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UNIQUE SOLVABILITY OF INITIAL-BOUNDARY VALUE PROBLEM FOR ONE-DIMENSIONAL EQUATIONS OF POLYTROPIC FLOWS OF MULTICOMPONENT VISCOUS COMPRESSIBLE FLUIDS

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Abstract. We consider the initial-boundary value problem governing unsteady polytropic motions of viscous compressible multifluids. We prove the existence and uniqueness of a strong solution to the problem.

Keywords: multifluid, mixture of fluids, viscous compressible fluid, existence, uniqueness, initial-boundary value problem, unsteady motion.

The paper is devoted to the analysis of the solvability of the equations of motion of multicomponent viscous compressible fluids (mixtures of fluids, multifluids). Concerning the origin of the model and its physical interpretation, we refer the reader to [12], [13]. An overview of the options for formulating the model and the known results can be found in [6], [8], [14]. Related multi-velocity models of multifluids are considered in [2], [21], [26]. As the first results on the well-posedness of the multidimensional equations of multifluids, we can refer to [3], [4], [5].

Weak solutions for multidimensional barotropic problems for the model considered in the paper are constructed in the steady version in [9], [25] (polytropic case), and then in [10], [15] (general case); in the unsteady version in [11] (polytropic case), and then in [20] (general case). Similar results for the heat-conductive model are obtained in [7], [16]. For a number of reasons, including the purpose of constructing more regular solutions, one-dimensional formulations are of interest. A detailed

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discussion of these formulations and a review of the results can be found in [17], [18]. Solvability for related models is shown in [19], [23], [24].

The specificity of the paper is that we consider a variant of the model with an average velocity in the transport operator.

1. Statement of the problem and the main result

We consider the system governing the one-dimensional flow of multicomponent viscous compressible fluids without taking into account chemical reactions:

$$\partial_t\rho_i + \partial_x(\rho_i v) = 0, \qquad \rho_i\left(\partial_t u_i + v\partial_x u_i\right) = \partial_x P_i, \qquad i = 1,\dots,N.$$

Here $N \geqslant 2$ is the number of constituents, ρ_i is the density of the i-th constituent, u_i is the velocity of the i-th constituent, $v = \dfrac{1}{N}\sum\limits_{i=1}^{N} u_i$ is the average velocity of the multifluid, P_i are the stresses. Let us accept generalized Newton's hypothesis for P_i:

$$P_i = -p + \sum_{j=1}^{N}\mu_{ij}\partial_x u_j,$$

where p is the pressure, and the viscosity coefficients $\{\mu_{ij}\}_{i,j=1}^{N}$ form a symmetric matrix $\mathbf{M}$, and $\mathbf{M} > 0$, i. e. $(\mathbf{M}\boldsymbol{\xi},\boldsymbol{\xi}) \geqslant C_0(\mathbf{M})|\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi}\in\mathbb{R}^N$ with some constant $C_0(\mathbf{M}) > 0$.

The written equations together with the constitutive equation

$$p = K\rho^\gamma, \quad \rho = \sum_{i=1}^{N}\rho_i, \quad K = \text{const} > 0, \quad \gamma = \text{const} > 1$$

form the closed system

$$\partial_t\rho_i + \partial_x(\rho_i v) = 0, \quad v = \frac{1}{N}\sum_{i=1}^{N} u_i, \tag{1}$$

$$\rho_i\left(\partial_t u_i + v\partial_x u_i\right) + K\partial_x\rho^\gamma = \sum_{j=1}^{N}\mu_{ij}\partial_{xx}u_j, \quad i = 1,\dots,N, \quad \rho = \sum_{i=1}^{N}\rho_i. \tag{2}$$

We are to consider this system in the rectangular Q_T (here and below $Q_t = (0,1)\times(0,t)$) with an arbitrary finite height T, $0 < T < \infty$, equipped with the following initial and boundary conditions ($i = 1,\dots,N$):

$$\rho_i|_{t=0} = \rho_{0i}(x), \quad u_i|_{t=0} = u_{0i}(x), \quad x\in[0,1], \tag{3}$$

$$u_i|_{x=0} = u_i|_{x=1} = 0, \quad t\in[0,T]. \tag{4}$$

Definition 1. By a strong solution to the problem (1)–(4) we mean $2N$ functions $(\rho_1,\dots,\rho_N,u_1,\dots,u_N)$ that satisfy equations (1), (2) almost everywhere in Q_T, the initial conditions (3) for almost all $x\in(0,1)$, and the boundary conditions (4) for almost all $t\in(0,T)$, and such that the following relations hold ($i = 1,\dots,N$)

$$\begin{gathered}\rho_i > 0, \quad \rho_i \in L_\infty\big(0,T;W_2^1(0,1)\big), \quad \partial_t\rho_i\in L_\infty\big(0,T;L_2(0,1)\big),\\ u_i\in L_\infty\big(0,T;W_2^1(0,1)\big)\bigcap L_2\big(0,T;W_2^2(0,1)\big), \quad \partial_t u_i\in L_2(Q_T).\end{gathered} \tag{5}$$

The main result of the paper is formulated as follows.

Theorem 2. Let the initial data in (3) be such that

$$\rho_{0i} \in W_2^1(0,1), \quad \rho_{0i} > 0, \quad u_{0i} \in \overset{\circ}{W}{}_2^1(0,1), \quad i = 1,\dots,N \quad (N \geqslant 2),$$

the symmetric viscosity matrix $\mathbf{M}$ is positive definite, the polytropic exponent $\gamma > 1$, and the constants $0 < K, T < \infty$. Then there exists a sufficiently small number $t_0 \in (0,T)$ such that for all $t \in (0,t_0)$ there exists a unique strong solution to the problem (1)–(4) in the sense of Definition 1 with t_0 instead of T.

The proof of Theorem 2 is given in Sections 2–5.

2. Construction of approximate solutions

In this section, we prove the time-local solvability of the initial-boundary value problem obtained from the problem (1)–(4) by applying the Galerkin method (in the spatial variable x) to the momentum equation (2).

Lemma 3. Under the assumptions of Theorem 2, for any $m \in \mathbb{N}$ there is a time-interval $(0,t_m) \subset (0,T)$ (cf. (15) below), where there exists a solution[1] to the problem (here $k = 1,\dots,m$, $i = 1,\dots,N$)

$$\partial_t \rho_i + \partial_x(\rho_i v) = 0, \quad v = \frac{1}{N}\sum_{i=1}^N u_i, \tag{6}$$

(7)
$$\int_0^1 \Big(\rho_i\,(\partial_t u_i + v\partial_x u_i) + K\partial_x \rho^\gamma - \sum_{j=1}^N \mu_{ij}\partial_{xx}u_j\Big)\omega_k(x)\,dx = 0 \ \forall t \in (0,t_m),\ \rho = \sum_{i=1}^N \rho_i,$$

$$\rho_i|_{t=0} = \rho_{0i}(x), \tag{8}$$

$$u_i = \sum_{k=1}^m \psi_{ik}(t)\omega_k(x), \quad u_i|_{t=0} = \sum_{k=1}^m \psi_{0ik}\omega_k(x), \tag{9}$$

where $\omega_k(x) = \sin(\pi k x)$, $\psi_{ik}(0) = \psi_{0ik} = 2\int\limits_0^1 u_{0i}(x)\omega_k(x)\,dx$, and we have

$$\begin{gathered}\rho_i \in L_\infty\big(0,t_m; W_2^1(0,1)\big) \bigcap W_\infty^1\big(0,t_m; L_2(0,1)\big), \quad \rho_i > 0,\\ u_i \in C^{1,\infty}([0,t_m]\times[0,1]).\end{gathered} \tag{10}$$

Proof of Lemma 3. We fix $t_m \in (0,T]$. In the space $(C[0,t_m])^{mN}$, we consider the set

$$B = \left\{\boldsymbol{\psi} \in (C[0,t_m])^{mN} \mid \ \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0,\ \|\boldsymbol{\psi}\|_{(C[0,t_m])^{mN}} \leqslant b\right\},$$

where

$$\boldsymbol{\psi} = (\boldsymbol{\psi}_1,\dots,\boldsymbol{\psi}_N), \quad \boldsymbol{\psi}_0 = (\boldsymbol{\psi}_{01},\dots,\boldsymbol{\psi}_{0N}), \quad \boldsymbol{\psi}_i = (\psi_{i1},\dots,\psi_{im}),$$

[1]Till the beginning of the fourth Section we omit the index m in the notations of solutions.

$$\boldsymbol{\psi}_{0i} = (\psi_{0i1}, \dots, \psi_{0im}), \quad i = 1, \dots, N, \quad b^2 = e^{\frac{\max\limits_{1\leqslant i\leqslant N}\left\{\sup\limits_{[0,1]}\rho_{0i}\right\}}{\min\limits_{1\leqslant i\leqslant N}\left\{\inf\limits_{[0,1]}\rho_{0i}\right\}}}\|\boldsymbol{\psi}_0\|^2_{\mathbb{R}^{mN}} + 1.$$

We construct the operator $\Lambda : B \to (C[0,t_m])^{mN}$, $\mathrm{Im}\,\Lambda \subset (C^1[0,t_m])^{mN}$, $\Lambda(\boldsymbol{\psi}) = \boldsymbol{\Psi}$, where $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_1, \dots, \boldsymbol{\Psi}_N)$, $\boldsymbol{\Psi}_i = (\Psi_{i1}, \dots, \Psi_{im})$, $i = 1, \dots, N$, by the following algorithm.

At the first step, we find $\rho_i \in L_\infty\big(0,t_m; W_2^1(0,1)\big) \bigcap W_\infty^1\big(0,t_m; L_2(0,1)\big)$, $\rho_i > 0$, $i = 1, \dots, N$, as solutions to the Cauchy problem (6), (8), where u_i, $i = 1, \dots, N$, are given by (9). Moreover, for all $i = 1, \dots, N$ we have the inequalities

$$\left(\inf_{[0,1]}\rho_{0i}\right)\exp\left\{-\frac{1}{N}\sum_{i=1}^N\int_0^t\sup_{[0,1]}|\partial_x u_i|\,d\tau\right\} \leqslant \rho_i(x,t) \leqslant \tag{11}$$
$$\leqslant \left(\sup_{[0,1]}\rho_{0i}\right)\exp\left\{\frac{1}{N}\sum_{i=1}^N\int_0^t\sup_{[0,1]}|\partial_x u_i|\,d\tau\right\},$$

which, due to $\boldsymbol{\psi} \in B$, give the estimates

(12)
$$\left(\inf_{[0,1]}\rho_{0i}\right)\exp\left\{-\pi m^2 bt\right\} \leqslant \rho_i(x,t) \leqslant \left(\sup_{[0,1]}\rho_{0i}\right)\exp\left\{\pi m^2 bt\right\}, \quad i = 1, \dots, N.$$

At the second step, we find $\boldsymbol{\Psi}$ from the Cauchy problem for the system of mN ordinary first order differential (linear) equations

$$\int_0^1\Big(\rho_i\partial_t U_i + \rho_i\left(\frac{1}{N}\sum_{j=1}^N u_j\right)\partial_x U_i + K\partial_x\rho^\gamma - \sum_{j=1}^N\mu_{ij}\partial_{xx}U_j\Big)\omega_k\,dx = 0, \tag{13}$$

$$k = 1, \dots, m, \quad i = 1, \dots, N,$$
$$\boldsymbol{\Psi}(0) = \boldsymbol{\psi}_0,$$

where $U_i = \sum\limits_{s=1}^m \Psi_{is}(t)\omega_s(x)$, $i = 1, \dots, N$, $\rho = \sum\limits_{i=1}^N \rho_i$. Since $\det A \neq 0$, where

$$A(t) = \begin{pmatrix} A_1(t) & 0 & \dots & 0 \\ 0 & A_2(t) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_N(t) \end{pmatrix},$$

$$A_i(t) = \left\{\int_0^1 \rho_i\omega_k\omega_s\,dx\right\}_{k,s=1}^m, \quad i = 1, \dots, N,$$

in view of the positivity of ρ_i, $i = 1, \dots, N$, we can solve the system (13) with respect to derivatives, which justifies the existence of $\boldsymbol{\Psi} \in (C^1[0,t_m])^{mN}$.

Thus, for arbitrary $t_m \in (0,T]$ we define the operator

$$\Lambda : B \to (C^1[0,t_m])^{mN} \subset (C[0,t_m])^{mN}, \qquad \Lambda(\boldsymbol{\psi}) = \boldsymbol{\Psi},$$

whose fixed point (if exists), together with the corresponding functions ρ_i, $i = 1, \dots, N$, is a solution to the problem (6)–(9).

We show that for sufficiently small t_m the operator Λ satisfies the assumptions of the Schauder theorem (cf. [1], P. 31), i. e.,

- B is a convex closed bounded set (which is obvious in the case under consideration);
- $\Lambda:\ B \to B$;
- Λ is a completely continuous operator.

We first show that $\Lambda(B) \subset B$. Multiplying (13) by $\Psi_{ik}(t)$, $k = 1, \dots, m$, $i = 1, \dots, N$, summarizing with respect to $k = 1, \dots, m$, $i = 1, \dots, N$, we obtain due to (6), that

$$\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{N}\int_0^1 \rho_i U_i^2\,dx\right) + \sum_{i,j=1}^{N}\mu_{ij}\int_0^1 (\partial_x U_i)(\partial_x U_j)\,dx = K\sum_{i=1}^{N}\int_0^1 \rho^\gamma \partial_x U_i\,dx,$$

and taking into account the inequalities (to obtain them, we use (12) and the fact $\mathbf{M} > 0$)

$$\sum_{i,j=1}^{N}\mu_{ij}\int_0^1 (\partial_x U_i)(\partial_x U_j)\,dx \geqslant C_0(\mathbf{M})\sum_{i=1}^{N}\int_0^1 |\partial_x U_i|^2\,dx,$$

$$K\sum_{i=1}^{N}\int_0^1 \rho^\gamma \partial_x U_i\,dx \leqslant \frac{C_0}{2}\sum_{i=1}^{N}\int_0^1 |\partial_x U_i|^2\,dx + B_1,$$

where[2] $B_1 = \dfrac{K^2 N^{2\gamma+1}}{2C_0}\left(\max\limits_{1\leqslant i\leqslant N}\left\{\sup\limits_{[0,1]}\rho_{0i}\right\}\right)^{2\gamma}\exp\left\{2\pi\gamma m^2 b t_m\right\}$, we obtain the estimate

$$\frac{d}{dt}\sum_{i=1}^{N}\int_0^1 \rho_i U_i^2\,dx + C_0\sum_{i=1}^{N}\int_0^1 |\partial_x U_i|^2\,dx \leqslant 2B_1,$$

which, in its turn, leads to

$$\sum_{i=1}^{N}\int_0^1 \rho_i U_i^2\,dx \leqslant \sum_{i=1}^{N}\int_0^1 \rho_{0i} U_{0i}^2\,dx + 2B_1 t_m, \tag{14}$$

where $U_{0i} = \sum\limits_{k=1}^{m}\Psi_{ik}(0)\omega_k(x) = \sum\limits_{k=1}^{m}\psi_{0ik}\omega_k(x)$, $i = 1, \dots, N$. Using (12) for the second time, we obtain from (14) the inequality

$$\|\mathbf{\Psi}\|^2_{(C[0,t_m])^{mN}} \leqslant e^{\pi m^2 b t_m}\frac{\max\limits_{1\leqslant i\leqslant N}\left\{\sup\limits_{[0,1]}\rho_{0i}\right\}}{\min\limits_{1\leqslant i\leqslant N}\left\{\inf\limits_{[0,1]}\rho_{0i}\right\}}\|\boldsymbol{\psi}_0\|^2_{\mathbb{R}^{mN}} + \frac{4B_1 e^{\pi m^2 b t_m}}{\min\limits_{1\leqslant i\leqslant N}\left\{\inf\limits_{[0,1]}\rho_{0i}\right\}}t_m.$$

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[2]Hereinafter, B with subscripts denotes positive quantities depending on arguments indicated in the brackets, initial data, physical constants and T.

Choosing

$$t_m < \min\left\{\frac{1}{\pi m^2 b}, \frac{\min\limits_{1\leqslant i\leqslant N}\left\{\inf\limits_{[0,1]}\rho_{0i}\right\}}{4eB_2}, T\right\}, \tag{15}$$

where $B_2 = \dfrac{K^2 N^{2\gamma+1}}{C_0}\left(\max\limits_{1\leqslant i\leqslant N}\left\{\sup\limits_{[0,1]}\rho_{0i}\right\}\right)^{2\gamma}\exp\{2\gamma\}$, we obtain that $B_1 \leqslant B_2$, and arrive at the required estimate $\|\mathbf{\Psi}\|_{(C[0,t_m])^{mN}} \leqslant b$. Thus, if (15) holds, the operator Λ maps the set B to itself.

Let us prove the compactness of the operator Λ. Multiplying (13) by $\dfrac{d\Psi_{ik}(t)}{dt}$, $k = 1,\dots,m$, $i = 1,\dots,N$, summarizing with respect to $k = 1,\dots,m$, $i = 1,\dots,N$, we get

$$\sum_{i=1}^N \int_0^1 \rho_i|\partial_t U_i|^2\,dx = \tag{16}$$

$$= \sum_{i=1}^N \int_0^1 \left(-\sum_{j=1}^N \mu_{ij}(\partial_{tx}U_i)(\partial_x U_j) - \frac{\rho_i}{N}\left[\sum_{j=1}^N u_j\right](\partial_x U_i)(\partial_t U_i) + K\rho^\gamma(\partial_{tx}U_i)\right)dx.$$

Let us estimate the terms on the right-hand side of (16) by using (12), the Cauchy inequality, and the inequalities $\|\boldsymbol{\psi}\|_{(C[0,t_m])^{mN}} \leqslant b$, $\|\mathbf{\Psi}\|_{(C[0,t_m])^{mN}} \leqslant b$, $\|\partial_x U_i\|_{L_2(0,1)} \leqslant B_3(m)\|U_i\|_{L_2(0,1)}$, $\|\partial_{tx}U_i\|_{L_2(0,1)} \leqslant B_3\|\partial_t U_i\|_{L_2(0,1)}$, $i = 1,\dots,N$:

$$\left|\sum_{i,j=1}^N \mu_{ij}\int_0^1(\partial_{tx}U_i)(\partial_x U_j)\,dx\right| \leqslant$$

$$\leqslant \frac{1}{6}\sum_{i=1}^N\int_0^1 \rho_i|\partial_t U_i|^2\,dx + B_4\left(B_3, \left\{\inf_{[0,1]}\rho_{0i}\right\}, \mathbf{M}, N, b, m, t_m\right),$$

$$\left|\frac{1}{N}\sum_{i=1}^N\int_0^1 \rho_i\left(\sum_{j=1}^N u_j\right)(\partial_x U_i)(\partial_t U_i)\,dx\right| \leqslant$$

$$\leqslant \frac{1}{6}\sum_{i=1}^N\int_0^1 \rho_i|\partial_t U_i|^2\,dx + B_5\left(B_3, \left\{\sup_{[0,1]}\rho_{0i}\right\}, N, b, m, t_m\right),$$

$$\left|K\sum_{i=1}^N\int_0^1 \rho^\gamma(\partial_{tx}U_i)\,dx\right| \leqslant$$

$$\leqslant \frac{1}{6}\sum_{i=1}^N\int_0^1 \rho_i|\partial_t U_i|^2\,dx + B_6\left(B_3, \left\{\inf_{[0,1]}\rho_{0i}\right\}, \left\{\sup_{[0,1]}\rho_{0i}\right\}, K, N, b, m, t_m, \gamma\right).$$

Thus, from (16) we get

$$\frac{1}{2}\sum_{i=1}^{N}\int_0^1 \rho_i|\partial_t U_i|^2\,dx \leqslant B_4+B_5+B_6.$$

Integrating the last inequality in time and applying (12), we obtain the estimate

$$\sum_{i=1}^{N}\|\partial_t U_i\|_{L_2(Q_{t_m})} \leqslant B_7\left(B_4,B_5,B_6,\left\{\inf_{[0,1]}\rho_{0i}\right\},b,m,t_m\right). \tag{17}$$

Thus, we have obtained the estimate for $\mathbf{\Psi}$ in $(W_2^1(0,t_m))^{mN}$. Consequently, Λ is a compact operator.

We establish the continuity of the operator Λ from B to $(C[0,t_m])^{mN}$. Let $\boldsymbol{\psi}^{(1,2)}\in B$, $\mathbf{\Psi}^{(1,2)}=\Lambda(\boldsymbol{\psi}^{(1,2)})$, $u_i^{(1,2)}=\sum\limits_{k=1}^{m}\psi_{ik}^{(1,2)}\omega_k$, $U_i^{(1,2)}=\sum\limits_{k=1}^{m}\Psi_{ik}^{(1,2)}\omega_k$, $i=1,\dots,N$. Further, let $\rho_i^{(1,2)}$, $i=1,\dots,N$, be solutions to the Cauchy problem (6), (8), with $v^{(1,2)}=\dfrac{1}{N}\sum\limits_{j=1}^{N}u_j^{(1,2)}$ instead of v. Denote $\rho_i=\rho_i^{(1)}-\rho_i^{(2)}$, $u_i=u_i^{(1)}-u_i^{(2)}$, $U_i=U_i^{(1)}-U_i^{(2)}$, $i=1,\dots,N$, $v=v^{(1)}-v^{(2)}$, $\rho=\rho^{(1)}-\rho^{(2)}$, where $\rho^{(1,2)}=\sum\limits_{j=1}^{N}\rho_j^{(1,2)}$, $\psi_{jk}=\psi_{jk}^{(1)}-\psi_{jk}^{(2)}$, $j=1,\dots,N$, $k=1,\dots,m$.

Differentiating[3] equations (6) for $\rho_i^{(1,2)}$, $i=1,\dots,N$ $\Big($i. e. the equations $\partial_t\rho_i^{(1,2)}+\partial_x\left(\rho_i^{(1,2)}v^{(1,2)}\right)=0$, $i=1,\dots,N\Big)$ with respect to x, multiplying by $\partial_x\rho_i^{(1,2)}$, $i=1,\dots,N$, using the initial data $\rho_i^{(1,2)}|_{t=0}=\rho_{0i}$, $i=1,\dots,N$, the inequality (12) and the Gronwall inequality we get

$$\left\|\partial_x\rho_i^{(1,2)}\right\|_{L_2(0,1)} \leqslant B_8\left(\|\rho_{0i}\|_{W_2^1(0,1)},b,m,t_m\right),\quad i=1,\dots,N. \tag{18}$$

Let us note that (6), (8) lead to the equalities

$$\partial_t\rho_i+\partial_x\left(\rho_i v^{(1)}\right)+\partial_x\left(\rho_i^{(2)}v\right)=0,\quad \rho_i|_{t=0}=0,\quad i=1,\dots,N. \tag{19}$$

[3]Cf. Remark 5 below.

Multiplying (19) by ρ_i, $i=1,\dots,N$, and integrating with respect to $x\in(0,1)$, we find for all $i=1,\dots,N$ that

$$\frac{1}{2}\frac{d}{dt}\int\limits_0^1 \rho_i^2\,dx = -\int\limits_0^1\left(\frac{1}{2}\rho_i^2\left(\partial_x v^{(1)}\right)+\rho_i^{(2)}\rho_i(\partial_x v)+\left(\partial_x\rho_i^{(2)}\right)\rho_i v\right)dx \leqslant$$

(20)
$$\leqslant \frac{1}{2}\left(\sup_{[0,1]}\left|\partial_x v^{(1)}\right|\int\limits_0^1\rho_i^2\,dx+\sup_{[0,1]}\rho_i^{(2)}\int\limits_0^1\left(\rho_i^2+(\partial_x v)^2\right)dx+\right.$$
$$\left.+\sup_{[0,1]}v^2\int\limits_0^1\left(\partial_x\rho_i^{(2)}\right)^2dx+\int\limits_0^1\rho_i^2\,dx\right)\leqslant$$
$$\leqslant B_9\left(B_8,\left\{\sup_{[0,1]}\rho_{0i}\right\},b,m,t_m\right)\left(\int\limits_0^1\rho_i^2\,dx+\sum_{j=1}^N\int\limits_0^1 u_j^2\,dx\right).$$

Here we have used obvious relations

(21)
$$\sum_{j=1}^N\int\limits_0^1 u_j^2\,dx=\frac{1}{2}\sum_{j=1}^N\sum_{k=1}^m\psi_{jk}^2(t),\quad \sup_{[0,1]}v^2\leqslant\frac{1}{N^2}\sum_{i,j=1}^N\sum_{k,l=1}^m|\psi_{jk}(t)\psi_{il}(t)|,$$
$$\int\limits_0^1|\partial_x v|^2\,dx=\frac{\pi^2}{2N^2}\sum_{i,j=1}^N\sum_{k=1}^m k^2\psi_{jk}(t)\psi_{ik}(t).$$

Basing on (20), applying the Gronwall inequality and the initial condition in (19), we obtain the inequality

$$\int\limits_0^1\rho_i^2\,dx\leqslant B_{10}(B_9,t_m)\sum_{j=1}^N\int\limits_{Q_t}u_j^2\,dxd\tau,\quad i=1,\dots,N \tag{22}$$

for all $t\in(0,t_m]$. Further, from the equations for $U_i^{(1,2)}$, $i=1,\dots,N$ (cf. (13)) due to (6), we get

(23)
$$\frac{1}{2}\sum_{i=1}^N\int\limits_0^1\rho_i^{(1)}U_i^2\,dx+\sum_{i,j=1}^N\mu_{ij}\int\limits_{Q_t}(\partial_x U_i)(\partial_x U_j)\,dxd\tau=$$
$$=K\sum_{i=1}^N\int\limits_{Q_t}\left(\left(\rho^{(1)}\right)^\gamma-\left(\rho^{(2)}\right)^\gamma\right)(\partial_x U_i)\,dxd\tau-\sum_{i=1}^N\int\limits_{Q_t}\rho_i U_i\left(\partial_t U_i^{(2)}\right)dxd\tau-$$
$$-\sum_{i=1}^N\int\limits_{Q_t}\rho_i^{(1)}vU_i\left(\partial_x U_i^{(2)}\right)dxd\tau-\sum_{i=1}^N\int\limits_{Q_t}\rho_i v^{(2)}U_i\left(\partial_x U_i^{(2)}\right)dxd\tau$$

for all $t \in (0, t_m]$. By (12), the first term on the left-hand side of (23) satisfies the estimate

$$\frac{1}{2}\sum_{i=1}^{N}\int_0^1 \rho_i^{(1)} U_i^2\,dx \geqslant \frac{1}{2}\min_{1\leqslant i\leqslant N}\left\{\inf_{[0,1]}\rho_{0i}\right\}\exp\left\{-\pi m^2 b t_m\right\}\sum_{i=1}^{N}\int_0^1 U_i^2\,dx. \tag{24}$$

For the second term we have

$$\sum_{i,j=1}^{N}\int_{Q_t}\mu_{ij}(\partial_x U_i)(\partial_x U_j)\,dxd\tau \geqslant C_0\sum_{i=1}^{N}\int_{Q_t}|\partial_x U_i|^2\,dxd\tau. \tag{25}$$

Applying the Cauchy inequality to the first term on the right-hand side of (23), we find

$$\begin{gathered} K\sum_{i=1}^{N}\int_{Q_t}\left(\left(\rho^{(1)}\right)^{\gamma}-\left(\rho^{(2)}\right)^{\gamma}\right)(\partial_x U_i)\,dxd\tau \leqslant \frac{C_0}{2}\sum_{i=1}^{N}\int_{Q_t}|\partial_x U_i|^2\,dxd\tau+ \\ +\frac{K^2N}{2C_0}\int_{Q_t}\left(\left(\rho^{(1)}\right)^{\gamma}-\left(\rho^{(2)}\right)^{\gamma}\right)^2\,dxd\tau, \end{gathered} \tag{26}$$

and hence, due to (cf. (12))

$$\left(\rho^{(1)}\right)^{\gamma}-\left(\rho^{(2)}\right)^{\gamma}=\gamma\left(\lambda\rho^{(1)}+(1-\lambda)\rho^{(2)}\right)^{\gamma-1}\rho,\quad \lambda\in[0,1],$$

$$\sum_{i=1}^{N}\left(\inf_{[0,1]}\rho_{0i}\right)\exp\left\{-\pi m^2 b t_m\right\}\leqslant \lambda\rho^{(1)}+(1-\lambda)\rho^{(2)}\leqslant$$

$$\leqslant \sum_{i=1}^{N}\left(\sup_{[0,1]}\rho_{0i}\right)\exp\left\{\pi m^2 b t_m\right\},$$

and (22), we get

$$\begin{gathered} K\sum_{i=1}^{N}\int_{Q_t}\left(\left(\rho^{(1)}\right)^{\gamma}-\left(\rho^{(2)}\right)^{\gamma}\right)(\partial_x U_i)\,dxd\tau \leqslant \\ \leqslant \frac{C_0}{2}\sum_{i=1}^{N}\int_{Q_t}|\partial_x U_i|^2\,dxd\tau + B_{11}\sum_{i=1}^{N}\int_{Q_t}u_i^2\,dxd\tau, \end{gathered} \tag{27}$$

where $B_{11}=B_{11}\left(C_0, B_{10}, \left\{\sup_{[0,1]}\rho_{0i}\right\}, K, N, \gamma, b, m, t_m\right)$. Using (17), (22) and the Cauchy inequality with small $\varepsilon > 0$, for the second term on the right-hand side

of (23) we get (cf. (21))
(28)

$$-\sum_{i=1}^{N}\int\limits_{Q_t}\rho_i U_i\left(\partial_t U_i^{(2)}\right)dxd\tau \leqslant \frac{B_{10}Nt}{2\varepsilon^2}\sum_{i=1}^{N}\int\limits_{Q_t}u_i^2\,dxd\tau+$$

$$+\frac{\varepsilon^2}{2}\left(\sum_{j=1}^{N}\sup_{Q_t}U_j^2\right)\left(\sum_{i=1}^{N}\left\|\partial_t U_i^{(2)}\right\|^2_{L_2(Q_t)}\right)\leqslant$$

$$\leqslant \frac{B_{10}Nt}{2\varepsilon^2}\sum_{i=1}^{N}\int\limits_{Q_t}u_i^2\,dxd\tau + m\varepsilon^2 B_7^2\sum_{i=1}^{N}\sup_{[0,t]}\int\limits_0^1 U_i^2\,dx =$$

$$\left[m\varepsilon^2 B_7^2=\frac{1}{4}\min_{1\leqslant i\leqslant N}\left\{\inf_{[0,1]}\rho_{0i}\right\}\exp\left\{-\pi m^2 bt_m\right\}\right]$$

$$= B_{12}\sum_{i=1}^{N}\int\limits_{Q_t}u_i^2\,dxd\tau + \frac{1}{4}\min_{1\leqslant i\leqslant N}\left\{\inf_{[0,1]}\rho_{0i}\right\}\exp\{-\pi m^2 bt_m\}\sum_{i=1}^{N}\sup_{[0,t]}\int\limits_0^1 U_i^2\,dx,$$

where $B_{12}=B_{12}\left(B_7,B_{10},\left\{\inf\limits_{[0,1]}\rho_{0i}\right\},N,b,m,t_m\right)$. The third term on the right-hand side of (23) is estimated with the help of (12), the Cauchy inequality, and the inequality $\|\mathbf{\Psi}^{(2)}\|_{(C[0,t_m])^{mN}}\leqslant b$ (cf. (21)):

$$(29)\quad -\sum_{i=1}^{N}\int\limits_{Q_t}\rho_i^{(1)}vU_i\left(\partial_x U_i^{(2)}\right)dxd\tau \leqslant B_{13}\left(\sum_{i=1}^{N}\int\limits_{Q_t}u_i^2\,dxd\tau+\sum_{i=1}^{N}\int\limits_{Q_t}U_i^2\,dxd\tau\right),$$

where $B_{13}=B_{13}\left(\left\{\sup\limits_{[0,1]}\rho_{0i}\right\},N,b,m,t_m\right)$. Finally, by (22) and the estimates $\|\boldsymbol{\psi}^{(2)}\|_{(C[0,t_m])^{mN}}\leqslant b$, $\|\mathbf{\Psi}^{(2)}\|_{(C[0,t_m])^{mN}}\leqslant b$, for the last term on the right-hand side of (23) we get

$$(30)\quad -\sum_{i=1}^{N}\int\limits_{Q_t}\rho_i v^{(2)}U_i\left(\partial_x U_i^{(2)}\right)dxd\tau \leqslant B_{14}\left(\sum_{i=1}^{N}\int\limits_{Q_t}u_i^2\,dxd\tau+\sum_{i=1}^{N}\int\limits_{Q_t}U_i^2\,dxd\tau\right),$$

where $B_{14}=B_{14}(B_{10},N,b,m,t_m)$. Thus, from (23) and (24)–(30) it follows that

$$\sum_{i=1}^{N}\int\limits_0^1 U_i^2\,dx \leqslant B_{15}\left(\sum_{i=1}^{N}\int\limits_{Q_t}u_i^2\,dxd\tau+\sum_{i=1}^{N}\int\limits_{Q_t}U_i^2\,dxd\tau\right),$$

where $B_{15}=B_{15}\left(B_{11},\ldots,B_{14},\left\{\inf\limits_{[0,1]}\rho_{0i}\right\},N,b,m,t_m\right)$, which, in view of the Gronwall inequality, implies the estimate

$$\sum_{i=1}^{N}\|U_i\|^2_{L_2(0,1)}\leqslant B_{16}(B_{15},t_m)\sum_{i=1}^{N}\|u_i\|^2_{L_2(Q_{t_m})},$$

and, consequently,

$$\|\mathbf{\Psi}\|_{(C[0,t_m])^{mN}} \leqslant B_{17}(B_{16}, t_m)\|\boldsymbol{\psi}\|_{(C[0,t_m])^{mN}}.$$

The last inequality justifies the continuity of Λ on B.

Since the operator Λ satisfies the assumptions of the Schauder theorem listed above, in B there exists a fixed point $\boldsymbol{\psi}$ of the operator Λ which, together with the corresponding functions ρ_i, $i = 1, \ldots, N$, is a solution to the problem (6)–(9). Lemma 3 is proved.

3. Uniform estimates of Galerkin approximations

We deal with a solution to the problem (6)–(9), which is defined for $t \in [0, \tau]$, where $\tau \in (0, T]$. For example, it can be the solution constructed in accordance with Lemma 3, and then $\tau \in [t_m, T]$, thus automatically $\rho_i > 0$, $i = 1, \ldots, N$.

Lemma 4. Under the assumptions of Theorem 2, there exist positive constants B_{18} and $t_0 \leqslant T$ (cf. (47) below) such that for any solution $(\rho_1, \ldots, \rho_N, u_1, \ldots, u_N)$ to the problem (6)–(9), which is defined for $t \in [0, \tau]$, the following estimate holds:

$$\sum_{i=1}^{N}\Big(\|u_i\|_{L_\infty\left(0,t_0;W_2^1(0,1)\right)} + \|u_i\|_{L_2\left(0,t_0;W_2^2(0,1)\right)} + \|\rho_i\|_{L_\infty\left(0,t_0;W_2^1(0,1)\right)} + \\ + \|\partial_t\rho_i\|_{L_\infty\left(0,t_0;L_2(0,1)\right)} + \|\partial_t u_i\|_{L_2(Q_{t_0})} + \|1/\rho_i\|_{L_\infty(Q_{t_0})}\Big) \leqslant B_{18}, \tag{31}$$

where

$$B_{18} = B_{18}\left(\left\{\|\rho_{0i}\|_{W_2^1(0,1)}\right\},\ \left\{\|u_{0i}\|_{W_2^1(0,1)}\right\},\ \left\{\inf_{[0,1]}\rho_{0i}\right\},\ K,\ \mathbf{M},\ N,\ T,\ \gamma\right),$$

and t_0 depends on the same quantities as B_{18}. The estimate (31) is valid for $t \in [0, \min\{\tau, t_0\}]$.

For $t_0 > \tau$ it is understood that the bound (31) holds only for $t \in [0, \tau]$. In other words, generally speaking, this estimate on the whole interval $[0, t_0]$ is a priori one.

Proof of Lemma 4. We set

$$\eta(t) = \sum_{i=1}^{N}\int_0^t\int_0^1 \left(\rho_i(\partial_t u_i)^2 + (\partial_{xx}u_i)^2\right)\,dxd\tau, \quad \eta'(t) \geqslant 0. \tag{32}$$

Equations (6) imply the inequalities (11) which, in turn, imply the estimate

$$B_{19}^{-1}\exp\{-B_{19}\eta(t)\} \leqslant \rho_i(x,t) \leqslant B_{19}\exp\{B_{19}\eta(t)\}, \quad i = 1, \ldots, N, \tag{33}$$

where $B_{19} = B_{19}\left(\left\{\sup_{[0,1]}\rho_{0i}\right\}, \left\{\inf_{[0,1]}\rho_{0i}\right\}, T\right)$. We note that (6) implies the identities

$$\rho_i\partial_{tx}(1/\rho_i) + \rho_i v\partial_{xx}(1/\rho_i) = \partial_{xx}v, \quad i = 1, \ldots, N, \tag{34}$$

which yield

$$\frac{d}{dt}\int_0^1 \rho_i(\partial_x(1/\rho_i))^2\,dx = 2\int_0^1 (\partial_{xx}v)(\partial_x(1/\rho_i))\,dx, \quad i = 1, \ldots, N. \tag{35}$$

From (35), integrating with respect to t and applying the Cauchy inequality, we get

$$\int_0^1 \rho_i(\partial_x(1/\rho_i))^2\,dx \leqslant \int_0^1 \rho_{0i}((1/\rho_{0i})')^2\,dx + \int_0^t\int_0^1 \rho_i(\partial_x(1/\rho_i))^2\,dxd\tau + \tag{36}$$
$$+\int_0^t\int_0^1 \frac{(\partial_{xx}v)^2}{\rho_i}\,dxd\tau, \quad i=1,\ldots,N.$$

Remark 5. In order to obtain (36), we need (in (34), (35)) an additional smoothness of ρ_i, $i=1,\ldots,N$, in comparison with (10), however the formulation of the relation (36) does not require any additional regularity. This means that (36) can be obtained via regularization of ρ_{0i}, $i=1,\ldots,N$, derivation of (36) for the solutions of the corresponding problems, and then the limit via the regularization parameter. The derivation of the relations (18) should be understood in a similar way.

Using (32), (33), and the Gronwall inequality, from (36) we get

$$\int_0^1 (\partial_x(1/\rho_i))^2\,dx + \int_0^1 (\partial_x\rho_i)^2\,dx \leqslant B_{20}\exp\{B_{20}\eta(t)\}, \quad i=1,\ldots,N, \tag{37}$$

where $B_{20} = B_{20}\left(B_{19}, \left\{\|\rho_{0i}\|_{W_2^1(0,1)}\right\}, \left\{\inf_{[0,1]}\rho_{0i}\right\}, T\right)$. Further, multiplying (7) by $\psi'_{ik}+\pi^2k^2\psi_{ik}$, summarizing with respect to $k=1,\ldots,m$, $i=1,\ldots,N$, and taking into account (6), (9), we obtain the relation[4]

$$\sum_{i=1}^N\int_0^1 \rho_i(\partial_t u_i)^2\,dx + \sum_{i,j=1}^N \mu_{ij}\int_0^1 (\partial_{xx}u_i)(\partial_{xx}u_j)\,dx + \frac{1}{2}\frac{d}{dt}\sum_{i=1}^N\int_0^1 \rho_i(\partial_x u_i)^2\,dx + \tag{38}$$
$$+\frac{1}{2}\frac{d}{dt}\sum_{i,j=1}^N \mu_{ij}\int_0^1 (\partial_x u_i)(\partial_x u_j)\,dx = \sum_{i=1}^N\int_0^1 \Big(-\rho_i v(\partial_t u_i)(\partial_x u_i) -$$
$$-K(\partial_x\rho^\gamma)(\partial_t u_i) + K(\partial_x\rho^\gamma)(\partial_{xx}u_i) + 2\rho_i v(\partial_x u_i)(\partial_{xx}u_i) - (\partial_x\rho_i)(\partial_t u_i)(\partial_x u_i)\Big)dx.$$

Since $\mathbf{M}>0$, we have

$$\text{the left-hand side of (38)} \geqslant B_{21}(C_0)\eta'(t) + \zeta'(t), \tag{39}$$

where

$$\zeta(t) = \frac{1}{2}\sum_{i=1}^N\int_0^1 \rho_i(\partial_x u_i)^2\,dx + \frac{1}{2}\sum_{i,j=1}^N \mu_{ij}\int_0^1 (\partial_x u_i)(\partial_x u_j)\,dx.$$

[4]Here, we also use the symmetry of the matrix $\mathbf{M}$.

We separately consider each term on the right-hand side of (38). By the Hölder inequality, the Cauchy inequality, and the estimate (33), we get

(40)
$$-\sum_{i=1}^{N}\int_0^1 \rho_i v(\partial_t u_i)(\partial_x u_i)\,dx \leqslant \sum_{i=1}^{N}\|v\|_{C[0,1]}\|\sqrt{\rho_i}\partial_t u_i\|_{L_2(0,1)}\|\sqrt{\rho_i}\partial_x u_i\|_{L_2(0,1)} \leqslant$$
$$\leqslant \frac{B_{21}}{10}\eta'(t) + B_{22}\,\zeta^2(t)\exp\{B_{22}\eta(t)\},$$

where $B_{22} = B_{22}(B_{19}, B_{21})$. By the Cauchy inequality, (33) and (37),

(41)
$$-K\sum_{i=1}^{N}\int_0^1 (\partial_x\rho^\gamma)(\partial_t u_i)\,dx \leqslant \frac{B_{21}}{10}\eta'(t) + B_{23}\exp\{B_{23}\eta(t)\},$$

where $B_{23} = B_{23}(B_{19}, B_{20}, B_{21}, K, N, \gamma)$. Similarly,

(42)
$$K\sum_{i=1}^{N}\int_0^1 (\partial_x\rho^\gamma)(\partial_{xx} u_i)\,dx \leqslant \frac{B_{21}}{10}\eta'(t) + B_{24}\exp\{B_{24}\eta(t)\},\ B_{24} = B_{24}(\text{arg. of } B_{23}).$$

Using the Hölder inequality, the Cauchy inequality, and the estimate (33), we find

(43)
$$2\sum_{i=1}^{N}\int_0^1 \rho_i v(\partial_x u_i)(\partial_{xx} u_i)\,dx \leqslant$$
$$\leqslant 2\sum_{i=1}^{N}\|v\|_{C[0,1]}\|\sqrt{\rho_i}\|_{C[0,1]}\|\sqrt{\rho_i}\partial_x u_i\|_{L_2(0,1)}\|\partial_{xx} u_i\|_{L_2(0,1)} \leqslant$$
$$\leqslant \frac{B_{21}}{10}\eta'(t) + B_{25}\,\zeta^2(t)\exp\{B_{25}\eta(t)\},\quad B_{25} = B_{25}(B_{19}, B_{21}).$$

Further, using the Cauchy inequality, the interpolation estimate

$$\|\partial_x u_i\|_{C[0,1]} \leqslant \sqrt{2}\|\partial_x u_i\|^{\frac12}_{L_2(0,1)}\|\partial_{xx} u_i\|^{\frac12}_{L_2(0,1)}$$

and the estimates (33), (37), we get

(44)
$$-\sum_{i=1}^{N}\int_0^1 (\partial_x\rho_i)(\partial_t u_i)(\partial_x u_i)\,dx \leqslant$$
$$\leqslant \sum_{i=1}^{N}\|\partial_x u_i\|_{C[0,1]}\|\partial_x\rho_i/\sqrt{\rho_i}\|_{L_2(0,1)}\|\sqrt{\rho_i}\partial_t u_i\|_{L_2(0,1)} \leqslant$$
$$\leqslant \frac{B_{21}}{10}\eta'(t) + B_{26}\,\zeta(t)\exp\{B_{26}\eta(t)\},\quad B_{26} = B_{26}(B_{19}, B_{20}, B_{21}).$$

Thus, from (40)–(44) it follows that

$$\text{the right-hand side of (38)} \leqslant \frac{B_{21}}{2}\eta'(t) + B_{27}(1+\zeta^2(t))\exp\{B_{27}\eta(t)\},$$
$$B_{27} = B_{27}(B_{22},\ldots,B_{25},B_{26}). \tag{45}$$

Combining (39) and (45), from (38) we obtain the inequality

$$\frac{B_{21}}{2}\eta'(t) + \zeta'(t) \leqslant B_{28}\exp\{B_{28}(B_{21}\eta(t)/2+\zeta(t))\}, \quad B_{28} = B_{28}(B_{21},B_{27}). \tag{46}$$

We take any $B_{29} > \zeta(0)$ (for example, $B_{29} = 2\zeta(0)$). Then for

$$t_0 = \min\left\{\frac{\exp\{-B_{28}\zeta(0)\} - \exp\{-B_{28}B_{29}\}}{B_{28}^2}, T\right\} \tag{47}$$

we derive from (46) that the estimate

$$\sup_{0\leqslant t\leqslant t_0}(\eta+\zeta) \leqslant \left(1+\frac{2}{B_{21}}\right)B_{30}, \quad B_{30} = \frac{1}{B_{28}}\ln\frac{1}{\exp\{-B_{28}\zeta(0)\} - B_{28}^2 t_0}$$

holds, which together with (6), (33) and (37) completes the of Lemma 4. We note that t_0 and B_{30} depend on the following values:

$$\left\{\|\rho_{0i}\|_{W_2^1(0,1)}\right\},\ \left\{\|u_{0i}\|_{W_2^1(0,1)}\right\},\ \left\{\inf_{[0,1]}\rho_{0i}\right\},\ \mathbf{M},\ N,\ T,\ K,\ \gamma.$$

4. Convergence of approximate solutions

Using Lemma 3, we construct solutions u_{im}, ρ_{im} to the problems (6)–(9) for all $m \in \mathbb{N}$, and extend them, if necessary, for $t \in (0,t_0)$ according to Lemma 4, and we can use the estimate (31) for them. Based on the estimate (31), we can extract a subsequence from u_{im}, ρ_{im}, $m \in \mathbb{N}$ (keeping the same notation) such that

$$\rho_{im} \to \rho_i \quad *-\text{weakly in } L_\infty\big(0,t_0;W_2^1(0,1)\big),$$

$$u_{im} \to u_i \quad *-\text{weakly in } L_\infty\big(0,t_0;W_2^1(0,1)\big) \text{ and weakly in } L_2\big(0,t_0;W_2^2(0,1)\big)$$

as $m\to\infty$ (for all $i=1,\ldots,N$). The other properties listed in (5) are also satisfied by this sequence in Q_{t_0} uniformly with respect to m. Consequently, the limit functions belong to the corresponding classes. We show that $(\rho_1,\ldots,\rho_N,u_1,\ldots,u_N)$ is a strong solution to the problem (1)–(4) on $(0,t_0)$.

By the Arzela–Ascoli theorem ([22], Theorem 1.70, P. 58) and the uniform estimates for ρ_{im}, u_{im} in $L_\infty\big(0,t_0;W_2^1(0,1)\big)$ and for $\partial_t\rho_{im}$, $\partial_t u_{im}$ in $L_2(Q_{t_0})$ (cf. (31)) we have (passing to a subsequence, if necessary, and keeping the same notation[5])

$$\rho_{im} \to \rho_i \text{ as } m\to\infty \text{ strongly in } C\big([0,t_0];L_2(0,1)\big), \quad i=1,\ldots,N, \tag{48}$$

$$u_{im} \to u_i \text{ as } m\to\infty \text{ strongly in } C\big([0,t_0];L_2(0,1)\big), \quad i=1,\ldots,N. \tag{49}$$

Since $\partial_t u_{im}$ are bounded in $L_2(Q_{t_0})$, an uniform estimate for $\partial_{tx}u_{im}$ in $L_2(0,t_0;W_2^{-1}(0,1))$ is valid, which, together with the estimate for $\partial_x u_{im}$ in $L_2(0,t_0;W_2^1(0,1))$ means via Lions–Aubin's lemma ([22], Theorem 1.71, P. 59)

$$\partial_x u_{im} \to \partial_x u_i \text{ as } m\to\infty \text{ strongly in } L_2(Q_{t_0}), \quad i=1,\ldots,N. \tag{50}$$

[5]Below this procedure is implied as necessary.

Hence (49) leads to

(51) $$u_{im} \to u_i \ \text{ as } \ m \to \infty \ \text{ strongly in } \ L_2\big(0, t_0; C[0,1]\big), \quad i = 1, \dots, N.$$

Thus, the limit functions ρ_i, u_i, $i = 1, \dots, N$, satisfy (almost everywhere in Q_{t_0}) the continuity equations (1), in which $v = \frac{1}{N}\sum\limits_{i=1}^{N} u_i$, the initial data (3) for almost all $x \in (0,1)$ and the boundary conditions (4) for almost all $t \in (0, t_0)$.

The boundedness of $\partial_t u_{im}$ in $L_2(Q_{t_0})$ implies the weak convergence of $\partial_t u_{im}$ to $\partial_t u_i$ in $L_2(Q_{t_0})$, which, together with (48) and the boundedness of $\rho_{im}\partial_t u_{im}$ in $L_2(Q_{t_0})$, implies the weak convergence of $\rho_{im}\partial_t u_{im}$ to $\rho_i \partial_t u_i$ in $L_2(Q_{t_0})$.

Further, from (48) and (50) it follows that

$$\rho_{im}\partial_x u_{im} \to \rho_i \partial_x u_i \ \text{ as } \ m \to \infty \ \text{ strongly in } \ L_2\big(0, t_0; L_1(0,1)\big), \quad i = 1, \dots, N,$$

and hence (51) leads to

(52)
$$(\rho_{im}\partial_x u_{im})u_{jm} \to (\rho_i \partial_x u_i)u_j \ \text{ as } \ m \to \infty \ \text{ strongly in } \ L_1(Q_{t_0}), \quad i, j = 1, \dots, N.$$

By (7), for any functions of the form

(53) $$\varphi_i = \sum_{k=1}^{M} \eta_k(t)\omega_k(x), \ \ \eta_k \in C[0, t_0], \ \ k = 1, \dots, M, \ \ M \leqslant m,$$

we have (for all $i = 1, \dots, N$) the inequalities

$$\int\limits_{Q_{t_0}} \Big(\rho_{im}\partial_t u_{im} + \rho_{im} v_m \partial_x u_{im} + K\partial_x \rho_m^{\gamma} - \sum_{j=1}^{N} \mu_{ij}\partial_{xx} u_{jm}\Big)\varphi_i \, dxdt = 0,$$

passing in which to the limit as $m \to \infty$ (by the proved convergences), we find (since the set of functions φ_i, $i = 1, \dots, N$ of the form (53) is everywhere dense in $L_2(Q_{t_0})$) the momentum equations (2) hold for the limit functions ρ_i, u_i, $i = 1, \dots, N$, almost everywhere in Q_{t_0}, with $\rho = \sum\limits_{i=1}^{N} \rho_i$. Thus, we have proved the existence of a strong solution to the problem (1)–(4) in small time.

5. Uniqueness of a strong solution

We assume that $\big(\rho_1^{(1)}, \dots, \rho_N^{(1)}, u_1^{(1)}, \dots, u_N^{(1)}\big)$ and $\big(\rho_1^{(2)}, \dots, \rho_N^{(2)}, u_1^{(2)}, \dots, u_N^{(2)}\big)$ are two strong solutions to the problem (1)–(4), $v^{(1,2)} = \frac{1}{N}\sum\limits_{i=1}^{N} u_i^{(1,2)}$, $\rho^{(1,2)} = \sum\limits_{i=1}^{N} \rho_i^{(1,2)}$. We set $\rho_i = \rho_i^{(1)} - \rho_i^{(2)}$, $u_i = u_i^{(1)} - u_i^{(2)}$, $i = 1, \dots, N$, $v = v^{(1)} - v^{(2)}$, $\rho = \rho^{(1)} - \rho^{(2)}$.

From (6), (8) we have (cf. (19))

(54) $$\partial_t \rho_i + \partial_x \left(\rho_i v^{(1)}\right) + \partial_x \left(\rho_i^{(2)} v\right) = 0, \quad \rho_i|_{t=0} = 0, \quad i = 1, \dots, N.$$

Multiplying (54) by $2\rho_i$, $i=1,\dots,N$, and integrating with respect to $x\in(0,1)$, we get

(55)
$$y_{1i}'(t)=-\int\limits_0^1\left(\rho_i^2\left(\partial_x v^{(1)}\right)+2\rho_i^{(2)}\rho_i(\partial_x v)+2\left(\partial_x\rho_i^{(2)}\right)\rho_i v\right)dx,\quad i=1,\dots,N,$$

where
$$y_{1i}(t)=\int\limits_0^1\rho_i^2\,dx,\quad i=1,\dots,N.$$

The terms on the right-hand side of (55) can be estimated as follows:
$$-\int\limits_0^1\rho_i^2\left(\partial_x v^{(1)}\right)dx\leqslant y_{1i}(t)\sum_{j=1}^N\|\partial_x u_j^{(1)}\|_{L_\infty(0,1)},\quad i=1,\dots,N,$$
$$-2\int\limits_0^1\rho_i^{(2)}\rho_i(\partial_x v)\,dx\leqslant\|\rho_i^{(2)}\|^2_{L_\infty(Q_T)}y_{1i}(t)+\sum_{j=1}^N\int\limits_0^1|\partial_x u_j|^2\,dx,\quad i=1,\dots,N,$$
$$-2\int\limits_0^1\left(\partial_x\rho_i^{(2)}\right)\rho_i v\,dx\leqslant y_{1i}(t)+\|v\|^2_{L_\infty(0,1)}\|\partial_x\rho_i^{(2)}\|^2_{L_\infty\left(0,T;L_2(0,1)\right)}\leqslant$$
$$\leqslant y_{1i}(t)+\|\partial_x\rho_i^{(2)}\|^2_{L_\infty\left(0,T;L_2(0,1)\right)}\sum_{j=1}^N\int\limits_0^1|\partial_x u_j|^2\,dx,\quad i=1,\dots,N.$$

By the inclusions
$$\rho_i^{(2)}\in L_\infty(Q_T),\quad \partial_x\rho_i^{(2)}\in L_\infty\big(0,T;L_2(0,1)\big),$$
$$\partial_x u_i^{(1)}\in L_2\big(0,T;L_\infty(0,1)\big),\quad i=1,\dots,N,$$
we obtain the estimates for the functions $y_{1i}(t)$, $i=1,\dots,N$:
$$y_{1i}'(t)\leqslant B_{31}(t)y_{1i}(t)+B_{32}\sum_{j=1}^N\int\limits_0^1|\partial_x u_j|^2\,dx,\quad i=1,\dots,N,$$
where $B_{31}=B_{31}\left(\left\{\|\partial_x u_j^{(1)}\|_{L_\infty(0,1)}\right\},\left\{\|\rho_j^{(2)}\|_{L_\infty(Q_T)}\right\}\right)$, $B_{31}\in L_2(0,T)$, $B_{32}=B_{32}\left(\left\{\|\partial_x\rho_j^{(2)}\|_{L_\infty\left(0,T;L_2(0,1)\right)}\right\}\right)$. Since $y_{1i}(0)=0$, $i=1,\dots,N$ (in view of the initial conditions in (54)), we can apply the Gronwall inequality to get

$$y_{1i}(t)\leqslant B_{33}\left(B_{32},\|B_{31}\|_{L_1(0,T)}\right)\sum_{j=1}^N\int\limits_0^t\int\limits_0^1|\partial_x u_j|^2\,dxd\tau,\quad i=1,\dots,N. \tag{56}$$

Further, from the equations (2) and boundary conditions (4) we get (cf. (23))
$$\frac12\sum_{i=1}^N\frac{d}{dt}\int\limits_0^1\rho_i^{(1)}u_i^2\,dx+\sum_{i,j=1}^N\mu_{ij}\int\limits_0^1(\partial_x u_i)(\partial_x u_j)\,dx=$$

$$= K \sum_{i=1}^{N} \int_0^1 \left(\left(\rho^{(1)} \right)^\gamma - \left(\rho^{(2)} \right)^\gamma \right) (\partial_x u_i) \, dx - \sum_{i=1}^{N} \int_0^1 \rho_i u_i \left(\partial_t u_i^{(2)} \right) dx -$$

$$- \sum_{i=1}^{N} \int_0^1 \rho_i^{(1)} v u_i \left(\partial_x u_i^{(2)} \right) dx - \sum_{i=1}^{N} \int_0^1 \rho_i v^{(2)} u_i \left(\partial_x u_i^{(2)} \right) dx,$$

which implies

(57)

$$y_2'(t) + y_3'(t) \leqslant K \sum_{i=1}^{N} \int_0^1 \left(\left(\rho^{(1)} \right)^\gamma - \left(\rho^{(2)} \right)^\gamma \right) (\partial_x u_i) \, dx - \sum_{i=1}^{N} \int_0^1 \rho_i u_i \left(\partial_t u_i^{(2)} \right) dx -$$

$$- \sum_{i=1}^{N} \int_0^1 \rho_i^{(1)} v u_i \left(\partial_x u_i^{(2)} \right) dx - \sum_{i=1}^{N} \int_0^1 \rho_i v^{(2)} u_i \left(\partial_x u_i^{(2)} \right) dx,$$

where

$$y_2(t) = \frac{1}{2} \sum_{i=1}^{N} \int_0^1 \rho_i^{(1)} u_i^2 \, dx, \quad y_3(t) = C_0 \sum_{i=1}^{N} \int_0^t \int_0^1 |\partial_x u_i|^2 \, dx d\tau.$$

We estimate the terms on the right-hand side of (57) as follows:

$$K \sum_{i=1}^{N} \int_0^1 \left(\left(\rho^{(1)} \right)^\gamma - \left(\rho^{(2)} \right)^\gamma \right) (\partial_x u_i) \, dx \leqslant \frac{C_0}{4} \sum_{i=1}^{N} \|\partial_x u_i\|_{L_2(0,1)}^2 +$$

$$+ B_{34} \left(C_0, K, N, \gamma, \left\{ \|\rho_j^{(1,2)}\|_{L_\infty(Q_T)} \right\} \right) \sum_{i=1}^{N} y_{1i}(t),$$

$$- \sum_{i=1}^{N} \int_0^1 \rho_i u_i \left(\partial_t u_i^{(2)} \right) dx \leqslant$$

$$\leqslant \frac{C_0}{4} \sum_{i=1}^{N} \|\partial_x u_i\|_{L_2(0,1)}^2 + B_{35} \left(C_0, \left\{ \|\partial_t u_j^{(2)}\|_{L_2(0,1)} \right\} \right) \sum_{i=1}^{N} y_{1i}(t),$$

$$- \sum_{i=1}^{N} \int_0^1 \rho_i^{(1)} v u_i \left(\partial_x u_i^{(2)} \right) dx \leqslant$$

$$\leqslant B_{36} \left(\left\{ \|\partial_x u_j^{(2)}\|_{L_\infty(0,1)} \right\}, \left\{ \sup_{[0,1]} \rho_{0j} \right\}, \left\{ \inf_{[0,1]} \rho_{0j} \right\} \right) y_2(t),$$

$$- \sum_{i=1}^{N} \int_0^1 \rho_i v^{(2)} u_i \left(\partial_x u_i^{(2)} \right) dx \leqslant B_{37} \left(\left\{ \|u_j^{(2)}\|_{L_\infty(Q_T)} \right\} \right) \sum_{i=1}^{N} y_{1i}(t) +$$

$$+ B_{38} \left(\left\{ \|\partial_x u_j^{(2)}\|_{L_\infty(0,1)} \right\}, \left\{ \|1/\rho_j^{(1)}\|_{L_\infty(Q_T)} \right\} \right) y_2(t),$$

where $B_{35}, B_{38} \in L_1(0,T)$, $B_{36} \in L_2(0,T)$. Here, in the third estimate, we have used the relations

$$\inf_{[0,1]} \frac{\rho_{0i}(x)}{\rho_0(x)} \leqslant \frac{\rho_i(x,t)}{\rho(x,t)} \leqslant \sup_{[0,1]} \frac{\rho_{0i}(x)}{\rho_0(x)} \leqslant 1 \quad \text{as} \quad (x,t) \in [0,1]\times[0,T], \quad i=1,\dots,N,$$

which can be easily verified via the transport equations for the concentrations ρ_i/ρ.

Hence, from (57), using the relation

$$\sum_{i=1}^{N} y_{1i}(t) \leqslant B_{39}(C_0, B_{33}, N) y_3(t)$$

proved above (cf. (56)), we deduce

$$y_2'(t) + \frac{1}{2} y_3'(t) \leqslant B_{40}(B_{34},\dots,B_{39}) \left(y_2(t) + \frac{1}{2} y_3(t) \right),$$

where $B_{40} \in L_1(0,T)$. Taking into account that $y_2(0) = y_3(0) = 0$, we obtain the identities $y_{1i} \equiv y_2 \equiv y_3 \equiv 0$, $i=1,\dots,N$, which complete the proof of Theorem 2.

In another paper we plan to obtain global-in-time a priori estimates of solutions, which would provide (basing on Theorem 2) the global solvability of the problem (1)–(4).

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