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CLASSIFICATION OF MAXIMAL SUBGROUPS OF ODD INDEX IN FINITE SIMPLE CLASSICAL GROUPS: ADDENDUM

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Abstract. A classification of maximal subgroups of odd index in finite simple groups was obtained by M. Liebeck and J. Saxl and, independently, by W. Kantor in 1980*s*. In the cases of alternating groups and classical groups of Lie type over fields of odd characteristics, the classification was not complete.

The classification was completed by the author in 2008. In the cases of finite simple classical groups of Lie type we referred to results obtained in P. Kleidman's PhD thesis. However, it turned out that the thesis contains a number of flaws that were corrected by J. Bray, D. Holt, and C. Roney–Dougal in 2013. Due to uncovered circumstances, in this note we provide a revision of our classification.

Keywords: primitive permutation group, finite simple group, classical group, maximal subgroup, odd index.

1. Introduction

M. Liebeck and J. Saxl [8] and, independently, W. Kantor [5] have obtained a classification of finite primitive permutation groups of odd degree. It is considered to be one of remarkable results in the theory of finite permutation groups.

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Both papers [8] and [5] contain certain lists of subgroups of finite simple groups that can be maximal subgroups of odd index. However, in the cases of alternating groups and classical groups of Lie type over fields of odd characteristics, neither in [8] nor in [5] it was described which of the specified subgroups are precisely maximal subgroups of odd index. Thus, the problem of the complete classification of maximal subgroups of odd index in finite simple groups remained open.

The classification was completed by the author in [9, 10]. In [9] we referred to results obtained by P. Kleidman [6] and by P. Kleidman and M. Liebeck [7]. However, there are a number of flaws in Kleidman's PhD thesis [6]. These flaws have been corrected in [2]. Due to uncovered circumstances, in this note we provide a revision of our classification of maximal subgroups of odd index in finite simple classical groups over fields of odd characteristics, which was obtained in [9]. In particular, we have made changes in statements of items (6), (10), and (21) of [9, Theorem 1].

The note is organized as follows. In Section 2 we give general definitions and some basic terminology. In Section 3 we formulate the main theorem. Section 4 is devoted to the proof of the main result.

2. Terminology and Notation

Our terminology and notation are mostly standard and can be found in [2, 4, 7, 8]. Throughout the paper we consider only finite groups; by a group everywhere we mean a finite group.

The cyclic group of order n is denoted by $\mathbb{Z}_n$ or (particularly when as a component of a group structure) just by n. An elementary abelian group of order p^n is denoted by E_{p^n} or just by p^n. By $[n]$ we denote a group of order n, of unspecified structure. For elementary abelian groups A we write A^{m+n} to mean a group with an elementary abelian normal subgroup A^m such that the quotient is isomorphic to A^n. For n even, D_n denotes the dihedral group of order n. If G and H are groups, then $G \times H$ is the direct product of G and H, and we write $G.H$ (or $G:H$) to denote an extension (a split extension, respectively) of G by H.

Denote by $Soc(G)$ the *socle* of a group G (i. e. the subgroup of G generated by the non-trivial minimal normal subgroups of G). Recall that G is *almost simple* if $Soc(G)$ is a nonabelian simple group.

We use the notation (m, n) for the greatest common divisor of integers m and n.

The greatest integer power of 2 dividing a positive integer k is called the 2-*part* of k and is denoted by k_2.

Let m and n be two non-negative integers with $m = \sum_{i=0}^{\infty} a_i \cdot 2^i$ and $n = \sum_{i=0}^{\infty} b_i \cdot 2^i$, where $a_i, b_i \in \{0,1\}$. We write $m \preceq n$ if $a_i \leq b_i$ for every i. It is clear that if $m \preceq n$, then $m \leq n$ and $m \neq n/2$. Moreover, $m \preceq n$ if and only if $n - m \preceq n$.

Let q be a positive integer power of an odd prime p. Assume that G is one of the simple classical groups $PSL_n(q)$, $PSU_n(q)$, $PSp_n(q)$, where n is even, $P\Omega_n(q)$, where n is odd, or $P\Omega_n^{\varepsilon}(q)$, where n is even and $\varepsilon \in \{+,-\}$. We denote by V the *natural projective module* of G, i.e. the vector space of dimension n over a field F with the corresponding bilinear form $\mathfrak{f}$, where $F = \mathbb{F}_{q^2}$ if G is unitary and $F = \mathbb{F}_q$ if G is linear, symplectic, or orthogonal. Note that if $\mathfrak{f}$ is non-degenerate, then for every non-degenerate subspace U of V we have (see [7, Ch. 2])

$$V = U \oplus U^{\perp}. \tag{1}$$

In the case of a non-degenerate symmetric bilinear form $\mathfrak{f}$ on V, the *discriminant* $D(V)$ is defined (the exact definition see in [7, Ch. 2]). We use the notation distinct from [7]: we write

$$D(V) = \begin{cases} 1 \text{ if D(V) is a square in } F, \\ -1 \text{ if } D(V) \text{ is a non-square in } F. \end{cases}$$

In the case of the group $P\Omega_n^{\varepsilon}(q)$ for even n, the parameter $\varepsilon \in \{+, -\}$ is called the *sign* of this group and the corresponding vector space V; this parameter is denoted by $sgn(V)$. For every non-degenerate subspace U of even dimension m of V, the sign $\upsilon = sgn(U)$ of U is defined, and the following formula holds (see [7, Proposition 2.5.10])

$$D(U) = D_m^{\upsilon}(q) = \begin{cases} 1 \text{ if } \upsilon = + \text{ and } (q-1)m/4 \text{ is even,} \\ 1 \text{ if } \upsilon = - \text{ and } (q-1)m/4 \text{ is odd,} \\ -1 \text{ otherwise.} \end{cases} \tag{2}$$

Moreover, it is known (see [7, Proposition 2.5.11]) that if U is a non-degenerate subspace of V, then

$$D(V) = D(U) \cdot D(U^{\perp}), \tag{3}$$

and

(4) $sqn(V) = sgn(U) \cdot sgn(U^{\perp})$ whenever the dimensions of U and V are even.

Using the classification of finite simple groups, M. Aschbacher described in [1] a large family of natural geometrically defined subgroups of simple classical groups. He has subdivided this family into eight classes C_i for $1 \leq i \leq 8$, which are now called Aschbacher classes, and has proved that if a maximal subgroup of a simple classical group does not belong to the union of Aschbacher classes of the group, then this maximal subgroup is almost simple (for details see [1]). For a given simple group G of dimension n and its subgroup $H \in \bigcup_{i=1}^{8} C_i(G)$, the normal structure of H and whether H is maximal in G can be found in [2] if $n \leq 12$ and in [7] if $n \geq 13$.

3. Main Result

We prove the following theorem.

Theorem. *Let q be a positive integer power of an odd prime p and G be one of the simple classical groups $PSL_n(q)$, where $n \geq 2$, $PSU_n(q)$, where $n \geq 3$, $PSp_n(q)$, where $4 \leq n$ is even, $P\Omega_n(q)$, where $7 \leq n$ is odd, or $P\Omega_n^{\varepsilon}(q)$, where $8 \leq n$ is even and $\varepsilon \in \{+, -\}$; and let V be the natural projective module of G. A subgroup H of G is maximal of odd index if and only if one of the following statements holds*:

(1) *$H = C_G(\sigma)$ for a field automorphism σ of odd prime order r of the group G, where $q = p^{kr}$ and $q_0 = p^k$ for some positive integer k*;

(2) *$G = PSL_n(q)$, H is the stabilizer of a subspace of dimension m of the space V, and $m \preceq n$*;

(3) $G = PSU_n(q)$ *or* $G = PSp_n(q)$, *H is the stabilizer of a non-degenerate subspace of dimension m of the space V, and* $m \preceq n$;

(4) $G = P\Omega_n(q)$, *H is the stabilizer of a non-degenerate subspace U of even dimension m of the space V,* $D(U) = 1$, *and* $m \preceq n$;

(5) $G = P\Omega_n^\varepsilon(q)$, *H is the stabilizer of a non-degenerate subspace U of dimension m of the space V, and one of the following conditions holds*:

(*i*) *m is odd,* $D(V) = -1$, *and* $m - 1 \preceq n - 2$;

(*ii*) *m is even,* $D(U) = D(V) = -1$, $m - 2 \preceq n - 2$, $(q, m, sgn(U)) \neq (3, 2, +)$, *and if* $m = n/2$, *then* $\varepsilon = -$;

(*iii*) *m is even,* $D(U) = D(V) = 1$, *and* $m \preceq n$;

(6) $G = PSL_n(q)$, *H is the stabilizer of a decomposition* $V = \bigoplus V_i$ *into a direct sum of subspaces of dimension m, and one of the following conditions holds*:

(*i*) $m = 2^w \geq 2$ *and* $(n, m, q) \neq (4, 2, 3)$;

(*ii*) $m = 1$, $q \equiv 1 \pmod 4$, $(n, q) \neq (4, 5)$, *and* $q \geq 13$ *in the case* $n = 2$;

(7) $G = PSU_n(q)$, *H is the stabilizer of an orthogonal decomposition* $V = \bigoplus V_i$ *into a direct sum of pairwise isometric subspaces* V_i *of dimension m, and one of the following conditions holds*:

(*i*) $m = 2^w \geq 2$;

(*ii*) $m = 1$, $q \equiv 3 \pmod 4$, *and* $(n, q) \neq (4, 3)$;

(8) $G = PSp_n(q)$, *H is the stabilizer of an orthogonal decomposition* $V = \bigoplus V_i$ *into a direct sum of pairwise isometric subspaces* V_i *of dimension m, and* $m = 2^w \geq 2$;

(9) $G = P\Omega_n(q)$, *H is the stabilizer of an orthogonal decomposition* $V = \bigoplus V_i$ *into a direct sum of pairwise isometric subspaces* V_i *of dimension* 1, *q is prime, and* $q \equiv \pm 3 \pmod 8$;

(10) $G = P\Omega_n^\varepsilon(q)$, *H is the stabilizer of an orthogonal decomposition* $V = \bigoplus V_i$ *into a direct sum of pairwise isometric subspaces* V_i *of dimension m,* $D(V) = 1$, *and one of the following conditions holds*:

(*i*) $m = 2^w \geq 2$, $D(V_i) = 1$, *and* $(m, q, sgn(V_i)) \notin \{(2, 3, -), (2, 5, +)\}$;

(*ii*) $(n, \varepsilon) \neq (8, +)$, $m = 1$, *q is prime, and* $q \equiv \pm 3 \pmod 8$;

(11) $G = PSL_2(q)$ *and* $H \cong PGL_2(q_0)$, *where* $q = p^{2k}$ *and* $q_0 = p^k$ *for some positive integer k*;

(12) $G = PSL_2(q)$ *and* $H \cong A_4$, *where q is prime and either* $q = 5$ *or* $q \equiv \pm 3, \pm 13 \pmod{40}$;

(13) $G = PSL_2(q)$ *and* $H \cong S_4$, *where q is prime and* $q \equiv \pm 7 \pmod{16}$;

(14) $G = PSL_2(q)$ *and* $H \cong A_5$, *where q is prime and* $q \equiv \pm 11, \pm 19 \pmod{40}$;

(15) $G = PSL_2(q)$ *and* $H \cong D_{q+1}$, *where* $7 < q \equiv 3 \pmod 4$;

(16) $G = PSU_3(5)$ *and* $H \cong M_{10}$;

(17) $G = PSL_4(q)$ *and* $H \cong 2^4.A_6$, *where q is prime and* $q \equiv 5 \pmod 8$;

(18) $G = PSL_4(q)$ *and* $H \cong PSp_4(q).2$, *where* $q \equiv 3 \pmod 4$;

(19) $G = PSU_4(q)$ *and* $H \cong 2^4.A_6$, *where q is prime and* $q \equiv 3 \pmod 8$;

(20) $G = PSU_4(q)$ *and* $H \cong PSp_4(q).2$, *where* $q \equiv 1 \pmod 4$;

(21) $G = PSp_4(q)$ *and* $H \cong 2^4.A_5$, *where q is prime and* $q \equiv \pm 3 \pmod 8$;

(22) $G = P\Omega_7(q)$ *and* $H \cong \Omega_7(2)$, *where q is prime and* $q \equiv \pm 3 \pmod 8$;

(23) $G = P\Omega_8^+(q)$ *and* $H \cong \Omega_8^+(2)$, *where q is prime and* $q \equiv \pm 3 \pmod 8$.

4. Proof of Theorem

Let G be a simple classical group over a field of odd characteristic, V be the natural projective module of dimension n of G, and H be a subgroup of G.

Using [7, 5], we can easily deduce the following lemma.

Lemma 1. *If H is a maximal subgroup of odd index in G, then one of the following statements holds*:

(a) $H = N_G(C_G(\sigma))$ *for a field automorphism σ of odd prime order r of the group G*;

(b) *H is the stabilizer of a non-degenerate* (*arbitrary if $G = PSL_n(q)$*) *subspace of dimension m of V*;

(c) *H is the stabilizer of an orthogonal* (*arbitrary if $G = PSL_n(q)$*) *decomposition* $V = \bigoplus_{i=1}^{n/m} V_i$ *with all V_i isometric of dimension m*;

(d) *G is $P\Omega_7(q)$ or $P\Omega_8^+(q)$ and H is $P\Omega_7(2)$ or $P\Omega_8^+(2)$, respectively, q is prime, and* $q \equiv \pm 3 \pmod 8$;

(e) *$G = PSL_2(q)$ and H is a dihedral group, A_4, S_4, A_5, or $PGL_2(q^{1/2})$*;

(f) *$G = PSU_3(5)$ and $H \cong M_{10}$.*

Remark. *Note that results of* [8] *and* [5] *were formulated for orthogonal groups, but not for their isomorphic copies of other types. The maximal subgroups of groups $PSL_4(q)$, $PSU_4(q)$, and $PSp_4(q)$ are known* [2, Tables 8.8–8.13], *and in order to prove Theorem we should consider these groups separately. However, it is not difficult to obtain similar results for the corresponding orthogonal groups.*

If H is a subgroup from statement (a) of Lemma 1, then $C_G(\sigma)$ is a maximal subgroup of G in view of [3], and $|G : H|$ is odd in view of [8]. Thus, $H = C_G(\sigma)$, and statement (1) of Theorem holds.

Further, we consider simple linear, unitary, symplectic, and orthogonal groups separately. For groups of each type we chose subgroups form items (b)–(f) of Lemma 1 which are maximal of odd index. At the same time, for linear, unitary, and symplectic groups of dimension 4 we consider all their maximal subgroups and verify whether they are of odd index. We use [2, Tables 8.1–8.87] to establish the maximality of subgroups in classical groups of dimension at most 12 and [7, Tables 3.5.A-F] in classical groups of dimension at least 13.

1. Assume that $G = PSL_n(q)$, where $n \geq 2$, q is odd, and $(n, q) \neq (2, 3)$.

1.1. If H is the stabilizer of a subspace of dimension m of V, then, in view of [9, Theorem 2], $|G : H|$ is odd if and only if $n \succeq m$. In view of Tables 8.1, 8.3, 8.8, 8.18, 8.24, 8.35, 8.44, 8.54, 8.60, 8.70, and 8.76 of [2] and Table 3.5.A of [7], H is maximal in G. Thus, statement (2) of Theorem holds.

1.2. If H is the stabilizer of a decomposition $V = \bigoplus_{i=1}^{n/m} V_i$ with all V_i of dimension m, then, in view of [9, Theorem 7], $|G : H|$ is odd if and only if either $m = 2^w \geq 2$ or $m = 1$ and $q \equiv 1 \pmod 4$. In view of Tables 8.1, 8.3, 8.8, 8.18, 8.24, 8.35, 8.44, 8.54, 8.60, 8.70, and 8.76 of [2] and Table 3.5.A of [7], H is non-maximal in G if and only if $n = 2$ and $q \leq 11$, or $m = 1$ and $q = 3$, or $(n, m, q) \in \{(4, 1, 5), (4, 2, 3)\}$. Thus, statement (6) of Theorem holds.

1.3. Consider additional possibilities for H when $n = 2$.

1.3.1. If $H \cong PGL_2(q_0)$, where $q = q_0^2$, then in view of [2, Tables 8.1, 8.7], H is maximal in G, and it is easy to see that $|G : H| = (q_0^2 + 1)/2$ is odd. Thus, statement (11) of Theorem holds.

1.3.2. Let $H \cong A_4$. In view of [2, Tables 8.1, 8.7], G contains a maximal subgroup H isomorphic to A_4 if and only if q is prime and either $q = 5$ or $q \equiv \pm 3, \pm 13 \pmod{40}$. It is clear that in this case $|G : H|$ is odd. Thus, statement (12) of Theorem holds.

1.3.3. Let $H \cong S_4$. In view of [2, Tables 8.1, 8.7], G contains a maximal subgroup H isomorphic to S_4 if and only if q is prime and $q \equiv \pm 1 \pmod 8$. It is clear that in this case $|G : H|$ is odd if and only if $q \equiv \pm 7 \pmod{16}$. Thus, statement (13) of Theorem holds.

1.3.4. Let $H \cong A_5$. In view of [2, Tables 8.2, 8.7], G contains a maximal subgroup H isomorphic to A_5 if and only if either q is prime and $q \equiv \pm 1 \pmod{10}$ or $q = p^2$, where p is prime and $p \equiv \pm 3 \pmod{10}$. It is clear that in the latter $|G : H|$ is even. If q is prime and $q \equiv \pm 1 \pmod{10}$, then $|G : H|$ is odd if and only if $q \equiv \pm 11, \pm 19 \pmod{40}$. Thus, statement (14) of Theorem holds.

1.3.5. If H is a dihedral group, then, in view of [2, Tables 8.1, 8.7], either $H \cong D_{q-1}$, and statement (6) of Theorem holds; or $H \cong D_{q+1}$. In the latter H is maximal in G if and only if $q \notin \{7, 9\}$. It is clear that $|G : H| = q(q-1)/2$ is odd if and only if $q \equiv 3 \pmod 4$. Thus, statement (15) of Theorem holds.

1.4. Consider possibilities for H when $n = 4$. In view of the Aschbacher theorem [1] and [2, Tables 8.8, 8.9], any maximal subgroup of G is either contained in $\bigcup_{i \in \{1,2,3,5,6,8\}} C_i(G)$ or almost simple.

1.4.1. $H \in C_1(G)$ and H is maximal in G. Then, in view of [2, Table 8.8], H is the stabilizer of a subspace of V, and statement (2) of Theorem holds.

1.4.2. $H \in C_2(G)$. In view of [2, Table 8.8], H is the stabilizer of a decomposition $V = \bigoplus_{i=1}^{n/m} V_i$ with all V_i of dimension m, and statement (6) of Theorem holds.

1.4.3. $H \in C_3(G)$. In view of [2, Table 8.8], the preimage of H in $SL_4(q)$ is isomorphic to $SL_2(q^2).\mathbb{Z}_{q+1}.\mathbb{Z}_2$, and

$$|G : H|_2 = \frac{(q^4-1)_2(q^3-1)_2(q^2-1)_2}{2(q^4-1)_2(q+1)_2} \geq 2.$$

1.4.4. $H \in C_5(G)$. In view of [2, Table 8.8], $H = C_G(\sigma)$ for a field automorphism σ of prime order r of the group G. If r is odd, then statement (1) of Theorem holds. If $r = 2$, then, in view of [2, Table 8.8], the preimage of H in $SL_4(q)$ is isomorphic to $SL_4(q_0).[(q_0+1, 4)]$ for $q = q_0^2$, and

$$|G : H|_2 = \frac{(q^4-1)_2(q^3-1)_2(q^2-1)_2}{(q_0^4-1)_2(q_0^3-1)_2(q_0^2-1)_2(q_0+1,4)} \geq 2.$$

1.4.5. $H \in C_6(G)$. In view of [2, Table 8.8], G contains a maximal subgroup $H \in C_6(G)$ if and only if q is prime and $q \equiv 1 \pmod 4$. In view of [7, Proposition 4.6.6],

$$H \cong \begin{cases} 2^4.S_6 \text{ if } q \equiv 1 \pmod 8, \\ 2^4.A_6 \text{ if } q \equiv 5 \pmod 8). \end{cases}$$

Note that we have

$$|G:H|_2 = \begin{cases} \frac{(q^4-1)_2(q^3-1)_2(q^2-1)_2}{2^{10}} > 2 \text{ if } q \equiv 1 \pmod 8, \\ \frac{(q^4-1)_2(q^3-1)_2(q^2-1)_2}{2^9} = 1 \text{ if } q \equiv 5 \pmod 8. \end{cases}$$

Thus, statement (17) of Theorem holds.

1.4.6. $H \in C_8(G)$. In view of [2, Table 8.8] and [7, Proposition 4.8.3], either H is isomorphic to $PSp_4(q).[\frac{(q-1,2)^2}{(q-1,4)}]$ or the preimage of H in $SL_4(q)$ is isomorphic to one of the following groups: $SO_4^\varepsilon(q).[(q-1,4)]$ for $\varepsilon \in \{+,-\}$, $SU_4(q_0).\mathbb{Z}_{(q_0-1,4)}$, where $q = q_0^2$.

In view of [2, Table 8.8], G always contains a maximal subgroup

$$H \cong PSp_4(q).\left[\frac{(q-1,2)^2}{(q-1,4)}\right],$$

and

$$|G:H|_2 = \frac{2(q^4-1)_2(q^3-1)_2(q^2-1)_2(q-1,4)}{(q^4-1)_2(q^2-1)_2(q-1,4)(q-1,2)^2} = \frac{(q-1)_2}{2}.$$

Now it is easy to see that $|G:H|_2 = 1$ if and only if $q \equiv 3 \pmod 4$. Thus, $H \cong PSp_4(q).\mathbb{Z}_2$, and statement (18) of Theorem holds.

If the preimage of H in $SL_4(q)$ is isomorphic to $SO_4^\varepsilon(q).[(q-1,4)]$ for $\varepsilon \in \{+,-\}$, then

$$|G:H|_2 = \frac{(q^4-1)_2(q^3-1)_2(q^2-1)_2}{(q-1,4)(q^2-\varepsilon 1)_2(q^2-1)_2} \geq \frac{(q^4-1)_2}{(q^2-\varepsilon 1)_2} \geq 2.$$

If $q = q_0^2$ and the preimage of H in $SL_4(q)$ is isomorphic to $SU_4(q_0).\mathbb{Z}_{(q_0-1,4)}$, then

$$|G:H|_2 = \frac{(q^4-1)_2(q^3-1)_2(q^2-1)_2}{(q_0^4-1)_2(q_0^3+1)_2(q_0^2-1)_2(q_0-1,4)} \geq \frac{(q_0^4+1)_2(q_0^3-1)_2(q_0^2+1)_2}{4} \geq 2$$

1.4.7. H is a maximal almost simple subgroup of G and $H \notin \bigcup_{i\in\{1,2,3,5,6,8\}} C_i(G)$. In view of [2, Table 8.9], $H \in \{PSL_2(7), A_7, PSU_4(2)\}$. So, $|H|_2 \leq 2^6$ in view of [4], and it is easy to see that

$$|G|_2 = \frac{(q^4-1)_2(q^3-1)_2(q^2-1)_2}{(q-1,4)} \geq 2^7.$$

Thus, $|G:H|$ is even.

2. Assume that $G = PSU_n(q)$, where $n \geq 3$ and q is odd.

2.1. Let H be the stabilizer of a non-degenerate subspace U of dimension m of V. Note that, in view of (1), H is the stabilizer of $U^\perp$, and we may assume that $m \leq n/2$. Moreover, if $m = n/2$, then H is contained in a subgroup which will be considered below, thus, we may assume that $m < n/2$. In view of [9, Theorem 3], $|G:H|$ is odd if and only if $n \succeq m$. Since $m \neq n/2$, in view of Tables 8.5, 8.10, 8.20, 8.26, 8.37, 8.46, 8.56, 8.62, 8.72, and 8.78 of [2] and Table 3.5.B of [7], H is maximal in G. Thus, statement (3) of Theorem holds for unitary groups.

2.2. If H is the stabilizer of an orthogonal decomposition $V = \bigoplus_{i=1}^{n/m} V_i$ with all V_i isometric of dimension m, then, in view of [9, Theorem 8], $|G:H|$ is odd if and only if either $m = 2^w \geq 2$ or $m = 1$ and $q \equiv 3 \pmod 4$. In view of Tables 8.5, 8.10, 8.20, 8.26, 8.37, 8.46, 8.56, 8.62, 8.72, and 8.78 of [2] and Table 3.5.B of [7], H is non-maximal if and only if $m = 1$ and $(n, q) \in \{(3, 5), (4, 3)\}$. Thus, statement (7) of Theorem holds.

2.3. Consider the additional possibility for H when $n = 3$. If $G = PSU_3(5)$ and $H \cong M_{10}$, then, in view of [4], H is a maximal subgroup of odd index in G. Thus, statement (16) of Theorem holds.

2.4. Consider possibilities for H when $n = 4$. In view of the Aschbacher theorem [1] and [2, Tables 8.10, 8.11], any maximal subgroup of G is either contained in $\bigcup_{i\in\{1,2,5,6\}} C_i(G)$ or almost simple.

2.4.1. $H \in C_1(G)$. In view of [2, Table 8.10], either H is the stabilizer of a non-degenerate subspace of V, and statement (3) of Theorem holds, or H is the stabilizer of a totally singular subspace of V of dimension m. In the last case $1 \le m \le 2$, and, in view of [2, Table 8.10], the preimage of H in $SU_4(q)$ is isomorphic to $E_q^{1+4}.SU_2(q) : \mathbb{Z}_{q^2-1}$ for $m = 1$ or to $E_q^4 : SL_2(q^2) : \mathbb{Z}_{q-1}$ for $m = 2$, and

$$|G : H|_2 = \begin{cases} \frac{(q^4-1)_2(q^3+1)_2(q^2-1)_2}{(q^2-1)_2^2} \ge 2 \text{ for } m = 1, \\ \frac{(q^4-1)_2(q^3+1)_2(q^2-1)_2}{(q^4-1)_2(q-1)_2} \ge 2 \text{ for } m = 2. \end{cases}$$

2.4.2. $H \in C_2(G)$. In view of [2, Table 8.10], either H is the stabilizer of an orthogonal decomposition $V = \bigoplus_{i=1}^{n/m} V_i$ with all V_i isometric of dimension m, and statement (7) of Theorem holds, or H is the stabilizer of a decomposition $V = V_1 \oplus V_2$, where V_1 and V_2 are totally singular subspaces of V of dimension 2. In the latter, in view of [2, Table 8.10], the preimage of H in $SU_4(q)$ is isomorphic to $SL_2(q^2).\mathbb{Z}_{q-1}.\mathbb{Z}_2$, and

$$|G : H|_2 = \frac{(q^4-1)_2(q^3+1)_2(q^2-1)_2}{2(q^4-1)_2(q-1)_2} \ge 2.$$

2.4.3. $H \in C_5(G)$. In view of [2, Table 8.10] and [7, Proposition 4.5.6], there are the following possibilities: $H = C_G(\sigma)$ for a field automorphism σ of prime odd order of the group G, and statement (1) of Theorem holds; H is isomorphic to $PSp_4(q).\left[\frac{(q+1,2)(q-1,2)}{(q+1,4)}\right]$; the preimage of H in $SU_4(q)$ is isomorphic to $SO_4^{\varepsilon}(q).[(q+1,4)]$ for $\varepsilon \in \{+,-\}$.

In view of [2, Table 8.10], G always contains a maximal subgroup

$$H \cong PSp_4(q).\left[\frac{(q+1,2)(q-1,2)}{(q+1,4)}\right],$$

and

$$|G : H|_2 = \frac{2(q^4-1)_2(q^3+1)_2(q^2-1)_2(q+1,4)}{(q+1,4)(q^4-1)_2(q^2-1)_2(q+1,2)(q-1,2)} = \frac{(q+1)_2}{2}.$$

Note that $|G : H|$ is odd if and only if $q \equiv 1 \pmod 4$. Thus, $H \cong PSp_4(q).\mathbb{Z}_2$, and statement (20) of Theorem holds.

If the preimage of H in $SU_4(q)$ is isomorphic to $SO_4^{\varepsilon}(q).[(q+1,4)]$ for $\varepsilon \in \{+,-\}$, then

$$|G : H|_2 = \frac{(q^4-1)_2(q^3+1)_2(q^2-1)_2}{(q+1,4)(q^2-\varepsilon 1)_2(q^2-1)_2} \ge \frac{(q^4-1)_2}{(q^2-\varepsilon 1)_2} \ge 2.$$

2.4.4. $H \in C_6(G)$. In view of [2, Table 8.10], G contains a maximal subgroup $H \in C_6(G)$ if and only if q is prime and $q \equiv 3 \pmod 4$. In view of [7, Proposition 4.6.6],

$$H \cong \begin{cases} 2^4.S_6 \text{ if } q \equiv 7 \pmod 8, \\ 2^4.A_6 \text{ if } q \equiv 3 \pmod 8. \end{cases}$$

Note that we have

$$|G : H|_2 = \begin{cases} \frac{(q^4-1)_2(q^3+1)_2(q^2-1)_2}{2^{10}} \geq 2 \text{ if } q \equiv 7 \pmod 8, \\ \frac{(q^4-1)_2(q^3+1)_2(q^2-1)_2}{2^9} = 1 \text{ if } q \equiv 3 \pmod 8. \end{cases}$$

Thus, statement (19) of Theorem holds.

2.4.5. H is an almost simple maximal subgroup of G and $H \notin \bigcup_{i\in\{1,2,5,6\}} C_i(G)$. Then, in view of [2, Table 8.11], $H \in \{PSL_2(7), A_7, PSL_3(4), PSU_4(2)\}$. So, $|H|_2 \leq 2^6$ in view of [4], and it is easy to see that

$$|G|_2 = \frac{(q^4-1)_2(q^3+1)_2(q^2-1)_2}{(q+1,4)} \geq 2^7.$$

Thus, $|G : H|$ is even.

3. Assume that $G = PSp_n(q)$, where $n \geq 4$, n is even, and q is odd.

3.1. Let H be the stabilizer of a non-degenerate subspace U of dimension m of V. Note that m is even. In view of (1), H is the stabilizer of $U^\perp$, and we may assume that $m \leq n/2$. Moreover, if $m = n/2$, then H is contained in a subgroup which will be considered below, thus, we may assume that $m < n/2$. In view of [9, Theorem 4], $|G : H|$ is odd if and only if $n \succeq m$. If $m \neq n/2$, then, in view of Tables 8.12, 8.28, 8.48, 8.64 and 8.80 of [2] and Table 3.5.C of [7], H is maximal in G. Thus, statement (3) of Theorem holds for symplectic groups.

3.2. If H is the stabilizer of an orthogonal decomposition $V = \bigoplus_{i=1}^{n/m} V_i$ with all V_i isometric of even dimension m, then, in view of [9, Theorem 9], $|G : H|$ is odd if and only if $m = 2^w \geq 2$. In view of Tables 8.12, 8.28, 8.48, 8.64 and 8.80 of [2] and Table 3.5.C of [7], H is maximal in G. Thus, statement (8) of Theorem holds.

3.3. Consider possibilities for H when $n = 4$. In view of the Aschbacher theorem [1] and [2, Tables 8.12, 8.13], any maximal subgroup of G is either contained in $\bigcup_{i\in\{1,2,3,5,6\}} C_i(G)$ or almost simple.

3.3.1. $H \in C_1(G)$. In view of [2, Table 8.12], H is the stabilizer of a totally singular subspace of V of dimension m, where $1 \leq m \leq 2$. In view of [2, Table 8.12], the preimage of H in $Sp_4(q)$ is isomorphic to $E_q^{1+2} : (\mathbb{Z}_{q-1} \times Sp_2(q))$ for $m = 1$ or to $E_q^3 : GL_2(q)$ for $m = 2$, and

$$|G : H|_2 = \frac{(q^4-1)_2(q^2-1)_2}{(q^2-1)_2(q-1)_2} \geq 2.$$

3.3.2. $H \in C_2(G)$. In view of [2, Table 8.12], either H is the stabilizer of an orthogonal decomposition $V = \bigoplus_{i=1}^{2} V_i$ with V_1 isometric to V_2, and statement (8) of Theorem holds, or H is the stabilizer of a decomposition $V = V_1 \oplus V_2$, where V_1

and V_2 are totally singular subspaces of V of dimension 2. In the latter, in view of [2, Table 8.12], the preimage of H in $Sp_4(q)$ is isomorphic to $GL_2(q).\mathbb{Z}_2$, and

$$|G:H|_2 = \frac{(q^4-1)_2(q^2-1)_2}{2(q^2-1)_2(q-1)_2} \geq 2.$$

3.3.3. $H \in C_3(G)$. In view of [2, Table 8.12], the preimage of H in $Sp_4(q)$ is isomorphic either to $Sp_2(q^2):\mathbb{Z}_2$ or to $GU_2(q).\mathbb{Z}_2$.

In the first case,

$$|G:H|_2 = \frac{(q^4-1)_2(q^2-1)_2}{2(q^4-1)_2} \geq 2.$$

In the latter,

$$|G:H|_2 = \frac{(q^4-1)_2(q^2-1)_2}{2(q^2-1)_2(q-1)_2} \geq 2.$$

3.3.4. $H \in C_5(G)$. In view of [2, Table 8.12], $H = C_G(\sigma)$ for a field automorphism σ of prime order r of the group G. If r is odd, then statement (1) of Theorem holds. If $r=2$, then, in view of [2, Table 8.12], the preimage of H in $Sp_4(q)$ is isomorphic to $Sp_4(q_0).\mathbb{Z}_2$ for $q=q_0^2$, and

$$|G:H|_2 = \frac{(q^4-1)_2(q^2-1)_2}{2(q_0^4-1)_2(q_0^2-1)_2} = 2.$$

3.3.5. $H \in C_6(G)$. In view of [2, Table 8.12], G contains a maximal subgroup $H \in C_6(G)$ if and only if q is prime. In view of [7, Proposition 4.6.9],

$$H \cong \begin{cases} 2^4.S_5 \text{ if } q \equiv \pm 1 \pmod 8, \\ 2^4.A_5 \text{ if } q \equiv \pm 3 \pmod 8, \end{cases}$$

and

$$|G:H|_2 = \begin{cases} \frac{(q^4-1)_2(q^2-1)_2}{2^8} \geq 2 \text{ if } q \equiv \pm 1 \pmod 8, \\ \frac{(q^4-1)_2(q^2-1)_2}{2^7} = 1 \text{ if } q \equiv \pm 3 \pmod 8. \end{cases}$$

Thus, statement (21) of Theorem holds.

3.3.6. H is an almost simple maximal subgroup of G and $H \notin \bigcup_{i\in\{1,2,3,5,6\}} C_i(G)$. In view of [2, Table 8.13], H is isomorphic either to $PSL_2(q)$ or to one of the following groups: A_6, S_6, or A_7. It is easy to see that $|G:H|$ is even.

4. Assume that $G = \Omega_n(q)$, where $n \geq 7$, n is odd, and q is odd.

4.1. Let H be the stabilizer of a non-degenerate subspace U of dimension m of V. In view of (1), we may assume that m is even. In view of [9, Theorem 5], $|G:H|$ is odd if and only if $n \succeq m$ and $D(U)=1$. In view of Tables 8.39, 8.58, and 8.74 of [2] and Table 3.5.D of [7], H is maximal in G if and only if $(q, m, sgn(U)) \neq (3, 2, +)$. However, $D_2^+(3) = -1$. Thus, statement (4) of Theorem holds.

4.2. Let H be the stabilizer of an orthogonal decomposition $V = \bigoplus_{i=1}^{n/m} V_i$ with all V_i isometric of dimension m. In view of Tables 8.39, 8.58, and 8.74 of [2] and Table 3.5.D of [7], H is maximal in G if and only if either $m > 1$ and $(m, q) \neq (3, 3)$

or $m = 1$ and q is prime. In view of [9, Theorem 10], $|G : H|$ is odd if and only if $m = 1$ and $q \equiv \pm 3 \pmod 8$. Thus, statement (9) of Theorem holds.

4.3. Consider the additional possibility for H when $n = 7$. If $H \cong P\Omega_7(2)$, then, in view of [2, Tables 8.39, 8.40], H is maximal in G if and only if q is prime. Note that $|P\Omega_7(2)|_2 = 2^9$,

$$|G : H| = \frac{(q^6-1)_2(q^4-1)_2(q^2-1)_2}{2^{10}},$$

and it is easy to see that $|G : H|$ is odd if and only if $q \equiv \pm 3 \pmod 8$. Thus, statement (22) of Theorem holds.

5. Assume that $G = \Omega_n^\varepsilon(q)$, where $\varepsilon \in \{+, -\}$, $n \geq 8$, n is even, and q is odd.

5.1. Let H be the stabilizer of a non-degenerate subspace U of dimension m of V. Note that if $m = n/2$ is odd and U and $U^\perp$ are not isometric, then, in view of [9, Theorem 12], $|G : H|$ is even. If $m = n/2$ and U and $U^\perp$ are isometric, then H is contained in a subgroup which will be considered below. Thus, if $m = n/2$, then we may assume that m is even and $\varepsilon = -$. With this assumption, in view of (1) and (4), Tables 8.50, 8.52, 8.66, 8.68, 8.82, and 8.84 of [2] and Tables 3.5.E and 3.5.F of [7], H is maximal in G if and only if $(q, m, sgn(U)) \notin \{(3, 2, +), (3, n-2, \varepsilon)\}$. Note that $D_2^+(3) = -1$ and, in view of (3), $D_{n-2}^\varepsilon(3) = -D_n^\varepsilon(3)$.

Assume that $D(V) = 1$. In view of [9, Theorem 6], $|G : H|$ is odd if and only if m is even, $D(U) = 1$, and $n \succeq m$. Note that if $n \succeq m$, then $m \neq n/2$. Thus, in the case $D(V)$=1 statement (5) of Theorem holds.

Assume that $D(V) = -1$. In view of (1) and (3), we can assume that $D(U) = -1$ if m is even. In view of [9, Theorem 6], $|G : H|$ is odd if and only if either m is odd and $n-2 \succeq m-1$ (note that in this case $m \neq n/2$) or m is even and $n-2 \succeq m-2$. Thus, in the case $D(V) = -1$ statement (5) of Theorem holds.

5.2. Let H be the stabilizer of an orthogonal decomposition $V = \bigoplus_{i=1}^{n/m} V_i$ with all V_i isometric of dimension m. Note that, in view of [7, Proposition 2.5.11], if m is odd, then $D(V) = 1$ (we have omitted this condition in [9, Theorem 1]), and, in view of (3) and (4), if m is even, then $D(V) = (D(V_i))^{n/m}$ and $\varepsilon = (sgn(V_i))^{n/m}$. Moreover, in view of [7, Proposition 2.5.12], if the aforementioned conditions hold, then the corresponding decompositions exist.

Assume that $D(V) = -1$. We conclude that m is even, n/m is odd, and $D(V_i) = -1$. Thus, in view of [9, Theorem 11], $|G : H|$ is even.

Assume that $D(V) = 1$. In view of [9, Theorem 11], $|G : H|$ is odd if and only if either $m = 2^w \geq 2$ and $D(V_i) = 1$ or $m = 1$ and $q \equiv \pm 3 \pmod 8$. In view of Tables 8.50, 8.52, 8.66, 8.68, 8.82, and 8.84 of [2] and Tables 3.5.E and 3.5.F of [7], H is maximal in G if and only if one of the possibilities holds: $m > 1$, $(m, q) \neq (3, 3)$, and $(m, q, sgn(V_i)) \notin \{(2, 3, \pm), (2, 5, +)\}$; $(n, \varepsilon) \neq (8, +)$, $m = 1$, and q is prime; $(n, q) = (8, +)$, $m = 1$, q is prime, and $q \equiv \pm 1 \pmod 8$. Thus, statement (10) of Theorem holds.

5.3. Consider the additional possibility for H when $n = 8$ and $\varepsilon = +$. If $H \cong P\Omega_8^+(2)$, then, in view of [2, Table 8.50], H is maximal in G if and only if q is prime. Note that $|P\Omega_8^+(2)|_2 = 2^{12}$,

$$|G : H| = \frac{(q^4-1)_2(q^6-1)_2(q^4-1)_2(q^2-1)_2}{2^{14}},$$

and it is easy to see that $|G : H|$ is odd if and only if $q \equiv \pm 3 \pmod 8$. Thus, statement (23) of Theorem holds. □

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