u^4 - 2u^2 + 1)(u^4 - u^2).$$

In P there is an element of order $(u^3 + 1)(u - 1)$ (see [33] or [34]), wherefrom $u^3 + 1 \mid (u + 1)(3u + 1)$, which is possible only if $u \leq 4$. If $u = 2$, then $q = 5$. In this case $P = {}^3D_4(2)$ and $L = S_4(5)$. But this case is impossible since in P there are elements of order $9 \notin \omega(L)$. The case $u = 3$ is impossible since $q^2 \neq 145$. Lastly, the case $u = 4$ is impossible since $q^2 \neq 481$.

Assume that $P = F_4(u)$, where u is odd. Then $n_2(P) = u^4 - u^2 + 1$. In $\omega_1(P)$ there are numbers $u^4 + 1$ and $(u^4 - 1)/2$ (see [35] or [34]), whose greatest common divisor equals 2, therefore Lemma 4 implies that $u^4 + u^2 - 1 < 4p$ and so, of course, $u^4 - u^2 + 1 < 4p$. Due to the equality $m = n_2(P)$ we have $q^2 < 8p - 1$, which is only possible if q equal to 5 or 7. Since u is odd, in neither of these cases the equality $m = n_2(P)$ holds.

Let $P = E_6(u)$. Then $n_2(P) = \dfrac{u^6 + u^3 + 1}{(3, u-1)}$. The spectrum $\omega_1(P)$ contains numbers $\dfrac{u^6 - 1}{(3, u-1)}$ and $u^5 - 1$ (see [36]), and their least common multiple is divisible by $\dfrac{(u^6 - 1)(u^5 - 1)}{(3, u-1)(u-1)}$. Using Lemma 4 we get $u^4 < p$, and due to the equality $m = n_2(P)$ this is only possible if $u = 2$. This in turn is impossible since $q^2 \neq 145$.

Suppose that $P = {}^2E_6(u)$, where $u > 2$. Then $n_2(P) = \dfrac{u^6 - u^3 + 1}{(3, u+1)}$. In $\omega_1(P)$ there are numbers $\dfrac{u^6 - 1}{(3, u+1)}$ and u^5+1 (see [36]), and their least common multiple is divisible by $\dfrac{(u^6-1)(u^5+1)}{(3, u+1)(u+1)}$. By Lemma 4 we have $u^4 - u^3 + u^2 \le p$. Using the equality $m = n_2(P)$ we get

$$2(u^6 - u^3 + 1) > q^2 \ge (u^4 - u^3 + u^2)^{2n},$$

which is impossible for $u > 2$.

Therefore, the simple groups P such that $s(P) = 2$ are considered. We proceed to the case $s(P) = 3$.

Let $P = L_2(u)$, where $u > 3$ is a power of an odd number r. We need to show that this case is only possible if $u = q^2$. Let $u \equiv \varepsilon \pmod 4$, $\varepsilon = \pm 1$. Then $\omega_1(P)$ contains $(u-\varepsilon)/2$, $n_2(P) = r$, and $n_3(P) = (u+\varepsilon)/2$. Suppose that $m = n_2(P)$. In this case $r \ge 13$, since $p \ge 5$. It follows from Lemma 4 that $(r^{2k}-1)/4r < p$, where $u = r^k$, wherefrom $r^{2k-1} < p^2$. This inequality is only possible if $n = k = 1$. If so, from Lemma 4 we have that $(r+1)/2$ divides $q^2(m-1) = (2r-1)(r-1)$, which is only possible when $r \le 11$; a contradiction. Assume now that $m = n_3(P)$. If $u \equiv -1 \pmod 4$, then $m = (u-1)/2 \ge 13$, wherefrom $u \ge 27$. The number $(u+1)/2$ divides $(u-2)(u-3)/2$, which is only possible when $u \le 11$; again we come to a contradiction. We are left with the case $u \equiv 1 \pmod 4$. In this case $(u+1)/2 = (q^2+1)/2$, therefore $u = q^2$.

Suppose that $P = L_2(u)$, where $u > 2$ is a power of 2. Then $n_2(P) = u - 1$, $n_3(P) = u + 1$. Since $m \ge 13$, we have $u \ge 16$. If $m = n_2(P)$, by Lemma 4 the number $u+1$ divides $q^2(m-1) = (u-2)(2u-3)$, which is possible only when $u \le 14$. If $m = n_3(P)$, then $u-1$ divides $u(2u+1)$, which is only possible if $u \le 4$. In either case we come to a contradiction.

The case $P = U_6(2)$ is impossible since in this case $n_2(P) = 7$ and $n_3(P) = 11$, whereas $m \ge 13$.

Let $P = O^-_{2r}(3)$, where $r = 2^l + 1$ is a prime. Then $n_2(P) = (3^{r-1}+1)/2$, $n_3(P) = (3^r+1)/4$. The case $m = n_2(P)$ is impossible since $p \ge 5$. Suppose that $m = n_3(P)$. The set $\omega_1(P)$ contains $(3^{r-2}+1)/2$, therefore by Lemma 4 we have $3^{r-3} < p$, which is only possible if $r \in \{3, 5\}$. If $r = 3$, then $n_3(P) = 7$, which is impossible since $m \ge 13$. If $r = 5$, then $q = 11$. The group $O^-_{10}(3)$ contains an element of order $41 \notin \omega(S_4(11))$, thus this case is also impossible.

Suppose that $P = G_2(u)$, where u is a power of 3. Then $n_2(P) = u^2 - u + 1$, $n_3(P) = u^2 + u + 1$. In $\omega_1(P)$ there is a number $u^2 - 1$. Assume that $m = n_2(P)$. Then Lemma 4 implies that $u^2 - 1 < p$, wherefrom $p^{2n-1} < 2$, which is impossible. Now let $m = n_3(P)$. Then by Lemma 4 we have $u < \sqrt{p} + 1$, which is possible only when $q = 5$. If so, $u = 3$. This case is impossible since $G_2(3)$ contains an element of order $8 \notin \omega(S_4(5))$.

Assume now that $P = {}^2G_2(u)$, where u is an odd power of 3 and $u > 3$. Then $n_2(P) = u - \sqrt{3u} + 1$, $n_3(P) = u + \sqrt{3u} + 1$. The spectrum of P contains $u - 1$. Let $m = n_2(P)$. Then Lemma 4 implies that $u - 1 < p$, wherefrom $p^{2n-1} < 2$, which is impossible. If $m = n_3(P)$, then since $u \ge 27$, by Lemma 4 we have $(u-1)/2 < p$, wherefrom $(u/4)^{2n} < 16$. Since $u \ge 27$, this inequality is impossible.

Let $P = F_4(u)$, where u is a power of 2. Then $n_2(P) = u^4+1$, $n_3(P) = u^4-u^2+1$. If $m = n_2(P)$, then $m-1 = u^4$, which is impossible since $m-1$ is divisible by 3. Suppose that $m = n_3(P)$. In $\omega_1(P)$ there is the number u^4-1, so by Lemma 4 we have $u^4-u^2+1 \leq p+1$, and so $q^2 \leq 2p+1$. This inequality cannot hold since $p \geq 5$.

Suppose now that $P = {}^2F_4(u)$, where u is an odd power of 2 and $u > 2$. Then $n_2(P) = u^2 - \sqrt{2u^3} + u - \sqrt{2u} + 1$, $n_3(P) = u^2 + \sqrt{2u^3} + u + \sqrt{2u} + 1$. The set $\omega_1(P)$ contains coprime numbers $u-\sqrt{2u}+1$ and $u+\sqrt{2u}+1$, the product of which equals u^2+1. Assume that $m = n_2(P)$. Then by Lemma 4 we have $u^2+1 < p$, which is impossible due to the equality $m = n_2(P)$. Let $m = n_3(P)$. Then since $u \geq 8$, using Lemma 4 we get $(u^2+1)/4 < p$. This inequality is also impossible due to $m = n_3(P)$.

If P is one of the groups $E_7(2)$, $E_7(3)$, then none of the equalities $m = n_2(P)$, $m = n_3(P)$ is possible. Thus, the simple groups P such that $s(P) = 3$ are dealt with.

Lastly, consider the case $s(P) > 3$. Suppose that $P = {}^2B_2(u)$, where u is an odd power of 2 and $u > 2$. Then $s(P) = 4$, $n_2(P) = u-1$, $n_3(P) = u - \sqrt{2u} + 1$, $n_4(P) = u + \sqrt{2u} + 1$. Using Lemma 4, we get $u - 1 < p$. Due to this inequality, none of the equalities $m = n_2(P)$, $m = n_3(P)$, $m = n_4(P)$ is possible.

Let $P = E_8(u)$. Then $4 \leq s(P) \leq 5$, $n_2(P) = \dfrac{u^{10}-u^5+1}{u^2-u+1}$, $n_3(P) = \dfrac{u^{10}+u^5+1}{u^2+u+1}$, $n_4(P) = u^8-u^4+1$. If $u \equiv 2, 3 \pmod 5$, then $s(P) = 4$. If $u \equiv 0, 1, 4 \pmod 5$, then $s(P) = 5$ and additionally $n_5(P) = \dfrac{u^{10}+1}{u^2+1}$. In either case by Lemma 4 we have $u^7 < p$. It can be checked directly that due to this none of the equalities $m = n_i(P)$, $i \geq 2$, can hold.

If P is one of the groups $L_3(4)$, ${}^2E_6(2)$, then none of the equalities $m = n_i(P)$, $i \geq 2$, holds. Thus we have proved that P is indeed isomorphic to either $S_4(q)$ or $L_2(q^2)$.

The fact that if $P \simeq S_4(q)$, the group K is trivial, follows from [37, Prop. 1.1]. Show that if $P \simeq L_2(q^2)$, then K is a p-group. Indeed, if so, the group P contains a Frobenius subgroup $M = FH$, where F is elementary abelian of order q^2 and H is cyclic of order $\dfrac{q^2-1}{(2, q-1)}$. Suppose that K is not a p-group, i. e. in $\pi(K)$ there is a prime r not equal to p. Since K is nilpotent, we can assume that K is an r-subgroup. Thus, the orders of K and F are coprime. Let M_1 be the preimage of M in G. Due to Proposition 1 the group M_1 meets the conditions of Lemma 2 and therefore G contains an element of order $r \cdot \dfrac{q^2-1}{(2, q-1)}$; a contradiction.

Finally, prove that the group G/K is an extension of P by a field automorphism of order 2^m, $m \geq 0$. If $P \simeq S_4(q)$, this follows from [38, Theorem 1] and [39].

Let $P \simeq L_2(q^2)$. Every outer automorphism of P is either a field automorphism, or a diagonal automorphism of order $(2, q-1)$, or the product of the two.

First show that if τ is a field automorphism of an odd prime order r of P, then $\omega(P\langle\tau\rangle) \not\subseteq \omega(S_4(q))$. For that we use the following Lemma.

Lemma 5. [12, Cor. 14] *Let q be a power of a prime, r a natural number and τ a field automorphism of order r of $L_n(q^r)$. Then*

$$\omega(L_n(q^r)\langle\tau\rangle)=\bigcup_{k|r}\frac{r}{k}\omega(L_n(q^k)).$$

Now let τ be a field automorphism of P of an odd prime order r and let q_0 be such that $q_0^r=q$. By Lemma 5 we have

$$\omega(P\langle\tau\rangle)=r\omega(L_2(q_0^2))\cup\omega(L_2(q^2)).$$

Put $m_0=\dfrac{q_0^2+1}{(2,q_0^2+1)}$. Since r is odd, the number m_0 divides m and consequently $m_0\notin\omega_1(L_2(q_0^2))$. If $r\notin\omega(S_4(q))$ then in $P\langle\tau\rangle$ there is an element of order $r\notin\omega(S_4(q))$. If $r\in\omega_1(S_4(q))$ then $P\langle\tau\rangle$ contains an element of order $rm_0\notin\omega(S_4(q))$. Lastly, if $r\in\omega_2(S_4(q))$ then $P\langle\tau\rangle$ contains an element of order $2r\notin\omega(S_4(q))$. Therefore, $\omega(P\langle\tau\rangle)\not\subseteq\omega(S_4(q))$.

Now assume that $p\neq 2$ and let θ be a diagonal automorphism of P such that $\theta^2\in P$. The group $P\langle\theta\rangle$ is isomorphic to $PGL_2(q^2)$ and so it contains an element of order $q^2-1\notin\omega(S_4(q))$. Thus it remains to check the case when P is being extended by an automorphism $\tau\theta$, where τ is a field automorphism. If the order of τ is divisible by an odd prime r, then there is an integer s such that $(\tau\theta)^s$ modulo P is a field automorphism of order r. Thus we can assume that the order or τ is a power of 2. Put $|\tau|=2^l$, $l>0$ and let q_0 be such that $q_0^{2^l}=q^2$. Arguing as in the proof of Lemma 3.3 from [40], we have

$$\omega(P\langle\tau\theta\rangle)=2^l\cdot\omega(PGL_2(q_0))\cup\bigcup_{k|2^l,\ k>1}\frac{2^l}{k}\cdot\omega(L_2(q_0^k)).$$

Note that $PGL_2(q_0)$ contains elements of order q_0-1 and q_0+1.

Here we are going to denote by $(x)_2$ the 2-part of an integer x, i. e. the maximal number that is a power of 2 that divides x. Let $((q^2-1)/2)_2=2^m$. In other words, 2^m is the biggest number that is a power of 2 that lies in $\omega(S_4(q))$. We have

$$q^2-1=q_0^{2^l}-1=(q_0^2-1)(q_0^2+1)(q_0^4+1)\dots(q_0^{2^{l-1}}+1).$$

Therefore, $q^2-1=(q_0^2-1)\cdot a$, where $(a)_2=2^{l-1}$, and consequently $(q_0^2-1)_2=2^{m-l+2}$. This means that the 2-part of one of the numbers q_0-1, q_0+1 is 2^{m-l+1}, from where we conclude that the set $2^l\cdot\omega(PGL_2(q_0))$ contains 2^{m+1}. Therefore, $\omega(P\langle\tau\theta\rangle)\not\subseteq\omega(S_4(q))$.

The theorem is proved.

4. Final remarks

1. The main theorem does not hold for q equal to 2 or 3. In particular, the group $S_4(2)$ is isospectral to a solvable Frobenius group of order $2^3\cdot 3\cdot 5^2$ ([6]) and the group $S_4(3)$ is isospectral to a solvable 2-Frobenius group of order $2^2{\cdot}3^{24}{\cdot}5$ ([20]).

There's also an exceptional case where (in the notation of the main theorem) G is isospectral to $S_4(5)$ and $P\simeq U_3(4)$. We will now show that in this case $G/K=P$, where K is a 2-group.

Note that $\mu(S_4(5))=\{12,13,20,30\}$ and $\mu(U_3(4))=\{4,10,13,15\}$. It follows from Lemma 1 that $P\leq G/K\leq\operatorname{Aut}P$ for some nilpotent normal $\pi_1(G)$-subgroup K of G. It also follows from [28] that if $P<H\leq\operatorname{Aut}P$, then H contains

an element of order 8. Therefore $P = G/K$. From inspecting the spectra of $S_4(5)$ and $U_3(4)$ it follows that $\pi(K) \subseteq \{2, 3\}$. Suppose that the order of K is divisible by 3. Let $K = S \times R$, where S is a Sylow 3-subgroup of K. Agree on the notation: $\overline{X}$ is the image of X under factorization by R. Then $\overline{K}$ is a 3-group and $\overline{G}/\overline{K} \simeq P$. Let a be an element of order 13 from P. Then $N = N_P(\langle a \rangle)$ is a Frobenius group with kernel of order 13 and complement of order 3. Let F be the preimage of N in $\overline{G}$. Lemma 2 implies that in this case$F$ contains an element of order 9.

It follows that K can only be a 2-group. Also, since $U_3(4)$ does not contain elements of order 6, the group K must contain an element of order 4, therefore it's not elementary abelian. Further investigation of this case requires a less trivial approach, thus we omit it here.

2. Suppose that $P \simeq S_4(q)$ (in the notation of the main theorem) and let τ be a field automorphism of order $2^m > 1$ of P. It follows from [38] and [39] that if $p \leq 3$, then $\omega(P\langle\tau\rangle) \neq \omega(P)$, and if $p > 3$, then $\omega(P\langle\tau\rangle) = \omega(P)$. Note that every group $P\langle\tau\rangle$ contains a section isomorphic to P thus the groups if this type are not critical. It follows from [41] that $S_4(q)$ is critical if $p \neq 2$. If $p = 2$, it contains a proper subgroup H isospectral to $S_4(q)$. This subgroup is isomorphic to $L_2(q^2)\langle\tau\rangle$, where τ is a field automorphism of order 2 of $L_2(q^2)$.

Now assume that $P \simeq L_2(q^2)$. In the previous paragraph we gave an example of a group $L_2(q^2)\langle\tau\rangle$ isospectral to $S_4(q)$ if $p = 2$. There are also examples of groups of this type for which $K > 1$. In [13] there is an example of a group isospectral to $S_4(q)$, that is a semidirect product $K \leftthreetimes P$, where $q = 2^n$ and $|K| = 2^{8n}$. In [10] there is a similar example with $q = 3^{2n}$ and $|K| = 3^{28n}$. Also in [10] there is an example of a semidirect product $K \leftthreetimes (P\langle\tau\rangle)$ isospectral to $S_4(q)$, where $q = p^n$, $p > 3$, τ is a field automorphism of order 2 of P and $|K| = p^{8n}$.

3. Prove that if $p = 2$, the group $L_2(q^2)\langle\tau\rangle$ is critical. Since $\omega(L_2(q^2)) \neq \omega(S_4(q))$, every subgroup H of $L_2(q^2)\langle\tau\rangle$ isospectral to $S_4(q)$ is an extension of some subgroup H_1 of $L_2(q^2)$ by an automorphism τ. Since the order of $L_2(q^2)\langle\tau\rangle$ is $2q^2(q^2-1)(q^2+1)$ and H contains elements of orders q^2-1 and q^2+1, it follows that the index $|L_2(q^2) : H_1|$ is a power of 2. There are no subgroups of such index in $L_2(q^2)$ (see [42, Tbl. 8.1, 8.2] for example). On the other hand, if some section $(L_2(q^2)\langle\tau\rangle)/N$ is isospectral to $S_4(q)$, the group N must be a normal 2-subgroup. It is clear that there are no such subgroups in $L_2(q^2)\langle\tau\rangle$. Therefore, $L_2(q^2)\langle\tau\rangle$ is critical.

4. The question of what can be the order of τ in the case when $P \simeq L_2(q^2)$, is still open. For example, show that if $q = 7$ then there is no extension of a 7-group by P isospectral to $S_4(q)$.

Lemma 6. [10, Lemma 10] *Let F be a finite field of order $q = p^n > 3$, where p is a prime, $W_i = W_i(q)$, $i = 0, 1, \dots, p-1$, be the space of homogeneous polynomials of degree i in variables x_1, x_2 over F. Let α be the Frobenius automorphism of F. For $j = 0, \dots, n-1$ turn W_i into an $SL_2(q)$-module $W_i^j = W_i^j(q)$, putting $f(x_1, x_2)A = f(a_{11}^{\alpha^j}x_1 + a_{12}^{\alpha^j}x_2, a_{21}^{\alpha^j}x_1 + a_{22}^{\alpha^j}x_2)$ for $f(x_1, x_2) \in W_i$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in SL_2(q)$. In particular, W_0^j is a trivial one-dimensional $SL_2(q)$-module.*

1. *The modules* $W(i_0,\dots,i_{n-1}) = \bigotimes_{j=0}^{n-1} W_{i_j}^j$ *make up the complete set of pairwise nonequivalent absolutely irreducible* $SL_2(q)$*-modules over a field of characteristic* p*. If* q *is odd then the center of* $SL_2(q)$ *acts trivially on* $W(i_0,\dots,i_{n-1})$ *(and therefore* $W(i_0,\dots,i_{n-1})$ *is an* $L_2(q)$*-module) if and only if* $i_0+\dots+i_{n-1}$ *is even.*
2. *If* λ*,* λ^{-1} *are the eigenvalues of* $A \in SL_2(q)$*, then the eigenvalues of* A *on* W_1^j *are* λ^{p^j}*,* λ^{-p^j}*, on* W_2^j *these are* 1*,* λ^{2p^j}*,* λ^{-2p^j}*, and on* $W(i_0,\dots,i_{n-1})$ *these are* $\lambda_{i_0}\dots\lambda_{i_{n-1}}$ *(where* λ_{i_j} *is an eigenvalue on* W_i^j*).*

Let $q = 7$. In this case $\mu(S_4(q)) = \{24, 25, 42, 56\}$ and $\mu(P) = \{7, 24, 25\}$.

Prove that no extension of a 7-group K by P is isospectral to $S_4(7)$. Let G be such extension. Consider a G-chief factor W in K. We can regard W as an absolutely irreducible P-module. By Lemma 6 this module is equivalent to a module $W(i_0, i_1)$ from Lemma 6, where $i_0, i_1 = 0, 1, \dots, 6$ and $i_0 + i_1$ is even. Moreover, if i_0 and i_1 are even, then by item 2 of Lemma 6 there is an element of order 5 in P with an eigenvalue 1 on $W(i_0, i_1)$. Therefore, we can assume that i_0 and i_1 are odd.

Let $a \in P$ be an element of order 5, A be its preimage of order 5 in $SL_2(49)$. It can be checked directly that among the modules $W(i_0, i_1)$, where $i_0, i_1 = 1, 3, 5$, only in $W(1, 1)$ the element A does not fix any nontrivial points. Thus we can assume that every G-chief factor of K as a P-module is equivalent to $W(1, 1)$.

Let a be an element of order 6 from P and A its preimage $SL_2(49)$. In this case $|A| = 12$. Again, it can be directly checked that A does not have nontrivial fixed points in $W(1, 1)$. It follows that elements of order 6 from G does not fix nontrivial points in K, therefore in G there are no elements of order 42. Thus, $\omega(G) \neq \omega(S_4(q))$.

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