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ON FREEDOM AND INDEPENDENCE IN HYPERGRAPHS OF MODELS OF THEORIES

B.SH. KULPESHOV, S.V. SUDOPLATOV

Abstract. The notions of freedom and independence for hypergraphs of models of a theory are introduced. We study properties of these notions in general case and for some natural classes of theories. We describe hypergraphs of models for theories of unary predicates, equivalence relations and study the properties of hypergraphs for ordered theories and for theories of unars.

Keywords: hypergraph of models, elementary theory, free set, independent sets, complete union of hypergraphs.

Hypergraphs of models of a theory are derived objects that enable us to obtain an essential structural information on both theories themselves and related semantical objects including graphs [1, 2, 3, 4, 5, 6, 7, 8, 9].

In the present paper the notions of freedom and independence for hypergraphs of models of a theory are introduced. Properties of these notions and their applications to some natural classes of theories are studied.

1. Preliminaries

Recall that a *hypergraph* is any pair of sets (X, Y) where Y is some subset of the Boolean $\mathcal{P}(X)$ of a set X. The set X is called the *universe* of the hypergraph (X, Y) and elements of Y are called *edges* of the hypergraph (X, Y).

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Let $\mathcal{M}$ be some model of a complete theory T. Following [5] we denote by $H(\mathcal{M})$ the family of all subsets N of the universe M of the structure $\mathcal{M}$ which are universes of elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$: $H(\mathcal{M}) = \{N \mid \mathcal{N} \preccurlyeq \mathcal{M}\}$. The pair $(M, H(\mathcal{M}))$ is called the *hypergraph of elementary submodels* of the model $\mathcal{M}$ and it is denoted by $\mathcal{H}(\mathcal{M})$.

For a cardinality λ we denote by $H_\lambda(\mathcal{M})$ and $\mathcal{H}_\lambda(\mathcal{M})$ respectively the restrictions of $H(\mathcal{M})$ and $\mathcal{H}(\mathcal{M})$ to the class of elementary submodels $\mathcal{N}$ of $\mathcal{M}$ such that $|N| < \lambda$.

By $\mathcal{H}_p(\mathcal{M})$, $\mathcal{H}_l(\mathcal{M})$, $\mathcal{H}_{\rm npl}(\mathcal{M})$, $\mathcal{H}_h(\mathcal{M})$, $\mathcal{H}_s(\mathcal{M})$ we denote the restrictions of the hypergraph $\mathcal{H}_{\omega_1}(\mathcal{M})$ to the class of elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$, that are prime over finite sets, limit, non-prime and non-limit, homogeneous, saturated respectively. Similarly, we denote the correspondent restrictions for $H_{\omega_1}(\mathcal{M})$ by $H_p(\mathcal{M})$, $H_l(\mathcal{M})$, $H_{\rm npl}(\mathcal{M})$, $H_h(\mathcal{M})$, and $H_s(\mathcal{M})$.

Definition [5, 11]. Let (X, Y) be a hypergraph and x_1, x_2 be distinct elements of X. We say that the element x_1 is *separated* or *separable* from the element x_2, or is T_0-*separable* if there is an $y \in Y$ such that $x_1 \in y$ and $x_2 \notin y$. The elements x_1 and x_2 are called *separable*, T_2-*separable*, or *Hausdorff separable* if there are disjoint $y_1, y_2 \in Y$ such that $x_1 \in y_1$ and $x_2 \in y_2$.

Theorem 1.1 [5]. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T and a and b be elements of $\mathcal{M}$. Then the following conditions are equivalent:*

(1) *the element a is separable from the element b in $\mathcal{H}(\mathcal{M})$;*
(2) *the element a is separable from the element b in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) $b \notin \mathrm{acl}(a)$.

Theorem 1.2 [5]. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T and a and b be elements of $\mathcal{M}$. Then the following conditions are equivalent:*

(1) *the elements a and b are separable in $\mathcal{H}(\mathcal{M})$;*
(2) *the elements a and b are separable in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \varnothing$.

Corollary 1.3 [5]. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements of $\mathcal{M}$, and there exists the prime model over a. Then the following conditions are equivalent:*

(1) *the element a is separable from the element b in $\mathcal{H}(\mathcal{M})$;*
(2) *the element a is separable from the element b in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) *the element a is separable from the element b in $\mathcal{H}_p(\mathcal{M})$;*
(4) $b \notin \mathrm{acl}(a)$.

Corollary 1.4 [5]. *Let $\mathcal{M}$ be an ω-saturated model of a countable complete theory T, a and b be elements of $\mathcal{M}$, and there exist prime models over a and b respectively. Then the following conditions are equivalent:*

(1) *the elements a and b are separable in $\mathcal{H}(\mathcal{M})$;*
(2) *the elements a and b are separable in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) *the elements a and b are separable in $\mathcal{H}_p(\mathcal{M})$;*
(4) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \varnothing$.

Definition [5]. Let (X, Y) be a hypergraph and X_1, X_2 be disjoint nonempty subsets of the set X. We say that the set X_1 is *separated* or *separable* from the set X_2, or T_0-*separable* if there is $y \in Y$ such that $X_1 \subseteq y$ and $X_2 \cap y = \varnothing$. The sets X_1 and X_2 are called *separable*, *T_2-separable*, or *Hausdorff separable* if there are disjunct $y_1, y_2 \in Y$ such that $X_1 \subseteq y_1$ and $X_2 \subseteq y_2$.

By using proofs of theorems 1.1 and 1.2, the following generalizations of these theorems are established.

Theorem 1.5 [5]. *Let $\mathcal{M}$ be a λ-saturated model of a complete theory T, $\lambda \geq \max\{|\Sigma(T)|, \omega\}$, A and B be nonempty sets in $\mathcal{M}$ having the cardinalities $< \lambda$. Then the following conditions are equivalent:*
(1) *the set A is separable from the set B in $\mathcal{H}(\mathcal{M})$;*
(2) *the set A is separable from the set B in $\mathcal{H}_\lambda(\mathcal{M})$;*
(3) $\mathrm{acl}(A) \cap B = \varnothing$.

Theorem 1.6 [5] *Let $\mathcal{M}$ be a λ-saturated model of a complete theory T, $\lambda \geq \max\{|\Sigma(T)|, \omega\}$, A u B be nonempty sets in $\mathcal{M}$ having the cardinalities $< \lambda$. Then the following conditions are equivalent:*
(1) *the sets A and B are separable in $\mathcal{H}(\mathcal{M})$;*
(2) *the sets A and B are separable in $\mathcal{H}_\lambda(\mathcal{M})$;*
(3) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

By analogy with corollaries 1.3 and 1.4 we obtain the following statements:
Corollary 1.7 [5]. *Let $\mathcal{M}$ be an ω-saturated model of a small theory T, A and B be finite nonempty sets in $\mathcal{M}$. The following are equivalent:*
(1) *the set A is separable from the set B in $\mathcal{H}(\mathcal{M})$;*
(2) *the set A is separable from the set B in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) *the set A is separable from the set B in $\mathcal{H}_p(\mathcal{M})$;*
(4) $\mathrm{acl}(A) \cap B = \varnothing$.

Corollary 1.8 [5]. *Let $\mathcal{M}$ be an ω-saturated model of a small theory T, A and B be finite nonempty sets in $\mathcal{M}$. Then the following conditions are equivalent:*
(1) *the sets A and B are separable in $\mathcal{H}(\mathcal{M})$;*
(2) *the sets A and B are separable in $\mathcal{H}_{\omega_1}(\mathcal{M})$;*
(3) *the sets A and B are separable in $\mathcal{H}_p(\mathcal{M})$;*
(4) $\mathrm{acl}(A) \cap \mathrm{acl}(B) = \varnothing$.

Definition [8]. Let $\mathcal{M}$ be a model of a theory T with a hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ of elementary submodels, A be an infinite definable set in $\mathcal{M}$, of arity n: $A \subseteq M^n$. The set A is called *$\mathcal{H}$-free* if for any infinite set $A' \subseteq A$, $A' = A \cap Z^n$ for some $Z \in H(\mathcal{M})$ containing parameters for A. Two $\mathcal{H}$-free sets A and B of arities m and n respectively are called *$\mathcal{H}$-independent* if for any infinite $A' \subseteq A$ and $B' \subseteq B$ there is $Z \in H(\mathcal{M})$ containing parameters for A and B and such that $A' = A \cap Z^m$ and $B' = B \cap Z^n$.

Note the following properties [8].

1. Any two tuples of a $\mathcal{H}$-free set A, whose distinct tuples do not have common coordinates, have same type.

Indeed, if there are tuples $\bar{a}, \bar{b} \in A$ with $\mathrm{tp}(\bar{a}) \neq \mathrm{tp}(\bar{b})$ then for some formula $\varphi(\bar{x})$ the sets of solutions of that formula and of the formula $\neg\varphi(\bar{x})$ divide the set A into two nonempty parts A_1 and A_2, where at least one part, say A_1, is infinite. Taking A_1 for A' we have $A_1 = A \cap Z^n$ for appropriate $Z \in H(\mathcal{M})$ and n. Then by the condition for tuples in A we have $A_2 \cap Z^n = \emptyset$ that is impossible since Z is the universe of an elementary submodel of $\mathcal{M}$.

Thus the formula $\varphi(\bar{x})$ defining A implies some complete type in $S^n(\emptyset)$ and if A is $\emptyset$-definable then $\varphi(\bar{x})$ is a principal formula.

In particular, if the set A is $\mathcal{H}$-free and $A \subseteq M$, then the formula defining A implies some complete type in $S^1(\emptyset)$.

2. If $A \subseteq M$ is a $\mathcal{H}$-free set then A does not have nontrivial definable subsets with parameters in A, i.e., subsets distinct from subsets defined by equalities and inequalities with elements in A.

Indeed, if $B \subset A$ is a nontrivial definable subset then B is defined by a tuple $\bar{a}$ of parameters in A, forming a *finite* set $A_0 \subset A$, and B is distinct from subsets of A_0 and from $A \setminus C$, where $C \subseteq A_0$. Then removing from A the elements of the set $B \setminus A_0$ or $(A \setminus B) \setminus A_0$ we obtain some $Z \in H(\mathcal{M})$ violating the satisfiability for B or its complement. It contradicts the condition that Z is the universe of an elementary submodel of $\mathcal{M}$.

3. If A and B are two $\mathcal{H}$-independent sets and $A \cup B$ does not have distinct tuples with common coordinates then $A \cap B = \emptyset$.

Indeed, if $A \cap B$ contains a tuple $\bar{a}$, then, choosing infinite sets $A' \subseteq A$ and $B' \subseteq B$ with $\bar{a} \in A'$ and $\bar{a} \notin B'$, we obtain $\bar{a} \in A' = A \cap Z^n$ for appropriate $Z \in H(\mathcal{M})$ and n, as so $\bar{a} \in B \cap Z^n = B'$. This contradiction means that $A \cap B = \emptyset$.

Definition [6]. The *complete union* of hypergraphs (X_i, Y_i), $i \in I$, is the hypergraph $\left(\bigcup\limits_{i \in I} X_i, Y\right)$, where $Y = \left\{\bigcup\limits_{i \in I} Z_i \mid Z_i \in Y_i\right\}$. If the sets X_i are disjoint, the complete union is called *disjoint* too. If the sets X_i form a $\subseteq$-chain, then the complete union is called *chain*.

By Property 3, we have the following theorem on decomposition of restrictions of hypergraphs $\mathcal{H}$ representable by unions of families of $\mathcal{H}$-independent sets.

Theorem 1.9 [8]. *A restriction of hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ to a union of a family of $\mathcal{H}$-free $\mathcal{H}$-independent sets $A_i \subseteq M$ is represented as a disjoint complete union of restrictions $\mathcal{H}_i$ of the hypergraph $\mathcal{H}$ to the sets A_i.*

Proof. Consider a family of $\mathcal{H}$-independent sets $A_i \subseteq M$. By Property 3, these sets are disjoint, and using the definition of $\mathcal{H}$-independence we immediately obtain that the union of restrictions $\mathcal{H}_i$ of $\mathcal{H}$ to the sets A_i is complete. □

Definition [10]. Let $\mathcal{M}$ be some model of a complete theory T, $(M, H(\mathcal{M}))$ be a hypergraph of elementary submodels of the model $\mathcal{M}$. Sets $N \in H(\mathcal{M})$ are called *elementarily submodel* or *elementarily substructural* in $\mathcal{M}$.

Proposition 1.10 [10]. *Let A be a definable set in an ω_1-saturated model $\mathcal{M}$ of a countable complete theory T. Then exactly one of the following conditions holds:*

(1) *The set A is finite and is contained in any elementarily substructural set in $\mathcal{M}$;*

(2) *The set A is infinite, has infinitely many different intersections with elementarily substructural sets in $\mathcal{M}$, and all these intersections are infinite; and the indicated intersections can be chosen so that to form an infinite chain/antichain by inclusion.*

Proposition 1.11 [10]. *Let A be a definable set in the countable saturated model $\mathcal{M}$ of a small theory T. Then exactly one of the following conditions holds:*

(1) *The set A is finite and is contained in any elementarily substructural set in $\mathcal{M}$;*

(2) *The set A is infinite, has infinitely many different intersections with elementarily substructural sets in $\mathcal{M}$, and all these intersections are infinite; and the indicated intersections can be chosen so that to form an infinite chain / antichain by inclusion.*

Note that the above concepts and statements by a natural manner are transferred to hypergraphs $\mathcal{H}_\lambda(\mathcal{M})$, $\mathcal{H}_p(\mathcal{M})$, $\mathcal{H}_l(\mathcal{M})$, $\mathcal{H}_{\rm npl}(\mathcal{M})$, $\mathcal{H}_h(\mathcal{M})$, and $\mathcal{H}_s(\mathcal{M})$.

Recall that a subset A of a linearly ordered structure $\mathcal{M}$ is called *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$. A *weakly o-minimal structure* is a linearly ordered structure $\mathcal{M} = \langle M, =, <, \ldots\rangle$ such that any definable (with parameters) subset of the structure $\mathcal{M}$ is a union of finitely many convex sets in $\mathcal{M}$.

In the following definitions $\mathcal{M}$ is a weakly o-minimal structure, $A, B \subseteq M$, $\mathcal{M}$ be $|A|^+$-saturated, $p, q \in S_1(A)$ be non-algebraic types.

Definition [14]. We say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there exist an A-definable formula $H(x, y)$, $\alpha \in p(\mathcal{M})$, and $\beta_1, \beta_2 \in q(\mathcal{M})$ such that $\beta_1 \in H(\mathcal{M}, \alpha)$ and $\beta_2 \notin H(\mathcal{M}, \alpha)$.

Definition [15]. We say that p is not *quite orthogonal* to q ($p \not\perp^q q$) if there exists an A-definable bijection $f : p(\mathcal{M}) \to q(\mathcal{M})$. We say that a weakly o-minimal theory is *quite o-minimal* if the notions of weak and quite orthogonality of 1-types coincide.

In the paper [16] the countable spectrum for quite o-minimal theories with non-maximal number of countable models has been described:

Theorem 1.12. *Let T be a quite o-minimal theory with non-maximum many countable models. Then T has exactly $3^k \cdot 6^s$ countable models, where k and s are natural numbers. Moreover, for any $k, s \in \omega$ there exists a quite o-minimal theory T having exactly $3^k \cdot 6^s$ countable models.*

Realizations of these theories with a finitely many countable models are natural generalizations of Ehrenfeucht examples obtained by expansions of dense linear orderings by a countable set of constants, and they are called theories of *Ehrenfeucht type*. Moreover, these realizations are representative examples for hypergraphs of prime models [1, 3, 5]. We consider operators for hypergraphs allowing on one hand to describe the decomposition of hypergraphs of prime models for quite o-minimal theories with few countable models, and on the other hand pointing out constructions leading to the building of required hypergraphs by some simplest ones.

Denote by $(M, H_{\rm dlo}(\mathcal{M}))$ a hypergraph of (prime) elementary submodels of a countable model $\mathcal{M}$ of the theory of dense linear order without endpoints.

Remark 1.13. The class of hypergraphs $(M, H_{\rm dlo}(\mathcal{M}))$ is closed under countable chain complete unions, modulo density and having an encompassing dense linear order without endpoints. Thus, any hypergraph $(M, H_{\rm dlo}(\mathcal{M}))$ is represented as a countable chain complete, modulo density, union of some its proper subhypergraphs.

Any countable model of a theory of Ehrenfeucht type is a disjoint union of some intervals, which are ordered both themselves and between them, and of some singletons. Dense subsets of the intervals form universes of elementary substructures. So, in view of Remark 1.13 we have:

Theorem 1.14 [6]. *A hypergraph of prime models of a countable model of a theory of Ehrenfeucht type is represented as a disjoint complete, modulo density, union of some hypergraphs in the form* $(M, H_{\rm dlo}(\mathcal{M}))$ *as well as singleton hypergraphs of the form* $(\{c\}, \{\{c\}\})$.

Remark 1.15. Taking into consideration links between sets of realizations of 1-types, which are not weakly orthogonal, as well as definable equivalence relations, the construction for the proof of Theorem 1.14 admits a natural generalization for an arbitrary quite o-minimal theory with few countable models. Here conditional complete unions should be additionally *coordinated*, i.e., considering definable bijections between sets of realizations of 1-types, which are not quite orthogonal.

2. On relative freedom and independence in hypergraphs of models of theories

As shown in Section 1, $\mathcal{H}$-free sets does not have non-trivial definable subsets. By this note at studying subsets A' of definable sets $A \subseteq M^n$ in structures $\mathcal{M}$ of a non-empty signature, where $A' = A \cap (M_1)^n$ for some $\mathcal{M}_1 \prec \mathcal{M}$, it is naturally instead of "absolute" $\mathcal{H}$-freedom to consider relative $\mathcal{H}$-freedom taking into account, as for dense linear orders, the specifics of subsets A' by some syntactical information taken from the complete diagram $D^*(\mathcal{M})$ of the system $\mathcal{M}$. In the following section we take into account this specific for ordered theories, and in this section we introduce general notions of relative $\mathcal{H}$-freedom and $\mathcal{H}$-independence, and we also establish links between distinct types of relativity.

Definition. Let $\mathcal{M}$ be some model of a theory T with a hypergraph of elementary submodels $\mathcal{H} = (M, H(\mathcal{M}))$, $D^*(\mathcal{M})$ be the complete diagram of the model $\mathcal{M}$, $\mathbf{D}$ be some set of diagrams $\Phi(A_0) \subseteq D^*(\mathcal{M})$ such that for some language $\Sigma \subseteq \Sigma(\mathcal{M})$ if $\varphi(\bar a)$ is a quantifier-free formula of the language Σ, $\bar a \in A_0$, then $\varphi(\bar a) \in \Phi(A_0)$ or $\neg\varphi(\bar a) \in \Phi(A_0)$. Here, the set A_0 is called the *universe* of the diagram $\Phi(A_0)$. We say that a set $A \subseteq M^n$ *satisfies* a diagram $\Psi \in \mathbf{D}$ if $\Psi = \Phi(A_0)$ for the set A_0 consisting of all the coordinates of tuples from A. The set $A \subseteq M^n$ is called *relatively $\mathcal{H}$-free*, $\mathcal{H}$-free modulo $\mathbf{D}$, or $(\mathcal{H}, \mathbf{D})$*-free* if for any set $A' \subseteq A$ satisfying some diagram of $\mathbf{D}$ the equality $A' = A \cap Z^n$ holds for some $Z \in H(\mathcal{M})$ containing parameters for A. Two $(\mathcal{H}, \mathbf{D})$-free sets A and B of arity m and n respectively are called *relatively $\mathcal{H}$-independent*, $\mathcal{H}$-independent modulo $\mathbf{D}$, or $(\mathcal{H}, \mathbf{D})$*-independent* if for any sets $A' \subseteq A$ and $B' \subseteq B$ satisfying some diagrams of $\mathbf{D}$ there exists $Z \in H(\mathcal{M})$ containing parameters for A and B and such that $A' = A \cap Z^m$ and $B' = B \cap Z^n$.

Note that when defining "absolute" $\mathcal{H}$-freedom and $\mathcal{H}$-independence for the empty language Σ, a set A is definable and infinite and diagrams Φ are taken either quantifier-free on all infinite sets $A' \subseteq A$ or as a result of adding the schemes of infinity of sets for A' to these quantifier-free diagrams.

Unlike definability in the case of type definability or non-definability of a set A under consideration of relative freedom and independence both a scheme of infinity and an infinity itself of sets A' may not be required. Indeed, for a theory T of unary predicates P_i with $P_{i+1} \subset P_i$, $i \in \omega$, the non-isolated type $p(x) = \{P_i(x) \mid i \in \omega\}$ can have the set of realizations of any, finite or infinite, cardinality. Thus, the set

$A = p(\mathcal{M})$, $\mathcal{M} \models T$, non-having non-trivial connections is $(\mathcal{H}, \mathbf{D}_p)$-free for a set of diagrams $\mathbf{D}_p$ describing realizability of the type $p(x)$ by elements of an arbitrary set $A' \subseteq A$.

If the theory T is expanded by unary predicates Q_i with conditions $Q_{i+1} \subset Q_i \subset \overline{P_0}$, $i \in \omega$, then the set $B = q(\mathcal{M})$, where $q(x) = \{Q_i(x) \mid i \in \omega\}$ is free relatively a set of diagrams $\mathbf{D}_q$ describing realizability of the type $q(x)$ by elements of an arbitrary set $B' \subseteq B$, will be $(\mathcal{H}, \mathbf{D}_p \cup \mathbf{D}_q)$-independent.

Since at extending diagrams of $\mathbf{D}$ a family of considered sets can only decrease the following hold:

Monotonicity properties. 1. If $\mathbf{D} \subseteq \mathbf{D}' \subseteq \mathcal{P}(D^*(\mathcal{M}))$ and a set A is $(\mathcal{H}, \mathbf{D}')$-free then A is $(\mathcal{H}, \mathbf{D})$-free.

2. If $\mathbf{D} \subseteq \mathbf{D}' \subseteq \mathcal{P}(D^*(\mathcal{M}))$, sets A and B are $(\mathcal{H}, \mathbf{D}')$-independent then A and B are $(\mathcal{H}, \mathbf{D})$-independent.

3. If diagrams of $\mathbf{D}$ are some restrictions/extensions of suitable diagrams from the set $\mathbf{D}' \subseteq \mathcal{P}(D^*(\mathcal{M}))$, with preservation of their universes, and the set A is $(\mathcal{H}, \mathbf{D})$-free, then A is $(\mathcal{H}, \mathbf{D}')$-free.

4. If diagrams of $\mathbf{D}$ are some restrictions/extensions of suitable diagrams of the set $\mathbf{D}' \subseteq \mathcal{P}(D^*(\mathcal{M}))$, with preservation of their universes, the sets A and B are $(\mathcal{H}, \mathbf{D})$-independent, then A and B are $(\mathcal{H}, \mathbf{D}')$-independent.

Inverse implications in monotonicity properties are not true. Indeed, if an infinite definable set $A \subseteq M$ is partitioned by a unary predicate P into two non-empty parts then A is not $\mathcal{H}$-free, although this set can be considered as $(\mathcal{H}, \mathbf{D}')$-free, for the language $\{P^{(1)}\}$, where $\mathbf{D}'$ consists of diagrams describing cardinalities $|A \cap P|$ and $|A \cap \overline{P}|$. The considered effect, at which two disjoint infinite definable sets A and B are partitioned by the predicate P into non-empty disjoint parts, shows that an independence of the sets A and B can be failed at transition from $(\mathcal{H}, \mathbf{D}')$ to $(\mathcal{H}, \mathbf{D})$.

Further on, for simplicity we will mostly consider the notions of relative freedom and independence for sets $A \subseteq M$, although these considerations can be adapted, for example, by the operation $\mathcal{M}^{\mathrm{eq}}$, for arbitrary sets $A \subseteq M^n$.

In connection with the introduced concepts, a series of natural questions and problems arises.

1. Is an arbitrary set in the given structure free relative to some set of diagrams $\mathbf{D}$?

2. Characterize the condition of $(\mathcal{H}, \mathbf{D})$-freedom of a set.

3. Characterize the condition of $(\mathcal{H}, \mathbf{D})$-independence of sets.

4. Is there a condition on sets of diagrams $\mathbf{D}'$ such that sets A are $(\mathcal{H}, \mathbf{D}')$-free, but not $(\mathcal{H}, \mathbf{D})$-free when $\mathbf{D} \subset \mathbf{D}'$? If yes, what a condition could it be?

5. Is there a condition on sets of diagrams $\mathbf{D}'$ such that sets A and B are $(\mathcal{H}, \mathbf{D}')$-independent, but not $(\mathcal{H}, \mathbf{D})$-independent when $\mathbf{D} \subset \mathbf{D}'$? If yes, what a condition could it be?

One of the ways to answer these questions is the considered below choice for sets A of diagrams $\Phi(A_0)$ with suitable sets A_0. However, this approach does not take into account the structural specificity of sets A and thus not in the full extent it reflects the real freedom of these sets, as well as their independence. This specificity is taken into account in the following sections, for some specific classes of theories.

Proposition 2.1. *For any set $A \subseteq M$ in a model $\mathcal{M}$ of a theory T there exists a set of diagrams $\mathbf{D}$ such that A is $(\mathcal{H}, \mathbf{D})$-free.*

Proof. It suffices to take $\mathbf{D}$ to be an arbitrary set of diagrams $\Phi \subseteq D^*(\mathcal{M})$ whose universes contain the set A. □

Proposition 2.2. *For any sets $A, B \subseteq M$ in a model $\mathcal{M}$ of a theory T there exists a set of diagrams $\mathbf{D}$ such that the sets A and B are $(\mathcal{H}, \mathbf{D})$-independent.*

Proof. It suffices to take $\mathbf{D}$ to be an arbitrary set of diagrams $\Phi \subseteq D^*(\mathcal{M})$ whose universes contain the set $A \cup B$. □

Propositions 1.10 and 1.11 imply the following statement.

Proposition 2.3. *For any set $A \subseteq M$ in an ω_1-saturated (countable saturated) model $\mathcal{M}$ of a countable (small) theory T exactly one of the following conditions holds:*

(1) *A is finite and $(\mathcal{H}, \mathbf{D})$-free only relative to the set of diagrams $\mathbf{D}$ whose universes contain the set A;*

(2) *A is infinite and $(\mathcal{H}, \mathbf{D})$-free relative to some infinite set of diagrams $\mathbf{D}$ whose universes have infinite distinct intersections with A, and these intersections can be chosen so that they form an infinite chain/antichain by inclusion.*

3. On freedom and independence in hypergraphs of models of theories with unary predicates and theories with equivalence relations

In this section we describe decompositions of hypergraphs $\mathcal{H}(\mathcal{M})$ for theories with unary predicates and some theories with equivalence relations.

Firstly we consider the theory T with unary predicates P_i, $i \in I$, and an equivalence relation coinciding with the equality relation. Since all language connections between elements are bounded by the condition of their uniformity, i.e., the coincidence of 1-types, a description of a hypergraph $\mathcal{H}(\mathcal{M})$ is reduced to the description of its restrictions on the sets of realizations of complete 1-types.

Due to the lack of connections between 1-types, based on Proposition 1.10 the following assertion holds for definable sets consisting of realizations of isolated types.

Proposition 3.1. *For any definable set A consisting of realizations in a model $\mathcal{M}$ of some isolated 1-type p and for the restriction $H(\mathcal{M}) \upharpoonright A$ of $H(\mathcal{M})$ to the set A, either $|H(\mathcal{M}) \upharpoonright A| = 1$ when $|A| < \omega$ or $|H(\mathcal{M}) \upharpoonright A| = 2^\lambda$ when $|A| = \lambda \geq \omega$. Here, infinite sets A are $\mathcal{H}$-free and $\mathcal{H}$-independent.*

By Theorem 1.9 and Proposition 3.1 the following holds

Corollary 3.2. *The restriction of a hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ to a union of any family of sets $A_j \subseteq M$, whose each member is the set of realizations of some isolated 1-type p_j, is represented in the form of disjoint complete union $\mathcal{H}_{\mathrm{isol}}$ of restrictions $\mathcal{H}_j$ of the hypergraph $\mathcal{H}$ to sets A_j.*

Now we consider restrictions of a hypergraph $\mathcal{H}(\mathcal{M})$ to the sets B_k of realizations of non-isolated types q_k. Since these types can be both omitted and realized by an arbitrary quantity of realizations, restrictions $\mathcal{H}(\mathcal{M}) \upharpoonright B_k$ are represented in the form of atomic Boolean lattices L_k.

If the considered types q_k can be omitted in aggregate (for example, if the theory has a prime model) then the family of lattices L_k compose their complete union also forming an atomic Boolean lattice:

Proposition 3.3. *Restrictions of a hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ to a union of any family of sets $B_k \subseteq M$, whose each member is the set of realizations of some non-isolated 1-type q_k, where the types q_k can be omitted in aggregate, are represented in the form of disjoint complete union $\mathcal{H}_{n\text{-isol}}$ of restrictions $\mathcal{H}_k$ of the hypergraph $\mathcal{H}$ to sets B_k. This disjoint complete union forms an atomic Boolean lattice.*

If we consider a restriction of a hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ to a union of a family of sets A_j and B_k and the types q_k can be omitted, then there exists a representation of this restriction in the form of a disjoint complete union of restrictions $\mathcal{H}_j$ and $\mathcal{H}_k$, and also in the form of disjoint complete union $\mathcal{H}_{\mathrm{isol}}$ and $\mathcal{H}_{n\text{-isol}}$:

Proposition 3.4. *The restriction of a hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ to a union of any family of sets $A_j \subseteq M$ whose each member is the set of realizations of some isolated 1-type p_j, and also any family of sets $B_k \subseteq M$ whose each member is the set of realizations of some non-isolated 1-type q_k, where the types q_k can be omitted in aggregate, is represented in the form of a disjoint complete union $\mathcal{H}'$ of restrictions $\mathcal{H}_j$ of the hypergraph $\mathcal{H}$ to sets A_j and restrictions $\mathcal{H}_k$ of the hypergraph $\mathcal{H}$ to sets B_k, i.e. in the form of disjoint complete union $\mathcal{H}_{\mathrm{isol}}$ and $\mathcal{H}_{n\text{-isol}}$. This disjoint complete union forms an atomic Boolean lattice modulo $\mathcal{H}_{\mathrm{isol}}$.*

To complete a description of decomposition of a hypergraph $\mathcal{H}(\mathcal{M})$ it remains to consider its restrictions on the sets C_l of realizations of non-isolated types r_l, non-omitted in aggregate. Such families of types arise, for example, in the theory of independent unary predicates having no isolated 1-types. In this case subsets C_l also can be varied arbitrarily, but with the condition of satisfaction of all consistent formulas by elements from A_j, B_k, C_l. This condition will be denoted by C. Thus, a *conditional* complete union or *C-union* of restrictions $\mathcal{H}_l$ of the hypergraph $\mathcal{H}$ on sets C_l arises:

Proposition 3.5. *The restriction of a hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ to a union of any family of sets $A_j \subseteq M$ whose each member is the set of realizations of some isolated 1-type p_j, any family of sets $B_k \subseteq M$ whose each member is the set of realizations of some non-isolated 1-type q_k, where the types q_k can be omitted in aggregate, and also any family of sets $C_l \subseteq M$ whose each member is the set of realizations of some non-isolated 1-type r_l, where the types r_l cannot be omitted in aggregate, is represented in the form of a C-union of restrictions $\mathcal{H}_j$ of the hypergraph $\mathcal{H}$ on sets A_j, restrictions $\mathcal{H}_k$ of the hypergraph $\mathcal{H}$ to sets B_k, and also restrictions $\mathcal{H}_l$ of the hypergraph $\mathcal{H}$ to sets C_l.*

Basing on statements 3.1, 3.2, 3.3, 3.4, and 3.5 we obtain the following theorem which describes the decomposition of a hypergraph $\mathcal{H}(\mathcal{M})$ by means of four types of hypergraphs.

Theorem 3.6. *For any model $\mathcal{M}$ of a theory T of some unary predicates the hypergraph $\mathcal{H}(\mathcal{M}) = (M, H(\mathcal{M}))$ is represented in the form of disjoint complete union of some of the following hypergraphs:*

1) *a hypergraph with the universe* M_0 *consisting of realizations of all algebraic* 1*-types, and having the only edge coinciding with* M_0*;*

2) *a disjoint complete union of* $\mathcal{H}$*-free hypergraphs whose universes consist of realizations of non-algebraic isolated* 1*-types;*

3) *a disjoint complete union of hypergraphs forming atomic Boolean lattices, whose universes consist of realizations of non-isolated* 1*-types omitted in aggregate;*

4) *a* C*-union of hypergraphs whose universes consist of realizations of non-isolated* 1*-types non-omitted in aggregate.*

Example 3.7. The hypergraphs of disjoint unions of the structures listed below can serve as examples including all the types of hypergraphs 1)–4) described in Theorem 3.6:

i) structures consisting of unique non-empty finite unary predicates;
ii) structures consisting of unique non-empty infinite unary predicates;
iii) structures consisting of countably many disjoint non-empty unary predicates;
iv) structures consisting of countably many independent unary predicates.

Now we consider theories T with equivalence relations E_i, $i \in I$.

If the relation E_i is unique then in the theory there is an information on the number and cardinalities of the equivalence classes. If the number of these classes is finite then all of them are presented in any elementary submodel $\mathcal{N}$ of a model $\mathcal{M} \models T$ and a hypergraph $\mathcal{H} = (M, H(\mathcal{M}))$ is represented in the form of disjoint complete union of its restrictions on E_i-classes. The same is related to finite E_i-classes with a finite number for given cardinality n. If the number of such E_i-classes is infinite then in elementary submodels $\mathcal{N}$ of the model $\mathcal{M}$ are included arbitrary families of n-element E_i-classes, i.e., the hypergraph $\mathcal{H}$ is *free* relative to E_i-classes.

Considering infinite E_i-classes we observe that each of them is $\mathcal{H}$-free and distinct E_i-classes are $\mathcal{H}$-independent. If the number of infinite E_i-classes is finite then each of them is presented in models $\mathcal{N}$, and if the number of infinite E_i-classes is infinite then the *$\mathcal{H}$-freedom property* for E_i-classes holds, i.e. any infinite subset of these E_i-classes together with given finite E_i-classes form an elementary submodel with the universe of $H(\mathcal{M})$.

The indicated $\mathcal{H}$-freedom and $\mathcal{H}$-independence is extended to theories with successively embedded equivalence relations under the condition of uniformity for E_i-classes. If in the uniform equivalence classes there exist structures with unary predicates then at representation of a hypergraph $\mathcal{H}$ in the form of disjoint complete union of restrictions on these classes also provides a decomposition described in Theorem 3.6.

In general case the problem of describing hypergraphs for theories with equivalence relations remains open.

4. On freedom and independence in hypergraphs of models of ordered theories

Note that if we consider an arbitrary non-algebraic isolated type $p \in S_1(\emptyset)$ in an arbitrary almost ω-categorical quite o-minimal theory T then in any model $\mathcal{M} \models T$ the set $p(\mathcal{M})$ will not be $\mathcal{H}$-free, since if we take as A' some closed interval $[a, b] \subset p(\mathcal{M})$, where $a < b$, then there is no $\mathcal{M}_1 \prec \mathcal{M}$ such that $A' = p(\mathcal{M}) \cap M_1$.

Another reason for the violation of $\mathcal{H}$-freedom is the possibility of taking an infinite set $A' \subset p(\mathcal{M})$ that is not dense, while the sets $p(\mathcal{M}) \cap M_1$ for models of the theory T must be dense.

An arbitrary open interval containing an element b is said to be a *neighbourhood* of the element b. Recall that an arbitrary subset A of a linearly ordered structure $\mathcal{M}$ is *open* if for any $b \in A$ there is a neighbourhood of the element b that is contained in A.

Definition. Let T be a weakly o-minimal theory, $\mathcal{M} \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type. We say that $p(\mathcal{M})$ is *relatively $\mathcal{H}$-free*, *$\mathcal{H}$-free relative to convex sets*, or $(\mathcal{H}, \mathrm{cs})$*-free* if for any open convex set $A' \subseteq p(\mathcal{M})$ the equality $A' = p(\mathcal{M}) \cap M_1$ holds for some $M_1 \in H(\mathcal{M})$.

We note that, in addition to the fact that $\mathcal{H}$ hypergraphs allow to select all infinite subsets of $\mathcal{H}$-free sets, the corresponding hypergraphs for $(\mathcal{H}, \mathrm{cs})$-free sets in addition to convex sets without endpoints make it possible to isolate dense sets without endpoints. For example, for a theory of dense linear order without endpoints any dense subset without endpoints is distinguished by such way.

We also introduce the following necessary definitions.

Definition [1, 21]. Let $p_1(x_1), \ldots, p_n(x_n) \in S_1(T)$. A type $q(x_1, \ldots, x_n) \in S(T)$ is called $(p_1, \ldots, p_n)$*-type* if $q(x_1, \ldots, x_n) \supseteq \bigcup_{i=1}^{n} p_i(x_i)$. The set of all $(p_1, \ldots, p_n)$-types of the theory T is denoted by $S_{p_1,\ldots,p_n}(T)$. A countable theory T is called *almost ω-categorical* if for any types $p_1(x_1), \ldots, p_n(x_n) \in S(T)$ there is only finitely many types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

Definition [22]. Let $\mathcal{M}$ be a weakly o-minimal structure, $A \subseteq M$, $\mathcal{M}$ be $|A|^+$-saturated, $p \in S_1(A)$ be a non-algebraic type.

(1) An A-definable formula $F(x, y)$ is *p-preserving* or *p-stable* if there are α, γ_1, $\gamma_2 \in p(\mathcal{M})$ such that $F(\mathcal{M}, \alpha) \setminus \{\alpha\} \neq \emptyset$ и $\gamma_1 < F(\mathcal{M}, \alpha) < \gamma_2$.

(2) A p-preserving formula $F(x, y)$ is *convex to right (left)* if there is $\alpha \in p(\mathcal{M})$ such that $F(\mathcal{M}, \alpha)$ is convex, α is the left (right) endpoint of the set $F(\mathcal{M}, \alpha)$ and $\alpha \in F(\mathcal{M}, \alpha)$.

Definition [23]. We say that a p-preserving convex to right (left) formula $F(x, y)$ is *equivalence-generating* if for any $\alpha, \beta \in p(\mathcal{M})$ such that $\mathcal{M} \models F(\beta, \alpha)$, the following holds:

$$\mathcal{M} \models \forall x[x \geq \beta \to [F(x, \alpha) \leftrightarrow F(x, \beta)]] \quad (\mathcal{M} \models \forall x[x \leq \beta \to [F(x, \alpha) \leftrightarrow F(x, \beta)]])$$

Definition [24]. Let T be a weakly o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model of T, and let $\phi(x)$ be an arbitrary M-definable formula with one free variable. The *convexity rank of the formula* $\phi(x)$ $(RC(\phi(x)))$ is defined as follows:

1) $RC(\phi(x)) \geq 1$ if $\phi(\mathcal{M})$ is infinite.

2) $RC(\phi(x)) \geq \alpha + 1$ if there are a parametrically definable equivalence relation $E(x, y)$ and infinitely many elements $b_i, i \in \omega$, such that:

- For any $i, j \in \omega$ whenever $i \neq j$ we have $\mathcal{M} \models \neg E(b_i, b_j)$;
- For each $i \in \omega$ $RC(E(x, b_i)) \geq \alpha$ and $E(\mathcal{M}, b_i)$ is a convex subset of $\phi(\mathcal{M})$.

3) $RC(\phi(x)) \geq \delta$ if $RC(\phi(x)) \geq \alpha$ for all $\alpha < \delta$ (δ is limit).

If $RC(\phi(x)) = \alpha$ for some α then we say that $RC(\phi(x))$ is defined. Otherwise (i.e. if $RC(\phi(x)) \geq \alpha$ for all α), we put $RC(\phi(x)) = \infty$.

The *convexity rank of an 1-type* p ($RC(p)$) is called the infimum of the set $\{RC(\phi(x)) \mid \phi(x) \in p\}$, i.e. $RC(p) := \inf\{RC(\phi(x)) \mid \phi(x) \in p\}$.

Lemma 4.1. *Let T be an almost ω–categorical quite o-minimal theory, $\mathcal{M} \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type. Then $p(\mathcal{M})$ is relatively $\mathcal{H}$-free $\Leftrightarrow RC(p) = 1$.*

Proof. ($\Rightarrow$) Let $p(\mathcal{M})$ is relatively $\mathcal{H}$-free. Assume the contrary: $RC(p) > 1$. By the binarity of T there is an $\emptyset$-definable equivalence relation $E(x, y)$ partitioning $p(\mathcal{M})$ into infinitely many infinite convex sets. Take an arbitrary $a \in p(\mathcal{M})$ and consider $E(a,\mathcal{M})$. Obviously, $E(a,\mathcal{M})$ is open convex set, and there is no an elementary submodel $\mathcal{M}_1$ of $\mathcal{M}$ such that $E(a,\mathcal{M}) = p(\mathcal{M}) \cap M_1$.

($\Leftarrow$) Let $RC(p) = 1$. We argue to show that $p(\mathcal{M})$ is indiscernible over $\emptyset$. By binarity of T it is sufficiently to prove that $p(\mathcal{M})$ is 2-indiscernible over $\emptyset$. Assume the contrary: there are $\langle a_1, a_2\rangle$, $\langle a'_1, a'_2\rangle \in [p(\mathcal{M})]^2$ such that $a_1 < a_2$, $a'_1 < a'_2$ and $tp(\langle a_1, a_2\rangle/\emptyset) \neq tp(\langle a'_1, a'_2\rangle/\emptyset)$. Then there exists $a''_2 \in p(\mathcal{M})$ such that $a_1 < a''_2$ and $tp(\langle a_1, a_2\rangle/\emptyset) \neq tp(\langle a_1, a''_2\rangle/\emptyset)$. Consequently, there is an $\emptyset$-definable formula $\phi(x, y)$ such that $\mathcal{M} \models \phi(a_1, a_2) \wedge \neg\phi(a_1, a''_2)$. By weak o-minimality we can assume that $\phi(a_1,\mathcal{M})$ is convex. Without loss of generality, we will also assume that $a_2 < a''_2$. Then consider the following formula:

$$F(x, a_1) := x \geq a_1 \wedge \exists y[\phi(a_1, y) \wedge x \leq y].$$

It is easy to see that $F(x, y)$ is a p-preserving convex to right. If $F(x, y)$ is equivalence-generating, we have a contradiction with $RC(p) = 1$. If $F(x, y)$ is not equivalence-generating, it contradicts to the almost ω-categoricity of T. Thus, $p(\mathcal{M})$ is indiscernible over $\emptyset$, whence for any open convex set $A' \subseteq p(\mathcal{M})$ there is an elementary submodel $\mathcal{M}_1$ of $\mathcal{M}$ such that $A' = p(\mathcal{M}) \cap M_1$. □

Example 4.2. Let $\mathcal{M} = \langle \mathbb{Q}, <, f^1\rangle$ be a linearly ordered structure, where $\mathbb{Q}$ is the set of rational numbers, $f(x) = x + 1$ is an unary function on $\mathbb{Q}$.

It is easily seen that $\mathcal{M}$ is an o-minimal structure, and $Th(\mathcal{M})$ is not almost ω-categorical. Note also that $p(x) := \{x = x\} \in S_1(\emptyset)$ is a non-algebraic isolated type, $RC(p) = 1$, but $p(\mathcal{M})$ is not relatively $\mathcal{H}$-free.

Definition. Let T be a weakly o-minimal theory, $\mathcal{M} \models T$, $p, q \in S_1(\emptyset)$ be non-algebraic isolated types, $RC(p) = RC(q) = 1$. We say that $p(\mathcal{M})$ and $q(\mathcal{M})$ are *relatively $\mathcal{H}$-independent*, *$\mathcal{H}$-independent with regard to convex sets*, or ($\mathcal{H}$, cs)-*independent* if for any open convex sets $A' \subseteq p(\mathcal{M})$ and $B' \subseteq q(\mathcal{M})$ there is $M_1 \in H(\mathcal{M})$ such that $A' = p(\mathcal{M}) \cap M_1$ and $B' = q(\mathcal{M}) \cap M_1$.

Let $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$ be non-algebraic types. We say that the family of types $\{p_1, \ldots, p_s\}$ is *orthogonal over* $\emptyset$ if for any sequence $(n_1, \ldots, n_s) \in \omega^s$, for any increasing tuples $\bar{a}_1, \bar{a}'_1 \in [p_1(\mathcal{M})]^{n_1}$, …, $\bar{a}_s, \bar{a}'_s \in [p_s(\mathcal{M})]^{n_s}$ such that $tp(\bar{a}_1/\emptyset) = tp(\bar{a}'_1/\emptyset)$, …, $tp(\bar{a}_s/\emptyset) = tp(\bar{a}'_s/\emptyset)$ we have $tp(\langle\bar{a}_1, \ldots, \bar{a}_s\rangle/\emptyset) = tp(\langle\bar{a}'_1, \ldots, \bar{a}'_s\rangle/\emptyset)$.

Lemma 4.3. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $p, q \in S_1(\emptyset)$ be non-algebraic isolated types, $RC(p) = RC(q) = 1$. Then $p(\mathcal{M})$ and $q(\mathcal{M})$ are relatively $\mathcal{H}$-independent $\Leftrightarrow p \perp^w q$.*

Proof. ($\Rightarrow$) Let $p(\mathcal{M})$ and $q(\mathcal{M})$ are relatively $\mathcal{H}$-independent. Assume the contrary: $p \not\perp^w q$. By quite o-minimality there is an $\emptyset$-definable bijection $f : p(\mathcal{M}) \to q(\mathcal{M})$.

Since $RC(p) = RC(q) = 1$, this bijection is strictly monotonic. Take an arbitrary open convex set $A' \subseteq p(\mathcal{M})$ and consider $f(A')$. By the strict monotonicity of f the image $f(A')$ is also an open convex set. Take arbitrary $a, b \in f(A')$ with the condition $a < b$. Then let $B' := \{c \in q(\mathcal{M}) \mid a < c < b\}$. Then there is no $\mathcal{M}_1 \prec \mathcal{M}$ such that $A' = p(\mathcal{M}) \cap M_1$ and $B' = q(\mathcal{M}) \cap M_1$.

($\Leftarrow$) Let $p \perp^w q$. Then by almost ω-categoricity of T the family $\{p, q\}$ is orthogonal over $\emptyset$, whence $p(\mathcal{M})$ and $q(\mathcal{M})$ are relatively $\mathcal{H}$-independent. □

Corollary 4.4. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type, $RC(p) = n$, where $n > 1$. Suppose that $E_1(x, y)$, $E_2(x, y)$, ..., $E_{n-1}(x, y)$ are $\emptyset$-definable equivalence relations partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes so that $E_1(a, \mathcal{M}) \subset E_2(a, \mathcal{M}) \subset \ldots \subset E_{n-1}(a, \mathcal{M})$ for any $a \in p(\mathcal{M})$. Then the following holds:*

1) *Every E_1-class is relatively $\mathcal{H}$-free;*
2) *Any two E_1-classes are relatively $\mathcal{H}$-independent;*
3) *For any $2 \le i \le n - 1$ every E_i-class is not relatively $\mathcal{H}$-free.*

Example 4.5. Let $\mathcal{M} = \langle \mathbb{Q} \times \mathbb{Q}; <, E^2, f^1 \rangle$ be a linearly ordered structure, where $\mathbb{Q} \times \mathbb{Q}$ is lexicographically ordered. The symbol E is interpreted by a binary relation defined as follows: $E(a, b) \Leftrightarrow n_1 = n_2$ for any $a = (n_1, m_1), b = (n_2, m_2) \in \mathbb{Q} \times \mathbb{Q}$. The symbol f is interpreted by a unary function defined by the equality $f((n, m)) = (n + 1, m)$ for all $(n, m) \in \mathbb{Q} \times \mathbb{Q}$.

Obviously, $E(x, y)$ is an equivalence relation partitioning M into infinitely many infinite convex classes.

It can be established that $Th(\mathcal{M})$ is a quite o-minimal theory, and it is not almost ω-categorical. Note that $p(x) := \{x = x\} \in S_1(\emptyset)$ is a non-algebraic isolated type, $RC(p) = 2$, every E-class is relatively $\mathcal{H}$-free, however $E(a, \mathcal{M})$ and $E(f(a), \mathcal{M})$ are not relatively $\mathcal{H}$-independent for each $a \in M$.

Definition. Let T be a weakly o-minimal theory, $\mathcal{M} \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type. Let $E(x, y)$ be an $\emptyset$-definable equivalence relation partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes. If $A \subseteq p(\mathcal{M})$ then we denote by A/E the set of representatives of E-classes having a nonempty intersection with A. We say that $p(\mathcal{M})$ is *relatively $(\mathcal{H}, E)$-free* if for any convex $A' \subseteq p(\mathcal{M})$ such that A'/E is an open set, the equality $A' = p(\mathcal{M}) \cap M_1$ holds for some $M_1 \in H(\mathcal{M})$.

Note that in the latter definition in case of dense ordering of $p(\mathcal{M})$ the convexity of A' is essential. Indeed, let $p(x) := \{U(x)\}$, $a_1, a_2 \in p(\mathcal{M})$ such that $\mathcal{M} \models E(a_1, a_2) \wedge a_1 < a_2$. Consider the following formula:

$$\phi(x, a_1, a_2) := U(x) \wedge [x \le a_1 \vee x \ge a_2].$$

Let $A' = \phi(\mathcal{M}, a_1, a_2)$. Obviously, $A' \subseteq p(\mathcal{M})$, A' is not convex, A'/E is open convex set, but there is no $\mathcal{M}_1 \prec \mathcal{M}$ such that $A' = p(\mathcal{M}) \cap M_1$.

Proposition 4.6. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $p \in S_1(\emptyset)$ be a non-algebraic isolated type, $E(x, y)$ be an $\emptyset$-definable equivalence relation partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes. Then $p(\mathcal{M})$ is relatively $(\mathcal{H}, E)$-free $\Leftrightarrow$ for any $\emptyset$-definable equivalence relation $E'(x, y)$ partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes we have $E'(a, \mathcal{M}) \subseteq E(a, \mathcal{M})$ for some $a \in p(\mathcal{M})$.*

Proof. ($\Rightarrow$) Let $p(\mathcal{M})$ be relatively $(\mathcal{H}, E)$-free. By almost ω-categoricity of T $RC(p) < \omega$, i.e. there is an $\emptyset$-definable equivalence relation $E^*(x, y)$ partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes, and for any $\emptyset$-definable equivalence relation $E'(x, y)$ partitioning $p(\mathcal{M})$ into infinitely many convex classes we have $E'(a, \mathcal{M}) \subseteq E^*(a, \mathcal{M})$ for some $a \in p(\mathcal{M})$. Then we assert that $E(x, y) \equiv E^*(x, y)$ on $p(\mathcal{M})$. If this is not true then either $E(a, \mathcal{M}) \subset E^*(a, \mathcal{M})$ or $E^*(a, \mathcal{M}) \subset E(a, \mathcal{M})$. In the first case we have a contradiction with relative $(\mathcal{H}, E)$-freedom of $p(\mathcal{M})$. In the second case we have a contradiction with that $E^*(x, y)$ is the greatest.

($\Leftarrow$) Let $E(x, y)$ be the greatest $\emptyset$-definable equivalence relation partitioning $p(\mathcal{M})$ into infinitely many convex classes. Then if as A' we take an arbitrary convex subset of $p(\mathcal{M})$ so that A'/E is open then we easily find $\mathcal{M}_1 \prec \mathcal{M}$ with the condition $A' = p(\mathcal{M}) \cap M_1$. □

Example 4.7. Let $\mathcal{M} = \langle M, <, E_i^2 \rangle_{i\in\omega}$ be a linearly ordered structure, where for each $i \in \omega$ $E_i(x, y)$ defines an equivalence relation partitioning M into infinitely many convex classes, and E_i partitions every E_{i+1}-class into infinitely many E_i-classes, every E_i-class is convex and open so that E_i-subclasses of each E_{i+1}-class are densely ordered without endpoints.

It can be established that $Th(\mathcal{M})$ is a quite o-minimal theory non-being almost ω-categorical. Obviously, M is 1-indiscernible, i.e. $p(x) := \{x = x\} \in S_1(\emptyset)$. It is not difficult to see that $p(\mathcal{M})$ is relatively $(\mathcal{H}, E_i)$-free for any $i \in \omega$.

Definition. Let T be a weakly o-minimal theory, $\mathcal{M} \models T$, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic isolated types. Let $E_1(x, y), E_2(x, y)$ be $\emptyset$-definable equivalence relations partitioning $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ respectively into infinitely many convex classes. Suppose that $p_1(\mathcal{M})$ is relatively $(\mathcal{H}, E_1)$-free and $p_2(\mathcal{M})$ is relatively $(\mathcal{H}, E_2)$-free. We say that $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ are *relatively* $(\mathcal{H}, E_1, E_2)$*-independent* if for any convex $A' \subseteq p_1(\mathcal{M})$ and $B' \subseteq p_2(\mathcal{M})$ such that A'/E_1 and B'/E_2 are open sets there is $M_1 \in H(\mathcal{M})$ such that $A' = p_1(\mathcal{M}) \cap M_1$ and $B' = p_2(\mathcal{M}) \cap M_1$.

Proposition 4.8. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic isolated types. Let $E_1(x, y)$, $E_2(x, y)$ be $\emptyset$-definable equivalence relations partitioning $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ respectively into infinitely many convex classes. Suppose that $p_1(\mathcal{M})$ is relatively $(\mathcal{H}, E_1)$-free, $p_2(\mathcal{M})$ is relatively $(\mathcal{H}, E_2)$-free. Then $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ are relatively $(\mathcal{H}, E_1, E_2)$-independent $\Leftrightarrow$ $p_1 \perp^w p_2$.*

Proof. Let $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ be relatively $(\mathcal{H}, E_1, E_2)$-independent. Assume the contrary: $p_1 \not\perp^w p_2$. By quite o-minimality there is an $\emptyset$-definable bijection $f : p_1(\mathcal{M}) \to p_2(\mathcal{M})$, whence $RC(p_1) = RC(p_2)$ and $f(E_1(a, \mathcal{M})) = E_2(f(a), \mathcal{M})$ for any $a \in p_1(\mathcal{M})$. Take an arbitrary convex set $A' \subseteq p_1(\mathcal{M})$ with open A'/E_1 and consider $f(A')$. Obviously, $f(A')$ is convex and $f(A')/E_2$ is open. Take arbitrary E_2-classes $C = E_2(a, \mathcal{M})$ and $D = E_2(b, \mathcal{M})$ for some $a, b \in p_2(\mathcal{M})$ with the condition $C < D$ lying in $f(A')$. Then let $B' := \{e \in p_2(\mathcal{M}) \mid C < e < D\}$. Obviously, B' will be also convex, and B'/E_2 will be open. It is easily to see that there is no $\mathcal{M}_1 \prec \mathcal{M}$ such that $A' = p_1(\mathcal{M}) \cap M_1$ and $B' = p_2(\mathcal{M}) \cap M_1$. □

Corollary 4.9. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M} \models T$, $p_1, p_2 \in S_1(\emptyset)$ be non-algebraic isolated types, and suppose that there exists an $\emptyset$-definable bijection $f : p_1(\mathcal{M}) \to p_2(\mathcal{M})$. Let $E_1(x, y)$ be an $\emptyset$-definable equivalence* *relation partitioning $p_1(\mathcal{M})$ into infinitely many convex classes. Define on the set $p_2(\mathcal{M})$ the relation $E_2(x,y)$ as follows:*

$$\text{for any } a,b\in p_2(\mathcal{M}) \quad E_2(a,b)\Leftrightarrow E_1(f^{-1}(a),f^{-1}(b)).$$

Then $p_1(\mathcal{M})$ is relatively $(\mathcal{H},E_1)$-free $\Leftrightarrow$ $p_2(\mathcal{M})$ is relatively $(\mathcal{H},E_2)$-free.

Further we extend definitions of relative $\mathcal{H}$-freedom, relative $\mathcal{H}$-independence, relative $(\mathcal{H},E)$-freedom, and relative $(\mathcal{H},E_1,E_2)$-independence on non-isolated 1-types.

Recall that if A is an arbitrary subset of a linearly ordered structure $\mathcal{M}$ then we denote by A^+ (and respectively by A^-) the sets of elements b of the considered structure with the condition $A<b$ ($b<A$).

Definition [14]. Let $\mathcal{M}$ be a weakly o-minimal structure, $A\subseteq M$, $p\in S_1(A)$ be a non-algebraic type. We say that p is *quasirational to right (left)* if there is an A-definable convex formula $U_p(x)\in p$ such that for any sufficiently saturated model $\mathcal{N}\succ\mathcal{M}$, $U_p(\mathcal{N})^+=p(\mathcal{N})^+$ ($U_p(\mathcal{N})^-=p(\mathcal{N})^-$). A non-isolated 1-type is called *quasirational* if it is either quasirational to right or quasirational to left. A non-quasirational non-isolated 1-type is called *irrational*.

Obviously, an 1-type being simultaneously quasirational to right and quasirational to left is isolated.

We say that a convex set A is *open to right (left)* if there is $a\in A$ such that for any $b>a$ ($b<a$) there exists a neighbourhood of the element b containing in A. Obviously, a set being simultaneously open to right and open to left is open.

Definition. Let T be a weakly o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model for T, $p\in S_1(\emptyset)$ be a non-isolated type, $RC(p)=1$. If p is quasirational to right (left) then we say $p(\mathcal{M})$ is *relatively $\mathcal{H}$-free* if for any open to right (left) convex $A'\subseteq p(\mathcal{M})$ the equality $A'=p(\mathcal{M})\cap M_1$ holds for some $M_1\in H(\mathcal{M})$. If p is irrational then it is sufficiently to take an arbitrary convex set as A'.

Lemma 4.10. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model for T, $p\in S_1(\emptyset)$ be a non-isolated type. Then $p(\mathcal{M})$ is relatively $\mathcal{H}$-free $\Leftrightarrow$ $RC(p)=1$.*

Proof. ($\Rightarrow$) Indeed, if $RC(p)>1$ then there is an $\emptyset$-definable equivalence relation $E(x,y)$ partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes. Obviously, there is no $\mathcal{M}_1\prec M$ such that $E(a,\mathcal{M})=p(\mathcal{M})\cap M_1$ for some $a\in p(\mathcal{M})$.

($\Leftarrow$) If $RC(p)=1$ then by analogy with proof of Lemma 4.1 it is established that $p(\mathcal{M})$ is indiscernible over $\emptyset$. Then if p is quasirational to right (left) then for any open to right (left) convex set $A'\subseteq p(\mathcal{M})$ there is $\mathcal{M}_1\prec\mathcal{M}$ such that $A'=p(\mathcal{M})\cap M_1$. If p is irrational then for any convex set $A'\subseteq p(\mathcal{M})$ (including the case when $A'=\{a\}$ for some $a\in p(\mathcal{M})$) there exists $\mathcal{M}_1\prec\mathcal{M}$ with $A'=p(\mathcal{M})\cap M_1$. □

Definition. Let T be a weakly o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model for T, $p,q\in S_1(\emptyset)$ be non-isolated types, $RC(p)=RC(q)=1$. We say that $p(\mathcal{M})$ and $q(\mathcal{M})$ are *relatively $\mathcal{H}$-independent* if for any convex sets $A'\subseteq p(\mathcal{M})$ and $B'\subseteq q(\mathcal{M})$ corresponding to p and q there exists $M_1\in H(\mathcal{M})$ such that $A'=p(\mathcal{M})\cap M_1$ and $B'=q(\mathcal{M})\cap M_1$.

Proposition 4.11. [14] *Let T be a weakly o-minimal theory, $\mathcal{M} \models T$, $A \subseteq M$, $p, q \in S_1(A)$ be non-algebraic types, $p \not\perp^w q$. Then:*
(1) *p is irrational $\Leftrightarrow$ q is irrational;*
(2) *p is quasirational $\Leftrightarrow$ q is quasirational.*

Lemma 4.12. *Let T be an almost ω–categorical quite o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model for T, $p, q \in S_1(\emptyset)$ be non-isolated types, $RC(p) = RC(q) = 1$. Then $p(\mathcal{M})$ and $q(\mathcal{M})$ are relatively $\mathcal{H}$-independent $\Leftrightarrow$ $p \perp^w q$.*

Proof. If $p \not\perp^w q$ then by Proposition 4.11 the types p and q are simultaneously either quasirational or irrational. Without loss of generality, suppose that p and q are quasirational. By quite o-minimality there is an $\emptyset$-definable bijection $f : p(\mathcal{M}) \to q(\mathcal{M})$. Since the convexity ranks of the types are equal to 1 then this bijection is strictly monotonic. For definiteness, suppose that p is quasirational to right. Then if f is strictly increasing (decreasing) then q will be quasirational to right (left). Take an arbitrary open to right convex set $A' \subseteq p(\mathcal{M})$ and consider $f(A')$. If f is strictly increasing (decreasing) then $f(A')$ will be also open to right (left) convex set. Take arbitrary $a, b \in f(A')$ with $a < b$. Then let $B' := \{c \in q(\mathcal{M}) \mid a < c < b\}$. Then there is no $\mathcal{M}_1 \prec \mathcal{M}$ such that $A' = p(\mathcal{M}) \cap M_1$ and $B' = q(\mathcal{M}) \cap M_1$. □

Definition. Let T be a weakly o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model for T, $p \in S_1(\emptyset)$ be a non-isolated type. Let $E(x, y)$ be an $\emptyset$-definable equivalence relation partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes. If p is quasirational to right (left) then we say $p(\mathcal{M})$ is *relatively* $(\mathcal{H}, E)$*-free* if for any convex $A' \subseteq p(\mathcal{M})$ such that A'/E is open to right (left) set the equality $A' = p(\mathcal{M}) \cap M_1$ holds for some $M_1 \in H(\mathcal{M})$. If p is irrational then it is sufficiently to take any open convex subset of $p(\mathcal{M})$ as A', leaving the type of the set A'/E for an arbitrariness.

Proposition 4.13. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model for T, $p \in S_1(\emptyset)$ be a non-isolated type, $E(x, y)$ be an $\emptyset$-definable equivalence relation partitioning $p(\mathcal{M})$ into infinitely many infinite convex classes. Then $p(\mathcal{M})$ is relatively $(\mathcal{H}, E)$-free $\Leftrightarrow$ $E(x, y)$ is the greatest $\emptyset$-definable equivalence relation partitioning $p(\mathcal{M})$ into infinitely many convex classes.*

Proof. ($\Rightarrow$) is proved similarly to Proposition 4.6.

($\Leftarrow$) If p is quasirational to right (left) then taking as A' an arbitrary convex subset of $p(\mathcal{M})$ with the condition that A'/E is open to right (left) we easily find $\mathcal{M}_1 \prec \mathcal{M}$ such that $A' = p(\mathcal{M}) \cap M_1$. If p is irrational then take as A' an arbitrary open convex subset of $p(\mathcal{M})$. □

Definition. Let T be a weakly o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model for T, $p_1, p_2 \in S_1(\emptyset)$ be non-isolated types. Let $E_1(x, y), E_2(x, y)$ be $\emptyset$-definable equivalence relations partitioning $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ respectively into infinitely many infinite convex classes. Suppose that $p_1(\mathcal{M})$ is relatively $(\mathcal{H}, E_1)$-free and $p_2(\mathcal{M})$ is relatively $(\mathcal{H}, E_2)$-free. We say that $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ are *relatively* $(\mathcal{H}, E_1, E_2)$*-independent* if for any convex $A' \subseteq p_1(\mathcal{M})$ and $B' \subseteq p_2(\mathcal{M})$ corresponding to p_1 and p_2 there is $M_1 \in H(\mathcal{M})$ such that $A' = p_1(\mathcal{M}) \cap M_1$ and $B' = p_2(\mathcal{M}) \cap M_1$.

Proposition 4.14. *Let T be an almost ω-categorical quite o-minimal theory, $\mathcal{M}$ be a sufficiently saturated model for T, $p_1, p_2 \in S_1(\emptyset)$ be non-isolated types. Suppose* *that $p_1(\mathcal{M})$ is relatively $(\mathcal{H}, E_1)$-free and $p_2(\mathcal{M})$ is relatively $(\mathcal{H}, E_2)$-free, where $E_1(x, y)$, $E_2(x, y)$ are $\emptyset$-definable equivalence relations partitioning $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ respectively into infinitely many infinite convex classes. Then $p_1(\mathcal{M})$ and $p_2(\mathcal{M})$ are relatively $(\mathcal{H}, E_1, E_2)$-independent $\Leftrightarrow$ $p_1 \perp^w p_2$.*

Proof. If $p_1 \not\perp^w p_2$ then by Proposition 4.11 the types p_1 and p_2 are simultaneously either quasirational or irrational. Without loss of generality, suppose that p_1 and p_2 are quasirational. By quite o-minimality there is an $\emptyset$-definable bijection $f : p_1(\mathcal{M}) \to p_2(\mathcal{M})$. For definiteness, let p_1 be quasirational to right. Then take an arbitrary convex $A' \subseteq p_1(\mathcal{M})$ with the condition that A'/E_1 is open to right. Obviously, $f(A')$ will be convex. If f is strictly increasing (decreasing) on $p_1(\mathcal{M})/E_1$ then $f(A')/E_2$ will be open to right (left). Taking arbitrary E_2-classes $E_2(a, \mathcal{M})$ and $E_2(b, \mathcal{M})$ for some $a, b \in p_2(\mathcal{M})$ with $E_2(a, \mathcal{M}) < E_2(b, \mathcal{M})$ lying in $f(A')$, and considering $B' := \{h \in p_2(\mathcal{M}) \mid E_2(a, \mathcal{M}) < h < E_2(b, \mathcal{M})\}$, we see that B' is convex, and B'/E_2 is open Obviously, there is no $\mathcal{M}_1 \prec \mathcal{M}$ such that $A' = p_1(\mathcal{M}) \cap M_1$ and $B' = p_2(\mathcal{M}) \cap M_1$. □

5. On freedom and independence in hypergraphs of models of theories of unars

In the case of unary theories, if sets $A_i = f^{-k_i}(a_i)$, $k_i > 0$, are $\mathcal{H}$-independent, then the restriction of $\mathcal{H}$ on $\bigcup_i A_i$ is represented in the form of disjoint complete union of restrictions $\mathcal{H}_i = \mathcal{H} \upharpoonright A_i$.

Example 5.1. Consider a connected free unar $\mathcal{M} = \langle M, f\rangle$, i.e. a connected unar non-having cycles and such that every element has infinitely many f-preimages. Consider also a hypergraph $\mathcal{H}$ of elementary subsystems of $\mathcal{M}$. Then for every element $a \in M$ and pairwise distinct elements $a_i \in f^{-k}(a)$, $k > 0$, the sets $A_i = f^{-k_i}(a_i)$ are $\mathcal{H}$-independent for any $k_i > 0$. The restriction of $\mathcal{H}$ on $\bigcup_i A_i$ is represented in the form of disjoint complete union of restrictions $\mathcal{H}_i = \mathcal{H} \upharpoonright A_i$.

On the other hand, the sets $f^{-k}(a)$ and $f^{-m}(b)$ for $k, m > 0$ and $b \in f^{-k}(a)$ are not $\mathcal{H}$-independent, since $b \notin Z \in H(\mathcal{M})$ implies $f^{-m}(b) \cap Z = \emptyset$.

Thus, $\mathcal{H}$-independence of the sets $f^{-k}(a)$ and $f^{-m}(b)$ is equivalent to their disjointness, and also the conditions $b \notin \triangle_f(a)$ and $a \notin \triangle_f(b)$, where $\triangle_f(a) = \bigcup_{n\in\omega} f^{-n}(a)$ is a lower cone of the root a [1]. The indicated description of $\mathcal{H}$-independence is naturally is extended on an arbitrary family of the sets $f^{-k_i}(a_i)$.

By Properties 1 and 3, of $\mathcal{H}$-freedom and $\mathcal{H}$-independence, in general case the $\mathcal{H}$-independence of the sets $A_i = f^{-k_i}(a_i)$ for $a_i \in f^{-k}(a)$, allowing to conduct an indicated decomposition, implies their infinity and presence of completeness of the types $p_i(x)$ over $\{a_i\}$ isolated by the formulas $f^{k_i}(x) \approx a_i$.

If the set $A_i = f^{-k_i}(a_i)$ is finite then its inclusion in $Z \in H(\mathcal{M})$ is equivalent to inclusion of a_i in Z.

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Beibut Shaiykovich Kulpeshov
International Information Technologies University,
Manas str. 34A/Zhandosov str. 8A,
050040, Almaty, Kazakhstan.
E-mail address: b.kulpeshov@iitu.kz

Sergey Vladimirovich Sudoplatov
Sobolev Institute of Mathematics,
Academician Koptyug avenue, 4
630090, Novosibirsk, Russia.
Novosibirsk State Technical University,
K. Marx avenue, 20
630073, Novosibirsk, Russia.
Novosibirsk State University,
Pirogova street, 1
630090, Novosibirsk, Russia.
E-mail address: sudoplat@math.nsc.ru