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СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 15, стр. 60–73 (2018) УДК 517.518
DOI 10.17377/semi.2018.15.008 MSC 28A80, 41A20, 65D05, 65D07

CONSTRAINED FRACTAL INTERPOLATION FUNCTIONS WITH VARIABLE SCALING

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Abstract. Fractal interpolant function (FIF) constructed through iterated function systems is more general than classical spline interpolant. In this paper, we introduce a family of rational cubic splines with variable scaling, where the numerators and denominators of rational function are cubic and linear polynomial respectively. FIFs with variable scaling offer more flexibility in fitting and approximation of many complicated phenomena than that of in FIF with constant scaling. The convergence result of the proposed rational cubic interpolant to data generating function in $\mathcal{C}^1$ is proven. When interpolation data is constrained by piecewise curves, we derive sufficient condition on the parameter of rational FIF so that it lies between them.

Keywords: fractals, rational splines, constrained interpolation, rational fractal interpolation function.

1. Introduction

The term fractal was coined by Mandelbrot [17] in 1975 to describe the geometry of nature such as the shape of clouds, forests, coastlines, leaves, flowers, galaxies, etc. Fractals are generally self-affine in nature and possess non-integer dimensions. Iterated function system(IFS) is an ideal tool for construction of fractals [16]. Based on IFS, Barnsley introduced the concept of FIF that has similarity with classical splines [1, 2]. The main distinctions of FIF with the classical interpolants include the construction by iteration of fractal interpolant instead of using an analytic expression and the presence of scaling factors, which offer flexibility in the choice of smooth or non-smooth interpolant in contrast to the unicity of a specific traditional

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Chand, A.K.B., Reddy, K.M., Constrained fractal interpolation functions with variable scaling.

Received October, 19, 2017, published January, 29, 2018.

interpolant. When the scaling factors are zero, a fractal interpolant reduces to the corresponding classical nonrecursive spline. Barnsley and Harrington [3] introduced the construction of a differentiable FIF or $\mathcal{C}^r$-FIF f that interpolates prescribed data if the values of $f^{(k)}$, $k = 1, 2, \dots, r$ are assigned at the initial end point of the interpolation interval. Fractal polynomial splines with general boundary conditions developed in constructive manner [4, 8, 20]. Using α-fractal functions, it is possible to generalize some classical piecewise interpolants, see for instance [18, 19, 20]. A data set under consideration for interpolation may have some fundamental shapes like irregularity, positivity, monotonicity, convexity (concavity). The notion of shape refers to the geometrical behavior of a function or approximant. The processes of evaluating interpolant that mimic the shape properties associated with a data set is called shape preserving or iso-geometric interpolation, and the corresponding interpolant is referred to as shape preserving interpolant [5, 10, 11, 15, 21, 23, 24]. Using fractal methodology, our group has initiated the shape preserving fractal splines in literature by restricting the scaling factors and shape parameters suitably [6, 7]. The shape preserving α-fractal function with constant scaling are developed recently in [26]. Constrained control of interpolating curve is a fundamental task and is an important subject that we face in applications including computer aided geometric design [13], data visualization, image analysis [14]. On the other hand, there are practical situations wherein interpolating curves that lie completely above or below a prefixed curve, for instance, a polygonal (piecewise linear and quadratic function) are sought-after. To provide additional flexibility and diversity, and to match intricate curves that show less self-similarity, FIF with variable scaling functions has been introduced by Wang and Fan [27]. Later Wang and Shan [28] studied analytical properties such as smoothness, stability and sensitivity of FIF with function vertical scaling factors. In this paper, we introduce a novel class of rational cubic FIFs with function vertical scaling factors and study its convergence, and constrained aspects when data set satisfies the same constraints.

The paper is organized as follows, In section 2, we discuss the construction of rational cubic FIF with variable scaling. In section 3, we derive an uniform error bound between the rational cubic FIF and the original function in $\mathcal{C}^1$ for convergence result. In section 4, we deduce the sufficient conditions for the proposed rational cubic spline with variable scaling so that it stays above or below of a piecewise spline. The theoretical results of previous section are implemented in section 5 to construct the desired constrained rational cubic spline FIFs with variable scaling.

2. Preliminaries

First, we discuss the construction of FIFs from a given data set which is generated from a complex function, and the details can be found in [1, 2, 3]. For a fixed $N \in \mathbb{N}$, we denote $\mathbb{N}_N$ as the set of first N natural numbers. Let the prescribed set of interpolation data be $\{(x_i, y_i) \in I \times \mathbb{R} : i \in \mathbb{N}_N\}$, where $I = [x_1,\ x_N],\ N > 2$. For $i \in \mathbb{N}_{N-1}$, let $L_i : I \to I_i = [x_i,\ x_{i+1}]$ be contraction homeomorphisms that obey

$$L_i(x_1) = x_i,\ L_i(x_N) = x_{i+1}. \tag{1}$$

Let $F_i : I \times \mathbb{R} \to \mathbb{R}$ be continuous maps such that

$$\left.\begin{array}{l} F_i(x_1, y_1) = y_i,\ F_i(x_N, y_N) = y_{i+1},\ i \in \mathbb{N}_{N-1}, \\ |F_i(x, y) - F_i(x, y')| \le \alpha_i |y - y'|,\ x \in I;\ y, y' \in \mathbb{R};\ 0 \le \alpha_i < 1. \end{array}\right\} \quad (2)$$

For $i \in \mathbb{N}_{N-1}$, define functions $w_i(x, y) = \big(L_i(x), F_i(x, y)\big)$. The collection $\{I \times \mathbb{R}; w_i : i \in \mathbb{N}_{N-1}\}$ is referred as an iterated function system (IFS). The following is the most fundamental theorem in the field of fractal interpolation.

Theorem 2.1. [1] *The IFS $\{I \times \mathbb{R};\ w_i, i \in \mathbb{N}_{N-1}\}$ defined above admits a unique attractor G, and G is the graph of a continuous function $f : I \to \mathbb{R}$ which obeys $f(x_i) = y_i$ for $i \in \mathbb{N}_N$.*

The above function f is called a FIF corresponding to the IFS and its construction is based on the following:
Let $\mathcal{F} = \{g \mid g : I \to \mathbb{R}$ is continuous, $g(x_1) = y_1$ and $g(x_N) = y_N\}$. Then $\mathcal{F}$, endowed with the uniform metric in a complete metric space. The Read-Bajraktarević operator $T : \mathcal{F} \to \mathcal{F}$ is defined by

$$(Tg)(x) = F_i(L_i^{-1}(x), g \circ L_i^{-1}(x)),\ x \in I_i,\ i \in \mathbb{N}_{N-1}.$$

T is a contract in map with contractivity factor $|\alpha|_\infty$=max$\{|\alpha_i| : i = 1,\ 2, \dots, N-1\}$. The fixed point of T is the FIF f whose functional equation is

$$f(x) = F_i(L_i^{-1}(x),\quad f \circ L_i^{-1}(x)) \ \text{ for } x \in I_i. \quad (3)$$

The most extensively studied FIFs so far in the literature stem from the IFS:

$$\left.\begin{array}{l} \{I \times \mathbb{R};\ w_i(x, f) = (L_i(x), F_i(x, f)), i \in \mathbb{N}_{N-1}\}, \\ L_i(x) = a_i x + b_i, \quad F_i(x, f) = \alpha_i f + q_i(x), \end{array}\right\} \quad (4)$$

where a_i is a horizontal scaling factor, α_i is a vertical scaling factor and α=$(\alpha_1, \alpha_2, \dots, \alpha_{N-1})$ is the scaling factor of the IFS. The functions $q_i(x)$, $i \in \mathbb{N}_{N-1}$ are suitable continuous functions such that the conditions prescribed in (2) are satisfied. Bransley[1] introduced generalization of a FIF by IFS (4) with $q_i(x)$=$f \circ L_i(x) - \alpha_i b(x)$, where b is a continuous function satisfy $b(x_1) = f(x_1)$, and $b(x_N) = f(x_N)$. The corresponding FIF is called α-fractal function of f with respect to α, base function b and partition Δ=$\{x_1, \dots, x_N\}$. The corresponding $\alpha-$fractal function $f^\alpha_{\Delta,b} = f^\alpha$ satisfy the functional equation[19]:

$$f^\alpha\big(L_i(x)\big) = \alpha_i f^\alpha(x) + f \circ L_i(x) - \alpha_i b(x), \quad x \in I,\ \ i \in N_{N-1}, \quad (5)$$

Viswanathan and Chand [26], developed α-fractal function f^α with family of base functions $\{b_i : i \in \mathbb{N}_{N-1}\}$ to introduce shape parameters in the structure of FIF as

$$f^\alpha(x) = f(x) + \alpha_i\big(f^\alpha - b_i\big)(L_i^{-1}(x)),\ x \in I_i,\ i \in \mathbb{N}_{N-1}, \quad (6)$$

If in order to accommodate variable scaling and shape parameters in α-FIF, we modify (6) as

$$f^{\boldsymbol{\alpha}}(L_i(x)) = \boldsymbol{\alpha}_i(x) f^{\boldsymbol{\alpha}}(x) + f(L_i(x)) - \boldsymbol{\alpha}_i(x) b_i(x), \quad x \in I,\ i \in \mathbb{N}_{N-1}, \quad (7)$$

where $\boldsymbol{\alpha}_i(x)$ is a Lipschitz function. An upper bound for the uniform error between f and its perturbation fractal function $f^{\boldsymbol{\alpha}}$ is calculated in the following:

$$\begin{aligned}(f^{\boldsymbol{\alpha}} - f)(x) &= \boldsymbol{\alpha}_i(L_i^{-1}(x))\big(f^{\boldsymbol{\alpha}})(L_i^{-1}(x)) - b_i(L_i^{-1}(x))\big),\ x \in I_i,\ i \in \mathbb{N}_{N-1}, \\ &= \boldsymbol{\alpha}_i(L_i^{-1}(x))\big(f^{\boldsymbol{\alpha}} - b_i\big)(L_i^{-1}(x)),\ x \in I_i,\ i \in \mathbb{N}_{N-1}.\end{aligned}$$

From above expression, we have

$$|(f^{\boldsymbol{\alpha}} - f)(x)| \leq ||\boldsymbol{\alpha}_i||_\infty \ \ ||f^{\boldsymbol{\alpha}} - b_i||_\infty,\ x \in I_i,\ i \in \mathbb{N}_{N-1}.$$

Hence, we obtain

$$\begin{aligned} ||f^{\boldsymbol{\alpha}} - f||_\infty &\leq \max\{||\boldsymbol{\alpha}_i||_\infty : i \in \mathbb{N}_{N-1}\} \quad \max\{||f^{\boldsymbol{\alpha}} - b_i||_\infty : i \in \mathbb{N}_{N-1}\}, \\ &\leq \max\{||\boldsymbol{\alpha}_i||_\infty : i \in \mathbb{N}_{N-1}\}\{||f^{\boldsymbol{\alpha}} - f||_\infty + \max\{||f - b_i||_\infty : i \in \mathbb{N}_{N-1}\}. \\ ||f^{\boldsymbol{\alpha}} - f||_\infty &\leq \frac{||\boldsymbol{\alpha}||_\infty}{1 - ||\boldsymbol{\alpha}||_\infty} \max\{||f - b_i||_\infty : i \in \mathbb{N}_{N-1}\}, \end{aligned} \tag{8}$$

where $||\boldsymbol{\alpha}||_\infty = \max\{||\boldsymbol{\alpha}_i||_\infty : i \in \mathbb{N}_{N-1}\}$.

3. Construction of rational cubic FIF with variable scalings

A $\mathcal{C}^1$-continuous rational cubic spline with linear denominator was introduced in [12], where each piece is defined over a subinterval. With a suitable change of variable, we can rewrite this interpolant as

$$f\big(L_i(x)\big) = \frac{(1-\theta)^3 r_i y_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta) W_i + \theta^3 t_i y_{i+1}}{(1-\theta) r_i + \theta t_i},\ i \in \mathbb{N}_{N-1}, \tag{9}$$

where $\theta = \frac{x - x_1}{x_N - x_1}$, $x \in I$, r_i, t_i are non-negative shape parameters and

$$V_i = (2r_i + t_i) y_i + r_i h_i d_i, \quad W_i = (r_i + 2t_i) y_{i+1} - t_i h_i d_{i+1}.$$

The fractal perturbation $f^{\boldsymbol{\alpha}}$ to be $\mathcal{C}^1$- continuous, it suffices to choose the variable scaling $\boldsymbol{\alpha}_i(x)$ such that $||\boldsymbol{\alpha}_i||_{\mathcal{C}^1(I)} < \frac{a_i}{2}$ where $||\boldsymbol{\alpha}||_{\mathcal{C}^1(I)} = \{||\boldsymbol{\alpha}^r||_\infty, r = 0, 1\}$ [25]. In order to construct α-FIF with variable scaling of f, we need the base functions $\{b_i : i \in \mathbb{N}_{N-1}\}$ so that each b_i agrees with f at the extremes of the interpolation interval up to the first derivative [26]. We choose

$$b_i(x) = \frac{B_{1i}(1-\theta)^3 + B_{2i}\theta(1-\theta)^2 + B_{3i}\theta^2(1-\theta) + B_{4i}\theta^3}{(1-\theta) r_i + \theta t_i}, \tag{10}$$

where the coefficients B_{1i}, B_{2i}, B_{3i}, and B_{4i} are prescribed as

$$\begin{aligned} &B_{1i} = r_i y_1,\ \ B_{2i} = (2r_i + t_i) y_1 + r_i d_1 (x_N - x_1), \\ &B_{3i} = (r_i + 2t_i) y_N - t_i d_N (x_N - x_1),\ \ B_{4i} = t_i y_N. \end{aligned}$$

Therefore, in view of (7), (9) and (10), the desired $\mathcal{C}^1$- continuous rational cubic fractal spline with variable scaling is

$$f^{\boldsymbol{\alpha}}\big(L_i(x)\big) = \boldsymbol{\alpha}_i(x) f^{\boldsymbol{\alpha}}(x) + \frac{E_i(x)}{F_i(x)}, \tag{11}$$

$$\begin{aligned} E_i(x) \ &= \ (y_i - \boldsymbol{\alpha}_i(x) y_1) r_i (1-\theta)^3 + (y_{i+1} - \boldsymbol{\alpha}_i(x) y_N) t_i \theta^3 + \big\{(2r_i + t_i) y_i \\ &\quad + r_i h_i d_i - \boldsymbol{\alpha}_i(x)[(2r_i + t_i) y_1 + r_i (x_N - x_1) d_1]\big\}\theta(1-\theta)^2 \\ &\quad + \big\{(r_i + 2t_i) y_{i+1} \\ &\quad - t_i h_i d_{i+1} - \boldsymbol{\alpha}_i(x)[(r_i + 2t_i) y_N - t_i (x_N - x_1) d_N]\big\}\theta^2(1-\theta), \\ F_i(x) \ &= \ (1-\theta) r_i + \theta t_i, \quad \theta = \frac{x - x_1}{x_N - x_1},\ ||\boldsymbol{\alpha}_i||_{\mathcal{C}^1(I)} < \frac{a_i}{2},\ i \in \mathbb{N}_{N-1}. \end{aligned}$$

Remark 3.1. *If shape parameters $r_i = t_i$ for all $i \in \mathbb{N}_{N-1}$, then the α-fractal rational cubic spline reduces to the $\mathcal{C}^1$-cubic Hermite FIF $S_1^{\boldsymbol{\alpha}}$ with variable scaling vectors:*

$$\begin{aligned} S_1^{\boldsymbol{\alpha}}\big(L_i(x)\big) &= \boldsymbol{\alpha}_i(x)S_1^{\boldsymbol{\alpha}}(x) + r_i\{(y_i - \boldsymbol{\alpha}_i(x)y_1)(1-\theta)^3 + (y_{i+1} - \boldsymbol{\alpha}_i(x)y_N)\theta^3 \\ &\quad + \big\{(3y_i + h_i d_i - \boldsymbol{\alpha}_i(x)[3y_1 + (x_N - x_1)d_1]\big\}\theta(1-\theta)^2 \\ &\quad + \big\{3y_{i+1} - t_i h_i d_{i+1} - \boldsymbol{\alpha}_i(x)[3y_N - t_i(x_N - x_1)d_N]\big\}\theta^2(1-\theta)\}, \\ &\quad \theta = \frac{x - x_1}{x_N - x_1},\ ||\boldsymbol{\alpha}_i||_{\mathcal{C}^1(I)} < \frac{a_i}{2},\ i \in \mathbb{N}_{N-1}. \end{aligned}$$

Remark 3.2. *If variable scaling vectors $\boldsymbol{\alpha}_i(x) = 0$ and shape parameters $r_i = t_i$ for all $i \in \mathbb{N}_{N-1}$, the α-fractal rational cubic spline is same as the classical cubic Hermite spline S_2, where*

$$S_2(x) = r_i\{y_i(1-\rho)^3 + y_{i+1}\rho^3 + (3y_i + h_i d_i)\rho(1-\rho)^2 + (3y_{i+1} - h_i d_{i+1})\rho^2(1-\rho)\},$$

where $\rho = \frac{x - x_i}{x_{i+1} - x_i}$, $x \in I_i$, $i \in \mathbb{N}_{N-1}$.

From (11), the derivative of $f^{\boldsymbol{\alpha}}$ satisfies the functional equation

$$(f^{\boldsymbol{\alpha}})'\big(L_i(x)\big) = \frac{\boldsymbol{\alpha}_i(x)(f^{\boldsymbol{\alpha}})'(x)}{a_i} + \frac{\boldsymbol{\alpha}_i'(x)f^{\boldsymbol{\alpha}}(x)}{a_i} - \frac{(x_N - x_1)\boldsymbol{\alpha}_i'(x)A_i(x) - D_i(x)}{h_i(F_i(x))^2}, \tag{12}$$

$$\begin{aligned} A_i(x) &= r_i^2 y_i(1-\theta)^4 + t_i^2 y_{i+1}\theta^4 + \big\{r_i^2[2y_1 + (x_N - x_1)d_1] + 2r_i t_i y_1\big\}\theta(1-\theta)^3 \\ &\quad + \big\{r_i^2 y_N + r_i t_i[2(y_1 + y_N) + (x_1 - x_N)(d_1 - d_N)] + t_i^2 y_1\big\}\theta^2(1-\theta)^2 \\ &\quad + \{2r_i t_i y_N + t_i^2[2y_N - (x_N - x_1)d_N]\}\theta^3(1-\theta), \end{aligned}$$

$$\begin{aligned} D_i(x) &= r_i^2[h_i d_i - \boldsymbol{\alpha}_i(x)(x_N - x_1)d_1](1-\theta)^3 + t_i^2[h_i d_{i+1} - \boldsymbol{\alpha}_i(x)(x_N - x_1)d_N] \\ &\quad \times \theta^3 + \big\{r_i^2\{2(y_{i+1} - y_i) - h_i d_i - \boldsymbol{\alpha}_i(x)[2(y_N - y_1) - (x_N - x_1)d_1]\} \\ &\quad + r_i t_i\{4(y_{i+1} - y_i) - 2h_i d_{i+1} - \boldsymbol{\alpha}_i(x)[4(y_N - y_1) - 2(x_N - x_1)d_N]\}\big\}\theta \\ &\quad \times (1-\theta)^2 + \big\{2r_i t_i\{2(y_{i+1} - y_i) - h_i d_i - \boldsymbol{\alpha}_i(x)[2(y_N - y_1) - (x_N - x_1)d_1]\} \\ &\quad + t_i^2\{2(y_{i+1} - y_i) - h_i d_{i+1} - \boldsymbol{\alpha}_i(x)[2(y_N - y_1) - (x_N - x_1)d_N]\}\big\}\theta^2(1-\theta), \\ F_i(x) &= (1-\theta)r_i + \theta t_i,\ \theta = \frac{x - x_1}{x_N - x_1},\ ||\boldsymbol{\alpha}_i||_{\mathcal{C}^1(I)} < \frac{a_i}{2},\ i \in \mathbb{N}_{N-1}. \end{aligned}$$

Suppose the interpolation data $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ is generated by an unknown function $\Phi \in \mathcal{C}^1(I)$ such that $\Phi'(x)$ is very irregular in nature. Since the rational FIF is not defined explicitly, it is difficult to get an error bound by direct procedures in numerical analysis. But using the corresponding classical rational cubic spline, we can estimate the error bound of f^{α} with Φ that can be used to get its convergence result. Now, we need the following result which proposed in [9].

Proposition 3.1 ([9]). *Let $f(x)$ be the classical rational cubic spline associated with the data $\{(x_i, y_i) : i \in \mathbb{N}_N\}$. Assume that the derivatives at the knots are given or estimated by some linear approximation methods. Then the point-wise error is given by*

$$|\Phi(x) - f(x)| \leq\ h_i\ c_i\|\Phi'\|_\infty,\ x \in I_i,$$

where $c_i = \frac{r_i\rho(1-\rho)^2(1+2\rho)+t_i\rho^2(1-\rho)(3-2\rho)}{r_i(1-\rho)+t_i\rho}$, $\rho = \frac{x-x_i}{x_{i+1}-x_i}$, $x \in I_i$, $i \in \mathbb{N}_{N-1}$.

Theorem 3.1. *Let* $f^{\boldsymbol{\alpha}}$ *be the corresponding rational cubic spline FIF for data* $\{(x_i, y_i) : i \in \mathbb{N}_N\}$*, which is generated from* $\Phi \in \mathcal{C}^1(I)$*. Then,*

$$\begin{aligned}\|\Phi - f^{\boldsymbol{\alpha}}\|_\infty \leq & \frac{||\boldsymbol{\alpha}||_\infty}{1-||\boldsymbol{\alpha}||_\infty}\Big[|y|_\infty + \max\{|y_1|, |y_N|\} + \frac{1}{4}\big(h|d|_\infty + \\ & |I|\max\{|d_1|, |d_N|\}\big)\Big] + ch\|\Phi'\|_\infty,\end{aligned} \tag{13}$$

where $c := \max\{c_i : i \in \mathbb{N}_{N-1}\}, |y|_\infty = \max\{|y_j| : j \in \mathbb{N}_N\}, |d|_\infty = \max\{|d_j| : j \in \mathbb{N}_N\}$, $h = \max\{h_i : i \in \mathbb{N}_{N-1}\}$ *, and* $|I| = x_N - x_1$.

Proof. From (9) and (10), it easy to compute the bounds for $||f||_\infty$ and $||b_i||_\infty$ as

$$\begin{aligned}\|f\|_\infty &\leq |y|_\infty + \frac{h}{4}|d|_\infty, \\ \|b_i\|_\infty &\leq \max\{|y_1|, |y_N|\} + \frac{|I|}{4}\max\{|d_1|, |d_N|\}.\end{aligned}$$

Using these estimates in (8), we have

$$\|f - f^{\boldsymbol{\alpha}}\|_\infty \leq \frac{||\boldsymbol{\alpha}||_\infty}{1-||\boldsymbol{\alpha}||_\infty}\big(|y|_\infty + \frac{h}{4}|d|_\infty + \max\{|y_1|, |y_N|\} + \frac{|I|}{4}\max\{|d_1|, |d_N|\}. \tag{14}$$

Using Proposition 3.1 and (14) in

$$||\Phi - f^{\boldsymbol{\alpha}}||_\infty \leq ||\Phi - f||_\infty + ||f - f^{\alpha}||_\infty,$$

we obtain the desired estimate (13) of Theorem 3.1. □

Convergence result: Since $||\boldsymbol{\alpha}_i||_\infty < \frac{a_i}{2} = \frac{h_i}{2(x_N - x_1)}$, we have $||\boldsymbol{\alpha}||_\infty < \frac{h}{2|I|}$ and hence $\frac{||\boldsymbol{\alpha}||_\infty}{1-||\boldsymbol{\alpha}||_\infty} < \frac{h}{2|I|-h}$. From (13), we get

$$||\Phi - f^{\boldsymbol{\alpha}}||_\infty = O(h) \text{ as } h \to 0.$$

Thus, $f^{\boldsymbol{\alpha}}$ converges uniformly to Φ as the norm of the partition tends to zero.

4. Constrained rational cubic fractal splines

In this section, we deal with the selection of parameters associated in the rational cubic spline FIF such that it satisfies the constrained nature of data with respect to a given piecewise curve. Given data set $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ and a function p (piecewise linear or quadratic with joints at the knots x_i) satisfying $y_i \geq p(x_i)$ (or $y_i \leq p(x_i)$), the problem is to construct a rational cubic spline FIF with variable scaling $f^{\boldsymbol{\alpha}}$ such that $f^{\boldsymbol{\alpha}}(x) \geq p(x)$ (or $f^{\boldsymbol{\alpha}}(x) \leq p(x)$) for all $x \in I$. Since $f^{\boldsymbol{\alpha}}$ in defined implicitly, obtaining conditions for which $f^{\boldsymbol{\alpha}}$ lies above (or below) a piecewise defined function is comparatively difficult than that in the corresponding classical counterpart. But, this can be solved by connecting $f^{\boldsymbol{\alpha}}$ with its classical counterpart $f^{\mathbf{0}} = f$, and performing the analysis on f rather than on $f^{\boldsymbol{\alpha}}$ itself. Let us commence by noting that

$$\begin{aligned}f^{\boldsymbol{\alpha}}(x) - p(x) &= f^{\boldsymbol{\alpha}}(x) - f(x) + f(x) - p(x), \\ &\geq \frac{||\boldsymbol{\alpha}||_\infty}{||\boldsymbol{\alpha}||_\infty - 1}M + f(x) - p(x),\end{aligned}$$

where $M = |y|_\infty + \max\{|y_1|, |y_N|\} + \frac{1}{4}\big(h|d|_\infty + |I| \max\{|d_1|, |d_N|\}\big)$.
It follows that for $f^\alpha(x) - p(x) \geq 0$, it is sufficient to take

$$f(x) \geq p(x) - K, \quad \text{where} \quad K = \frac{||\boldsymbol{\alpha}||_\infty}{||\boldsymbol{\alpha}||_\infty - 1} M.$$

If we replace $L_i(x)$ by x in (9), the classical $\mathcal{C}^1$-rational cubic spline can be written as $f(x) = \frac{R_i(x)}{S_i(x)}$, where

$$\begin{aligned} R_i(x) &= (1-\rho)^3 r_i y_i + \rho(1-\rho)^2[(2r_i + t_i)y_i + r_i h_i d_i] + \rho^2(1-\rho)[(r_i + 2t_i)y_{i+1} \\ &\quad - t_i h_i d_{i+1}] + \rho^3 t_i y_{i+1}, \\ S_i(x) &= (1-\rho)r_i + \rho t_i, \ \rho = \frac{x - x_i}{h_i}. \end{aligned}$$

Case - I Suppose the data set is constrained by a piecewise straight line $p(x)$ with joints at x_i, $i \in \mathbb{N}_N$. If $p_i = p(x_i)$ and $p_{i+1} = p(x_{i+1})$, then we have

$$p(x) = p_i(1-\rho) + p_{i+1}\rho, \ x \in I_i = [x_i, x_{i+1}].$$

Therefore $f(x) \geq p(x) - K$ for all $x \in I$ is satisfied if

$$\begin{aligned} &R_i(x) - [p_i(1-\rho) + p_{i+1}\rho - K]S_i(x) \geq 0, \quad x \in I_i, \ i \in \mathbb{N}_{N-1}. \\ &\Rightarrow R_i(x)[p_i(1-\rho) + p_{i+1}\rho]S_i(x) + KS_i(x) \geq 0, \quad x \in I_i, \ i \in \mathbb{N}_{N-1}. \end{aligned} \tag{15}$$

By using the technique of degree elevation, we are able to write

$$\begin{aligned} &[p_i(1-\rho) + p_{i+1}\rho]S_i(x) \\ &\qquad = [p_i(1-\rho) + p_{i+1}\rho][r_i(1-\rho)^2 + (r_i + t_i)\rho(1-\rho) + t_i\rho^2], \\ &\qquad = r_i p_i (1-\rho)^3 + [r_i(p_i + p_{i+1}) + p_i t_i]\rho(1-\rho)^2 \\ &\qquad\quad + [r_i p_{i+1} + t_i(p_i + p_{i+1})]\rho^2(1-\rho) + \rho^3 t_i p_{i+1}, \\ &S_i(x) \quad = (1-\rho)r_i + \rho t_i \\ &\qquad = r_i(1-\rho)^3 + (2r_i + t_i)(1-\rho)^2\rho + (r_i + 2t_i)\rho^2(1-\rho) + \rho^3 t_i. \end{aligned}$$

Using the above expressions, the condition (15) reduces to

$$\begin{aligned} &r_i(y_i - p_i + K)(1-\rho)^3 + \big[r_i(2y_i + h_i d_i - p_i - p_{i+1} + 2K) \\ &+ t_i(y_i - p_i + K)\big]\rho(1-\rho)^2 \\ &+ \big[r_i(y_{i+1} - p_{i+1} + K) + t_i(2y_{i+1} - h_i d_{i+1} - p_i - p_{i+1} + 2K)\big]\rho^2(1-\rho) \\ &+ t_i(y_{i+1} - p_{i+1} + K)\rho^3 \geq 0, \text{ for all } i \in \mathbb{N}_{N-1}. \end{aligned} \tag{16}$$

Since the canonical basis elements $(1-\rho)^3$, $\rho(1-\rho)^2$, $\rho^2(1-\rho)$, ρ^3 are non-negative, we need to impose conditions on parameters of IFS so that each coefficient of the cubic polynomial appearing in (16) is nonnegative. Given that $y_i \geq p_i$ for all $i \in \mathbb{N}_N$ and $M \geq 0$. Thus, we need

$$y_j - p_j + K \geq 0 \quad \Longleftrightarrow \quad ||\boldsymbol{\alpha}||_\infty \leq \frac{y_j - p_j}{y_j - p_j + M}, \ j = i, i+1,$$

for nonnegativity of the coefficients of $(1-\rho)^3$ and ρ^3. Additionally, we need to choose the shape parameters r_i and t_i such that the coefficients of $\rho(1-\rho)^2$ and $\rho^2(1-\rho)$ are to be nonnegative. The discussion we had until now is summarized in the following theorem.

Theorem 4.1. *Suppose that a data set $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ lies above a piecewise linear function p with joints at knots x_i such that $y_i \geq p_i = p(x_i)$. Then the rational cubic spline FIF $f^{\boldsymbol{\alpha}}$ stays above p, it is sufficient to choose the rational IFS parameters as*

(i) $||\boldsymbol{\alpha}||_{\mathcal{C}^1(I)} < \frac{a_i}{2}$ *for* $i \in \mathbb{N}_{N-1}$, *and* $||\boldsymbol{\alpha}||_\infty \leq \min\big\{\dfrac{y_i - p_i}{y_i - p_i + M} : i \in \mathbb{N}_N\big\}$,

(ii) $r_i(2y_i - p_{i+1} - p_i + h_i d_i + 2K) + t_i(y_i - p_i + K) \geq 0,\ i \in \mathbb{N}_{N-1}$,

(iii) $r_i(y_{i+1} - p_{i+1} + K) + t_i(2y_{i+1} - p_{i+1} - p_i - h_i d_{i+1} + 2K) \geq 0,\ i \in \mathbb{N}_{N-1}$,

where $M = |y|_\infty + \max\{|y_1|, |y_N|\} + \frac{1}{4}\big(h|d|_\infty + |I|\max\{|d_1|, |d_N|\}\big)$ *and* $K = \frac{||\boldsymbol{\alpha}||_\infty}{||\boldsymbol{\alpha}||_\infty - 1}M$.

If the given data set lies below a piecewise linear function $p^*(x)$, then we can proceed in the same manner but with a reverse inequality to get the following result:

Theorem 4.2. *Suppose that a data set $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ lies below a piecewise linear function p^* with joints at knots x_i such that $y_i \leq p^*_i = p^*(x_i)$. Then the rational cubic spline FIF $f^{\boldsymbol{\alpha}}$ stays below p^*, it is sufficient to choose the rational IFS parameters as*

(i) $||\boldsymbol{\alpha}||_{\mathcal{C}^1(I)} < \frac{a_i}{2}$ *for* $i \in \mathbb{N}_{N-1}$, *and* $||\boldsymbol{\alpha}||_\infty \leq \min\big\{\frac{p^*_i - y_i}{p^*_i - y_i + M} : i \in \mathbb{N}_N\big\}$,

(ii) $r_i(p^*_{i+1} + p^*_i - 2y_i - h_i d_i - 2K) + t_i(p^*_i - y_i - K) \geq 0,\ i \in \mathbb{N}_{N-1}$,

(iii) $r_i(p^*_{i+1} - y_{i+1} - K) + t_i(p^*_{i+1} + p^*_i - 2y_{i+1} + h_i d_{i+1} - 2K) \geq 0,\ i \in \mathbb{N}_{N-1}$,

where $M = |y|_\infty + \max\{|y_1|, |y_N|\} + \frac{1}{4}\big(h|d|_\infty + |I|\max\{|d_1|, |d_N|\}\big)$, *and* $K = \frac{||\boldsymbol{\alpha}||_\infty}{||\boldsymbol{\alpha}||_\infty - 1}M$.

Case II Now let us consider μ to be a piecewise quadratic polynomial with joints at x_i, $i \in \mathbb{N}_N$. If $\mu(x_i) = \mu_i$, $\mu'(x_i) = \mu'_i$ and $\rho = \frac{x - x_i}{h_i}$, then we have $\mu(x) = \mu_i(1-\rho)^2 + (2\mu_i + \mu'_i h_i)\rho(1-\rho) + \mu_{i+1}\rho^2,\ x \in [x_i, x_{i+1}]$. Therefore $f(x) \geq \mu(x) - K$ for all $x \in I$ is satisfied if

$$R_i(x) - [\mu_i(1-\rho)^2 + (2\mu_i + \mu'_i h_i)\rho(1-\rho) + \mu_{i+1}\rho^2 - K]S_i(x) \geq 0,\ x \in I_i,\ i \in \mathbb{N}_{N-1}, \tag{17}$$

$$\begin{aligned}
&[\mu_i(1-\rho)^2 + (2\mu_i + \mu'_i h_i)\rho(1-\rho) + \mu_{i+1}\rho^2 - K]S_i(x) \\
&\quad = [\mu_i(1-\rho)^2 + (2\mu_i + \mu'_i h_i)\rho(1-\rho) + \mu_{i+1}\rho^2 - K][1-\rho)r_i + \rho t_i] \\
&\quad = r_i\mu_i(1-\rho)^3 + [r_i(2\mu_i + \mu'_i h_i) + \mu_i t_i]\rho(1-\rho)^2 \\
&\qquad + [r_i\mu_{i+1} + t_i(2\mu_i + \mu'_i h_i)]\rho^2(1-\rho) + \rho^3 t_i\mu_{i+1}.
\end{aligned} \tag{18}$$

Using the above expression and the cubic degree elevation form of $S_i(x)$ in (17), we will get following inequality for a cubic polynomial :

$$\begin{aligned}
r_i(y_i - \mu_i + K)(1-\rho)^3 &+ \big[r_i(2y_i + h_i d_i - 2\mu_i - \mu' h_i + K) + t_i(y_i - \mu_i + K)\big]\rho(1-\rho)^2 \\
&+ \big[r_i(y_{i+1} - \mu_{i+1} + K) + t_i(2y_{i+1} - h_i d_{i+1} - 2\mu_i - \mu'_i h_i + K)\big]\rho^2(1-\rho) \\
&+ t_i(y_{i+1} - \mu_{i+1} + K)\rho^3 \geq 0, \text{ for all } i \in \mathbb{N}_{N-1}.
\end{aligned}$$

Now (17) is true for all values of $x \in I$, when the coefficients of $(1-\rho)^3$, $\rho(1-\rho)^2$, $\rho^2(1-\rho)$, ρ^3 are non-negative in the above expression. This leads to conditions on the parameters of the rational IFS as described in the following:

Theorem 4.3. *Suppose that a data set $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ lies above a a piecewise quadratic function μ such that $y_i \geq \mu_i = \mu(x_i), i \in \mathbb{N}_N$. Then the rational cubic spline FIF $f^{\boldsymbol{\alpha}}$ stays above μ if the IFS parameters satisfy the following inequalities:*

(i) $||\boldsymbol{\alpha}_i||_{\mathcal{C}^1(I)} < \frac{a_i}{2}$ *for* $i \in \mathbb{N}_{N-1}$, *and* $||\boldsymbol{\alpha}||_\infty \leq \min\Big\{\dfrac{y_i - \mu_i}{y_i - \mu_i + M} : i \in \mathbb{N}_N\Big\}$,

(ii) $r_i(2y_i - 2\mu_i + h_i d_i - h_i \mu_i' + 2K) + t_i(y_i - \mu_i + K) \geq 0, \quad i \in \mathbb{N}_{N-1}$,

(iii) $r_i(y_{i+1} - \mu_{i+1} + K) + t_i(2y_{i+1} - 2\mu_i - h_i d_{i+1} - h_i \mu_i' + 2K) \geq 0, \quad i \in \mathbb{N}_{N-1}$,

where $M = |y|_\infty + \max\{|y_1|, |y_N|\} + \frac{1}{4}\big(h|d|_\infty + |I|\max\{|d_1|, |d_N|\}\big)$ *and* $K = \frac{||\boldsymbol{\alpha}||_\infty}{||\boldsymbol{\alpha}||_\infty - 1} M$.

Similarly the following theorem gives the restrictions on rational IFS parameters such that the fractal curve lies below the piecewise quadratic curve.

Theorem 4.4. *Suppose that a data set $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ lies above a a piecewise quadratic function μ^* such that $y_i \leq \mu_i^* = \mu^*(x_i)$. Then the rational cubic spline FIF f^α stays below μ^* if the IFS parameters satisfy the following inequalities:*

(i) $||\boldsymbol{\alpha}_i||_{\mathcal{C}^1(I)} < \frac{a_i}{2}$ *for* $i \in \mathbb{N}_{N-1}$, *and* $||\boldsymbol{\alpha}||_\infty \leq \min\Big\{\frac{\mu_i^* - y_i}{\mu_i^* - y_i + M} : i \in \mathbb{N}_N\Big\}$,

(ii) $r_i(2\mu_i^* - 2y_i - h_i d_i + h_i(\mu_i^*)' - 2K) + t_i(\mu_i^* - y_i - K) \geq 0, \quad i \in \mathbb{N}_{N-1}$,

(iii) $r_i(\mu_{i+1}^* - y_{i+1} - K) + t_i(2\mu_i^* - 2y_{i+1} + h_i d_{i+1} + h_i(\mu_i^*)' - 2K) \geq 0, \quad i \in \mathbb{N}_{N-1}$,

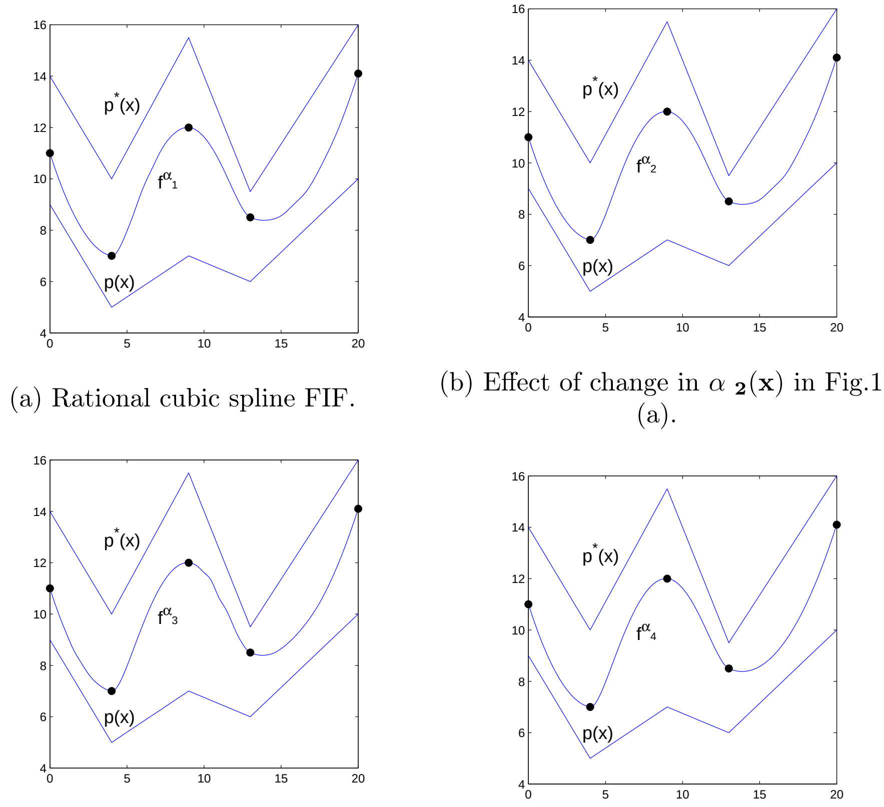
where $M = |y|_\infty + \max\{|y_1|, |y_N|\} + \frac{1}{4}\big(h|d|_\infty + |I|\max\{|d_1|, |d_N|\}\big)$ *and* $K = \frac{||\boldsymbol{\alpha}||_\infty}{||\boldsymbol{\alpha}||_\infty - 1} M$.

5. Numerical Examples

In this section, we implement our rational cubic spline FIFs that are contained by piecewise linear or quadratic curves. Consider the dataset $\{(x_i, y_i, d_i) : i = 1,2,3,4,5\} = \{(0, 11, -1.8889), (4, 7, -0.1111), (9, 12, -0.0417), (13, 8.5, -0.2711), (20, 14, 1.8425)\}$. Note that the prescribed data set lies above the piecewise linear function p with nodes at $\{(x_i, p_i)\}_{i=1}^5 = \{(0, 9), (4, 5), (9, 7), (13, 6), (20, 10\}$ and data set lies below the piecewise linear function p^* with nodes at $\{(x_i, p_i^*)\}_{i=1}^5 = \{(0, 14), (4, 10), (9, 15.5), (13, 9.5), (20, 15)\}$ are given by

$$p(x) := \begin{cases} -x + 9 & \text{if } 0 \leq x \leq 4, \\ \frac{2x+17}{5} & \text{if } 4 \leq x \leq 9, \\ \frac{-x+37}{4} & \text{if } 9 \leq x \leq 13, \\ \frac{4x-10}{7} & \text{if } 13 \leq x \leq 20. \end{cases}, \quad p^*(x) := \begin{cases} 14 - x & \text{if } 0 \leq x \leq 4, \\ \frac{11x+56}{10} & \text{if } 4 \leq x \leq 9, \\ \frac{58-3x}{2} & \text{if } 9 \leq x \leq 13, \\ \frac{13x-36}{14} & \text{if } 13 \leq x \leq 20. \end{cases}$$

respectively. By selecting the variable scaling and shape parameter according to the conditions prescribed in Theorems 4.1-4.2 (See Tables 1-2), a rational cubic spline FIF lying between $p(x)$ and $p^*(x)$ is generated in Fig. 1(a). To illustrate the effects of scaling factors, only the variable scaling $\alpha_2(x)$ is changed with respect to the parameters of Fig.1 (a). The corresponding constrained rational cubic spline FIF between the piecewise lies in plotted in Fig. 1(b). Next, we have taken a different



(a) Rational cubic spline FIF.

(b) Effect of change in $\alpha\ _{\mathbf{2}}(\mathbf{x})$ in Fig.1 (a).

(c) Effect of change in all $\alpha(\mathbf{x})$in Fig.1 (a)

(d) Classical rational cubic spline FIF.

Fig. 1. Rational cubic spline FIFs lying above a piecewise linear functions p and p^*

variable scale vector to generate a constrained rational cubic FIF in Fig. 1(c). Finally, we have retrieved the classical constrained rational cubic FIF in Fig. 1(d) by taking $\alpha_i(x)$ as a zero function. Due to $||\boldsymbol{\alpha}||_{\mathcal{C}^1(I)} < \frac{a_i}{2} < 1$, the difference in the graphs of rational cubic FIFs are not visible. Denoting $f_1^{\boldsymbol{\alpha}}$, $f_2^{\boldsymbol{\alpha}}$, $f_3^{\boldsymbol{\alpha}}$ and $f_4^{\boldsymbol{\alpha}}$ as rational cubic FIFs in Figs. 1(a), 1(b), 1(c) and 1(d) respectively, we have calculated the uniform norms $||f_1^{\boldsymbol{\alpha}} - f_2^{\boldsymbol{\alpha}}|| = 0.0783$, $||f_1^{\boldsymbol{\alpha}} - f_3^{\boldsymbol{\alpha}}||= 0.1412$ and $||f_1^{\boldsymbol{\alpha}} - f_4^{\boldsymbol{\alpha}}||$=0.1426. It is clear that if $f_1^{\boldsymbol{\alpha}}$ is an original function for given data, then $f_2^{\boldsymbol{\alpha}}$ is a better approximant to it in compare with the classical interpolant $f_4^{\boldsymbol{\alpha}}$.

Next, we consider the same interpolation data set lies above the piecewise quadratic function μ with nodes at $\{(x_i, \mu_i)\}_{i=1}^5$=$\{(0, 8), (4, 6.25), (9, 12), (13, 8.5), (20, 11\}$ and interpolation data set lies below the piecewise quadratic function μ^* with nodes at $\{(x_i, \mu_i^*)\}_{i=1}^5$=$\{(0, 12.5), (4, 10.75), (9, 11.3), (13, 11), (20, 15)\}$ defined as follows.

TABLE 1. Variable Scaling factors used in the construction of rational cubic FIFs.

Figures	Variable scaling $\boldsymbol{\alpha}(x)$
Fig.1a, 2a	$\left(\frac{x}{(20)^{2.9}},\ \frac{sin(x)}{(40)^{1.6}},\ -1+e^{(x/(20)^{2.88})},\ \frac{sin(x)}{(28)^{1.534}}\right)$
Fig.1b, 2c	$\left(\frac{x}{(20)^{2.9}},\ \frac{\lvert log(1/(x+2))\rvert}{20^{1.97}},\ -1+e^{(x/(20)^{2.88})},\ \frac{sin(x)}{(28)^{1.534}}\right)$
Fig.1c, 2d,	$\left(\frac{cos(x+2)}{400},\ \frac{sech(x+2.222)}{20},\ \frac{sin(x+1)}{20^{1.59}},\ \frac{1}{x+64.9}\right)$
Fig.1d, 2g	$(0,\ 0,\ 0,\ 0)$

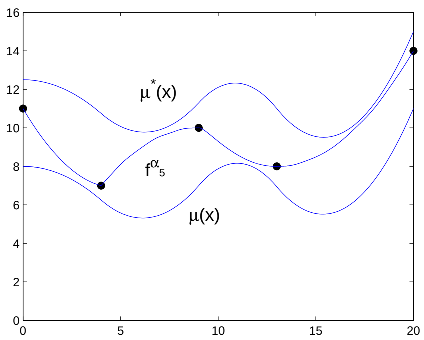
TABLE 2. Shape parameters used in the construction of rational cubic FIFs.

	Shape parameters	
Figures	r	t
Fig.1a-d	100(0.00002, 0.1, 1, 1)	40(0.0001, 1, 1, 2)
Fig. 2a-h	(1, 1, 0.01, 11)	(10, 2000 , 0.1, 8)

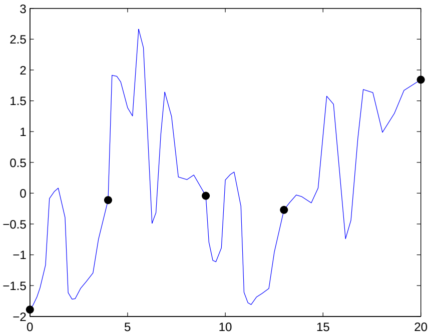
$$\mu(x) := \begin{cases} \frac{-7}{64}x^2 + 8 & \text{if } 0 \le x \le 4, \\ \frac{41}{200}(x-4)^2 - \frac{7}{8}(x-4) + \frac{25}{4} & \text{if } 4 \le x \le 9, \\ \frac{-119}{400}(x-9)^2 + \frac{47}{40}(x-9) + 7 & \text{if } 9 \le x \le 13, \\ \frac{51}{200}(x-13)^2 - \frac{241}{200}(x-13) + \frac{347}{50} & \text{if } 13 \le x \le 20, \end{cases}$$

$$\mu^*(x) := \begin{cases} \frac{-7}{64}x^2 + \frac{25}{2} & \text{if } 0 \le x \le 4, \\ \frac{197}{1000}(x-4)^2 - \frac{7}{8}(x-4) + \frac{43}{4} & \text{if } 4 \le x \le 9, \\ \frac{-117}{400}(x-9)^2 + \frac{219}{200}(x-9) + \frac{113}{10} & \text{if } 9 \le x \le 13, \\ \frac{417}{1607}(x-13)^2 - \frac{249}{200}(x-13) + 11 & \text{if } 13 \le x \le 20. \end{cases}$$

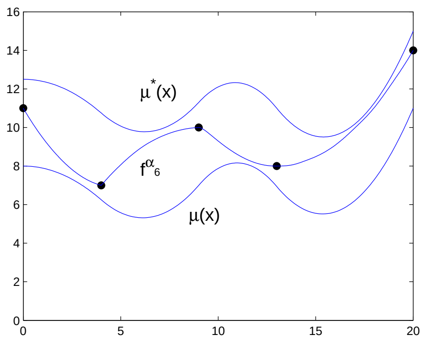
By selecting the variable scaling and shape parameter according to the conditions prescribed in Theorems 4.3 and 4.4 the constrained rational cubic FIFs are constructed in Figs. 2a, 2c, 2e lying below and above the quadratic splines μ and μ^*. The same choice of scaling factors of previous examples are satisfying the requirement in Theorems 4.3-4.4 (See Table 1). But the shape parameters are taken differently as per our requirement (See Table 2). When the scale vector is null, i.e. $\boldsymbol{\alpha}(x)$=(0,0,0,0), we retrieve the classical rational cubic spline plotted in Fig. 2g. The effects of scaling parameters on the proposed rational cubic FIFs can be seen by comparing Fig 1a vs Fig. 2a, Fig. 1b vs Fig. 2c and Fig. 1c vs Fig. 2e. Denoting the constrained



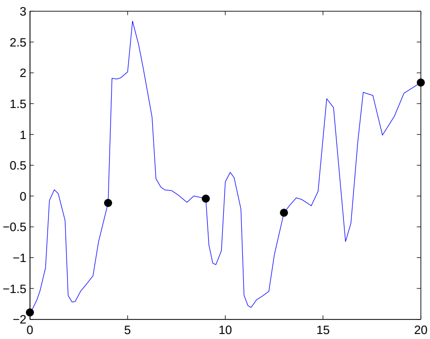
(a) Rational cubic spline FIF.



(b) Derivative of Rational cubic spline FIF.



(c) Effect of change in $\alpha_{\mathbf{2}}(\mathbf{x})$ in Fig.2(a).

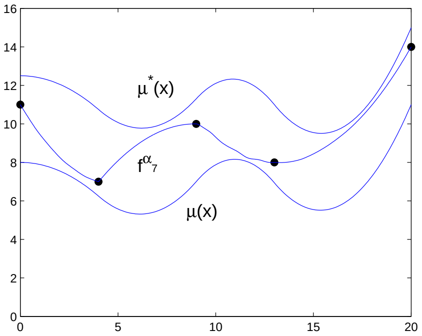


(d) Derivative of Fig.2(c).

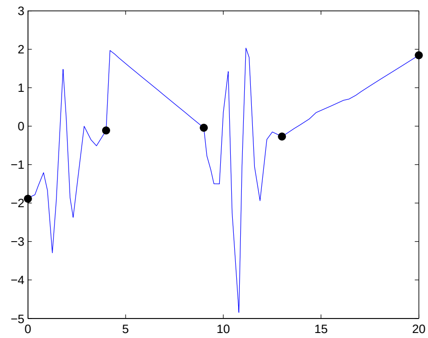
rational cubic FIFs lying below and above the prescribed quadratic splines graphed in Figs. 2a, 2c, 2e, 2g as $f_5^{\boldsymbol{\alpha}}$, $f_6^{\boldsymbol{\alpha}}$, $f_7^{\boldsymbol{\alpha}}$ and $f_8^{\boldsymbol{\alpha}}$ respectively, we have estimated the uniform norms $||f_5^{\boldsymbol{\alpha}} - f_6^{\boldsymbol{\alpha}}|| = 0.0019$, $||f_5^{\boldsymbol{\alpha}} - f_7^{\boldsymbol{\alpha}}||= 0.0509$ and $||f_5^{\boldsymbol{\alpha}} - f_8^{\boldsymbol{\alpha}}||$=0.0183 to show the difference between these fractal curves. In order to notice the difference between the shapes of constrained rational cubic FIFs in fractality, we have plotted the derivative of them in Figs. 2b, 2d, 2f, 2h. Note that the first derivatives of these rational cubic FIFs are typical fractal functions which are very much different in shapes.

6. Conclusions

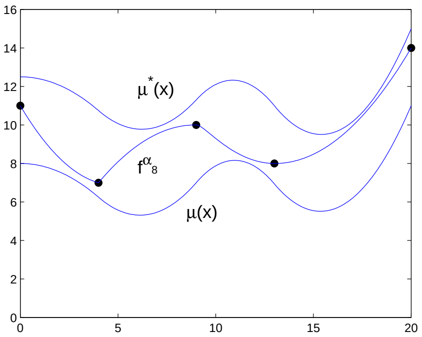
In this paper, we have proposed a novel rational constrained fractal function with variable scaling.The parameters of the FIF are chosen such a way that the graph of fractal interpolant lies above or below associated with rational FIF can a prescribed piecewise linear or quadratic curve. This may find potential applications in various nonlinear and non-equilibrium phenomena where the derivative of the associated variable is irregular in nature.



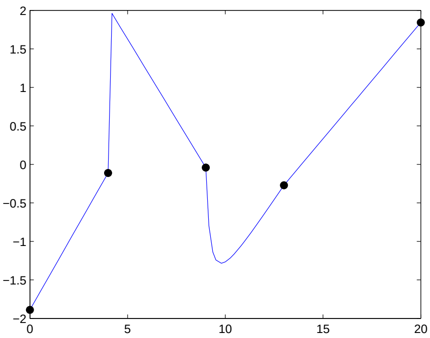
(e) Effect of change in all $\alpha(\mathbf{x})$in Fig.2(a).



(f) Derivative of Fig. 2(e).



(g) Classical rational cubic spline FIF.



(h) Derivative of Classical rational cubic spline FIF.

FIG. 2. Rational cubic spline FIFs lying between piecewise quadratic curves μ and μ^*.

REFERENCES

[1] Barnsley, M. F, *Fractal functions and interpolation*, Constr. Approx., **2**:4 (1986), 303–329. MR0892158

[2] Barnsley, M. F, *Fractals Everywhere*, Academic Press, Orlando, Florida, (1988). MR0977274

[3] Barnsley, M. F, Harrington, A. N, *The calculus of fractal interpolation functions*, J. Approx. Theory, **57**:1 (1989), 14–34. MR0990800

[4] Chand, A. K. B, Kapoor, G. P, *Generalized cubic spline fractal interpolation functions*, SIAM J. Numer. Anal., **44**:2 (2006), 655–676. MR2218964

[5] Brodlie, K. W, Butt, S, *Preserving convexity using piecewise cubic interpolation*, Computers and Graphics , **15**:1 (1991), 15–23.

[6] Chand, A. K. B, Vijender, N, *Monotonicity preserving rational quadratic fractal interpolation functions*, Adv. Num. Anal., (2014), Art ID 504825. MR3166698

[7] Chand, A. K. B, Vijender, N, Navascués, M. A, *Shape preservation of scientific data through rational fractal splines*, Calcolo, **51**:2 (2014), 329–362. MR3210161

[8] Chand, A. K. B, Viswanathan, P, *A constructive approach to cubic Hermite fractal Interpolation function and its constrained aspects*, BIT Numer. Math., **53**:4 (2013), 841–865. MR3129835

[9] Chand, A. K. B, Viswanathan, P, Reddy, K. M, *Towards a more general type of univariate constrained interpolation with fractal splines*, Fractals, **23**:4 (2015), 1550040. MR3433405

[10] Delbourgo, R, Gregory, J. A, C^2*-rational quadratic spline interpolation to monotonic data*, IMA J. Numer. Anal., **3**:2 (1983), 141–152. MR0716458

[11] Delbourgo, R, Gregory, J. A, *Shape preserving piecewise rational interpolation*, SIAM J. Sci. Stat. Comp., **6**:4 (1985), 967–976. MR0801184

[12] Duan Q., Xu G., Liu A. Wang, X. and Cheng F., *Constrained interpolation using rational Cubic spline with linear denominators*, Korean J. Comput. Appl. Math., **6** (1999), 203–215. MR1670844

[13] Farin, G, *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*, Computer Science and Scientific Computing, Academic Press, San Diego, Calif, USA, 1990. MR1058011

[14] Fisher, Y, *Fractal Image Compression*, Springer, New York, NY, USA, 1995. MR1313035

[15] Han, X, *Convexity-preserving piecewise rational quartic interpolation*, SIAM. J. Numer. Anal., **46**:2 (2008), 920–929. MR2383216

[16] Hutchinson, J. E, *Fractals and self similarity*, Indiana Univ. Math. J., **30** (1981), 713–747. MR0625600

[17] Mandelbrot, B, *The Fractal Geometry of Nature*, Freeman, San Francisco, 1982. MR0665254

[18] Navascués, Chand, A. K. B, *Fundamental sets of fractal functions*, Acta Applicandae Mathematicae, **100**:3 (2008), 247–261. MR2369080

[19] Navascués, M. A, *Fractal polynomial interpolation*, Z. Anal. Anwend, **25**:2 (2005), 401–418. MR2174271

[20] Navascués, M. A, Sebasstián , M. V, *Smooth fractal interpolation*, J. Inequal. Appl., (2006), Article ID 78734, 1–20. MR2253425

[21] Sarfraz, M, Al-Mulhem, M, Ashraf, F, *Preserving monotonic shape of the data using piecewise rational cubic functions*, Comput. Graph., **21**:1 (1997), 5–14.

[22] Schimdt, J. W, Heβ, W, *Positivity of cubic polynomials on intervals and positive spline interpolation*, BIT Numer. Math., **28**:2 (1988), 340–352. MR0938398

[23] Schumaker, L. L, *On shape preserving quadratic spline interpolation*, SIAM J. Numer. Anal., **20** (1983), 854–864. MR0708462

[24] Tian, M, *Monotonicity preserving piecewise rational cubic interpolation*, Int. J. Math. Anal., **5** (2011), 99–104. MR2776547

[25] Viswanathan, P, Navascués, M. A, Chand, A. K. B., *Associate fractal functions in* $\mathcal{L}^P$*-spaces and in one-sided uniform approximation*, J. Math. Anal. Appl., **433** (2016), 862–876. MR3398741

[26] Viswanathan, P, Chand, A. K. B, α*-fractal rational splines for constrained Interpolation*, Electron. Trans. Numer. Anal., **41** (2014), 420–442. MR3284416

[27] Wang, H.Y, Fan, Z. L, *Analytical characteristics of fractal interpolation functions with function vertical scaling factors*, Acta Math. Sinica, **54**:1 (2011), 147–158 (in Chinese). MR2830077

[28] Wang, H.Y, Shan, Yu, J, *Fractal interpolation functions with variable parameters and their analytical properties*, J. Approx. Theory, **175** (2013), 1–18. MR3101056

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