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FINDING $2^{\aleph_0}$ COUNTABLE MODELS FOR ORDERED THEORIES

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Abstract. The article is focused on finding conditions that imply small theories of linear order have the maximum number of countable non-isomorphic models. We introduce the notion of *extreme triviality* of non-principal types, and prove that a theory of order, which has such a type, has $2^{\aleph_0}$ countable non-isomorphic models.

Keywords: countable model, linear order, omitting types.

1. Introduction and preliminaries

The description of cases, in which complete theories have the maximal, that is $2^{\aleph_0}$, number of countable non-isomorphic models, is an important question in studying the countable spectrum of those theories. For instance at first, L. Mayer sufficient conditions an o-minimal theory to have the maximal number of countable non-isomorphic model of; and only after that she moved to proving the Vaught conjecture for o-minimal theories [1]. Another example is the work [2] by S. Sudoplatov and B. Kulpeshov, in which the authors indicated the conditions of maximality of countable spectrum, and proved the Vaught conjecture for quite o-minimal theories.

Like the study of o-minimality, we restrict to theories whose models are linearly ordered. But rather than the global hypothesis that all definable subsets are definable with just the order, we posit conditions on particular types and on the underlying linear order which imply the existence of continuum many countable models.

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In the article [3] M. Rubin investigated theories of pure linear orders and its expansions by finite or countable set of unary predicates. He proved that number of countable non-isomorphic models of such a theory T is either finite or 2^{ω}, and if the language of T is finite, then T is either ω-categorical, or it has $2^{\aleph_0}$ countable non-isomorphic model. Thus M. Rubin solved the Vaught Conjecture for linear orders expanded by unary predicates. In our paper there will be no restriction on language.

Further in the article we will consider small theories, that is, theories T with $|\bigcup\limits_{n<\omega} S_n(T)| = \omega$. By Gothic letters ($\mathfrak{M}$, $\mathfrak{N}$, etc.) we will denote structures, and by capital letters (M, N, etc.) – universes of those structures. Given a finite subset $A \subseteq M$ of a model $\mathfrak{M} \models T$, we will denote $T(A) := Th(\mathfrak{M}, a)_{a\in A}$. Note that if a theory T is small, then the theory $T(A)$ is small as well. Also the condition of T being small implies existence of a prime model, $\mathfrak{M}(A)$, of T over the finite set A, and of a countably saturated model of T [4]. If $\bar{a}_1$, $\bar{a}_2$, ..., $\bar{a}_n \in M$, $n \geq 1$, are some tuples of elements of M, then $M(\bar{a}_1, \bar{a}_2, ..., \bar{a}_n)$ will stand for a prime model of T over the set of all elements belonging to those tuples. Given a type p, by $p(M)$ we will denote the set of elements $\bar{\gamma} \in M$ with $\bar{\gamma} \models p$ in $\mathfrak{M}$.

2. Variants of triviality

Definition 2.1. *Let T be a small complete theory, $p(\bar{x})$ be a non-principal type over a finite subset A of some model of T.*

1) The type p is **extremely trivial***, if for every natural number $n \geq 1$ and every sequence of realizations $\bar{\beta}_1, \bar{\beta}_2, ...\bar{\beta}_n$ of p, $p(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})) = \{\bar{\beta}_1, \bar{\beta}_2 ... \bar{\beta}_n\}$, where $\bar{a}$ is some enumeration of the set A.*

2) The type p is **almost extremely trivial***, if for every $n \geq 1$ and every sequence of realizations $\bar{\beta}_1, \bar{\beta}_2, ...\bar{\beta}_n$ of p, $p(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}))$ is finite.*

3) The type p is **eventually extremely trivial***, if for every $n \geq 1$ there exist $m \geq n$ and realizations $\bar{\beta}_1, \bar{\beta}_2, ...\bar{\beta}_m$ of p such that $p(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m, \bar{a})) = \{\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m\}$.*

It is obvious that every extremely trivial type is almost extremely trivial, and every almost extremely trivial type is eventually extremely trivial.

Example 2.2. Let $\mathcal{L} = \{=, P_i\}_{i<\omega}$, where the P_i are unary, and T be an $\mathcal{L}$-theory and that the P_i are a decreasing sequence of sets with each $P_i - P_{i+1}$ infinte. It can be axiomtized as follows.

(1) $\forall x \left(P_{i+1}(x) \rightarrow P_i(x)\right)$ for all $i < \omega$;
(2) $\exists^{\geq n} x \left(P_i(x) \wedge \neg P_{i+1}(x)\right)$ for all $n < \omega$, $i < \omega$.

Then the type $p(x) := \{P_i(x) \mid i < \omega\}$ is extremely trivial, and the theory T has $\aleph_0$ countable models.

Example 2.3. Let $\mathcal{L} = \{=, P_i, R\}_{i<\omega}$ with the P_i unary and R binary, $k \geq 2$ be an integer, and T_k be an $\mathcal{L}$-theory that asserts the P_i's are a descending sequence of set and R is an equivalence relation with infinitely many classes, all of cardinality k and such no equivalence class can be split by a P_i. Axioms:

(1) $\forall x \left(P_{i+1}(x) \rightarrow P_i(x)\right)$ for all $i < \omega$;
(2) $\exists^{\geq n} x \left(P_i(x) \wedge \neg P_{i+1}(x)\right)$ for all $n < \omega$, $i < \omega$;
(3) $\forall x\, R(x, x)$;
(4) $\forall x \forall y \left(R(x, y) \rightarrow R(y, x)\right)$;

(5) $\forall x \forall y \forall z \ \Big(\big(R(x,y) \wedge R(y,z)\big) \to R(x,z)\Big)$;
(6) $\forall x \exists^{=k} y \ R(x,y)$;
(7) $\forall x \forall y \ \Big(\big(R(x,y) \wedge P_i(x)\big) \to P_i(y)\Big)$ for all $i < \omega$.

Let $p(x) := \{P_i(x) \mid i < \omega\}$.The type $p(x)$ is almost extremely trivial, but is not extremely trivial. This theory has $\aleph_0$ countable models: for every natural number n, a model with exactly kn realizations of p.

Example 2.4. Let $\mathcal{L} = \{=; <; P_i\}_{i<\omega}$, with the P_i unary and T be an $\mathcal{L}$-theory with the following axioms:

(1) $<$ is a dense linear order without endpoints;
(2) P_i's are dense codense disjoint predicates.

The type $p(x) := \{\neg P_i(x) \mid i < \omega\}$ is extremely trivial. This theory has $2^{\aleph_0}$ countable non-isomorphic models.

Question 2.5. Are there any examples of theories with an eventually extremely trivial type, which is not almost extremely trivial?

The following example including a unary function shows that our results extend those of [3].

Example 2.6. Modify Example 2.4 by adding a constant symbol 0 and a unary function f satisfying $f^2(x) = x$, $f(0) = 0$ and $x > y > 0$ implies $f(x) < f(y) < 0$.

The type $p(x) := \{\neg P_i(x) \mid i < \omega\}$ is extremely trivial. By Theorem 3.6 this theory has $2^{\aleph_0}$ countable non-isomorphic models.

Definition 2.7. *1) An A-definable formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a})$, $\bar{a} \in A$, is said to be p-n-**preserving**, if for every realizations $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$ of the type p, $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}) \vdash p(\bar{x})$.*

*2) Let $q(\bar{y}_1, ..., \bar{y}_n)$ $(n < \omega)$ be an A-type such that $\bigcup_{1 \le i \le n} p(\bar{y}_i) \cup \{\bigwedge_{1 \le i \ne j \le n} \bar{y}_i \ne \bar{y}_j\} \subseteq q$. An A-definable formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a})$, $\bar{a} \in A$, is said to be p-q-**preserving**, if for every realizations $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$ of the type p, we have: $tp(\bar{\beta}_1, ..., \bar{\beta}_n) = q$ implies $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}) \vdash p(\bar{x})$.*

*3) A p-n-preserving (p-q-preserving) formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a})$ is **non-trivial**, if for every model $\mathfrak{M} \models T$ and every realizations $\bar{\beta}_i$, $1 < i < n$, of the type p in $\mathfrak{M}$ (with $tp(\bar{\beta}_1, ..., \bar{\beta}_n/A) = q$) the set $\varphi(M, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})$ contains at least one element other than $\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n$.*

Proposition 2.8. *Let T be a countable complete theory, $p(\bar{x}) \in S(A)$ be a non-principal type over a finite subset A of some model of T. Then the type p is extremely trivial if and only if for every $n \ge 1$ every p-n-preserving A-definable formula is trivial.*

Proof. Further by $\bar{a}$ we denote a tuple enumerating the set A.

($\Rightarrow$) Let p be extremely trivial, $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$ $(n \ge 1)$ be realizations of p, and $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a})$ be a p-n-preserving A-definable formula. Directly from the definitions it follows that

$$\varphi\big(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}), \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}\big) \subseteq$$
$$p\big((M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})\big) = \{\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n\}.$$

Therefore, the formula φ is trivial.

($\Leftarrow$) Now suppose that for every $n \geq 1$ every p-n-preserving A-definable formula is trivial. Take a finite number of arbitrary realizations of p, namely, $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$. Towards a contradiction let us suppose that there exists a realization $\bar{\beta} \in p((M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}))$ other than $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$. Let $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})$ be an isolating formula of the principal type $p'(\bar{x}) := tp(\bar{\beta}/\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})$. Since $p(x) \subseteq p'(x)$, φ is p-n-preserving. And since $(\bigwedge_{1\leq i\leq n} \bar{x} \neq \bar{\beta}_i) \in p'(x)$, φ is non-trivial. This is a contradiction. $\square$

Proposition 2.9. *Let T be a countable complete theory, $p(\bar{x}) \in S(A)$ be a non-principal type over a finite subset A of some model of T. Then the following statements are equivalent:*

1) The type p is almost extremely trivial;

2) For every $n \geq 1$, and every A-type $q(\bar{y}_1, ..., \bar{y}_n)$ such that $\bigcup_{1\leq i\leq n} p(\bar{y}_i) \cup \{\bigwedge_{1\leq i\neq j\leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, there exists no more than finite number of non-equivalent non-trivial p-q-preserving A-formulas, and for every realizations $\bar{\beta}_1, ..., \bar{\beta}_n$ with $tp(\bar{\beta}_1, ..., \bar{\beta}_n/A) = q$, and every p-q-preserving A-formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a})$, the formula $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})$ is algebraic;

3) For every $n \geq 1$, and every A-type $q(\bar{y}_1, ..., \bar{y}_n)$ such that $\bigcup_{1\leq i\leq n} p(\bar{y}_i) \cup \{\bigwedge_{1\leq i\neq j\leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, there exist $m \geq n$ and a type $q'(\bar{y}_1, ..., \bar{y}_m) \supseteq q$ such that for every $\bar{\beta}_1$, ..., $\bar{\beta}_m \models q'$, $p(M(\bar{\beta}_1, ..., \bar{\beta}_m, \bar{a})) = \{\bar{\beta}_1, ..., \bar{\beta}_m\}$.

Proof. Further by $\bar{a}$ we denote a tuple enumerating the set A.

1) $\Rightarrow$ 2) Let p be almost extremely trivial. Let $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a})$ be a non-trivial p-q-n-preserving A-definable formula ($n \geq 1$), where $q(\bar{y}_1, ..., \bar{y}_n)$ is some A-type such that $\bigcup_{1\leq i\leq n} p(\bar{y}_i) \cup \{\bigwedge_{1\leq i\neq j\leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, and $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$ be arbitrary realizations of p. Since

$$\varphi(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}), \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}) \subseteq p\big(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})\big),$$

and p is almost extremely trivial, this set is finite, and $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})$ is an algebraic formula.

Now towards a contradiction suppose that there exist $n \geq 1$, an A-type $q(\bar{y}_1, ..., \bar{y}_n)$ with $\bigcup_{1\leq i\leq n} p(\bar{y}_i) \cup \{\bigwedge_{1\leq i\neq j\leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, and an infinite family Φ of pairwise non-equivalent non-trivial p-q-preserving A-definable formulas. Take arbitrary n realizations, $\bar{\beta}_1$, $\bar{\beta}_2$, ..., $\bar{\beta}_n$, of the type q. For every $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a}) \in \Phi$ we have

$$\varphi(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}), \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}) \subseteq p\big(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})\big).$$

And since the set Φ is infinite, and all the formulas from Φ are pairwise non-equivalent, $p\big(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})\big)$ should be infinite, what is impossible because of almost extreme triviality of p.

2) $\Rightarrow$ 3) Let n and q be as in 3), and $\bar{\beta}_1$, ..., $\bar{\beta}_n$ be realizations of q. If every p-q-preserving formula is trivial, then the desired type q' is q itself, and the proof is done. If not, then take an arbitrary element $\bar{\gamma} \in p(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})) \setminus \{\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n\}$. Denote by $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})$ an isolating formula of the principal type

$tp(\bar{\gamma}/\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})$. Since $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}) \vdash p$, for every formula $\psi(\bar{x}, \bar{a}) \in p$ we have $\models \forall\bar{x}\big(\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}) \to \psi(\bar{x}, \bar{a})\big)$. And therefore, the formula $\forall\bar{x}\big(\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a}) \to \psi(\bar{x}, \bar{a})\big)$ belongs to the type $tp(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})$. Since the last holds for every formula $\psi(\bar{x}, \bar{a})$ from the type p, we have that $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_n, \bar{a})$ is non-trivial p-q-preserving. By 2) this formula is algebraic, and therefore the set $\varphi\big(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}), \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}\big) \subseteq p\big(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})\big)$ is finite. This holds for every element $\bar{\gamma} \in p(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a}))\backslash\{\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n\}$, and since by 2) there exists only finite number of non-equivalent non-trivial p-q-preserving formulas, the set $p\big(M(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n, \bar{a})\big)$ is finite, and is equal to $\{\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m\}$, where $m > n$, and $\bar{\beta}_i \models p$ for all i, $n < i \leq m$. Denote $q' := tp(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m/\bar{a})$, it is easy to see that q' is the desired type.

3) $\Rightarrow$ 1) Let we are given arbitrary $n \geq 1$ and realizations $\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n$ of the type p. Denote by q the type $tp(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_n/\bar{a})$. By 3) there are $m \geq n$ and a type $q'(\bar{y}_1, ..., \bar{y}_m) \supseteq q$ such that for every $\bar{\beta}'_1$, ..., $\bar{\beta}'_m \models q'$, $p(M(\bar{\beta}'_1, ..., \bar{\beta}'_m, \bar{a})) = \{\bar{\beta}'_1, ..., \bar{\beta}'_m\}$. If $m = n$, then the proof for this n is finished. Now take arbitrary realizations $\bar{\beta}_{n+1}$, $\bar{\beta}_{n+2}$, ..., $\bar{\beta}_m \models p$ such that $\bar{\beta}_i \neq \bar{\beta}_j$ for all $1 \leq i \leq n$ and $n + 1 \leq j \leq m$. We have that $p(M(\bar{\beta}_1, ..., \bar{\beta}_n, \bar{a})) \subseteq p(M(\bar{\beta}_1, ..., \bar{\beta}_m, \bar{a})) = \{\bar{\beta}_1, ..., \bar{\beta}_m\}$, Therefore $p(M(\bar{\beta}_1, ..., \bar{\beta}_n, \bar{a}))$ is finite, and, since the proof is done for arbitrary n, p is an almost extremely trivial type. □

An obvious corollary from the proof of Proposition 2.9 is the following.

Proposition 2.10. *Let T be a countable complete theory, $p(\bar{x}) \in S(A)$ be a non-principal type over a finite subset A of some model of T. Then the following statements are equivalent:*

1) The type p is eventually extremely trivial;

2) For every $n \geq 1$, there exist m ($n \leq m$), and an A-type $q(\bar{y}_1, ..., \bar{y}_m)$ such that $\bigcup_{1\leq i\leq n} p(\bar{y}_i) \cup \{\bigwedge_{1\leq i\neq j\leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, there exists no more than finite number of non-equivalent non-trivial p-q-preserving A-formulas, and for every $\bar{\beta}_1, ..., \bar{\beta}_m$ with $tp(\bar{\beta}_1, ..., \bar{\beta}_m/A) = q$, for every p-q-preserving A-formula $\varphi(\bar{x}, \bar{y}_1, \bar{y}_2, ..., \bar{y}_m, \bar{a})$ the formula $\varphi(\bar{x}, \bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_m, \bar{a})$ is algebraic;

3) For every $n \geq 1$, there exists an A-type $q(\bar{y}_1, ..., \bar{y}_n)$ such that $\bigcup_{1\leq i\leq n} p(\bar{y}_i) \cup \{\bigwedge_{1\leq i\neq j\leq n} \bar{y}_i \neq \bar{y}_j\} \subseteq q$, there exist $m \geq n$ and a type $q'(\bar{y}_1, ..., \bar{y}_m) \supseteq q$ such that for every $\bar{\beta}_1$, ..., $\bar{\beta}_m \models q'$, $p(M(\bar{\beta}_1, ..., \bar{\beta}_m, \bar{a})) = \{\bar{\beta}_1, ..., \bar{\beta}_m\}$.

3. Number of countable models

Theorem 3.1. *Let T be a small complete theory. If there exists a finite subset A of some model of T and an eventually extremely trivial non-isolated type $p(\bar{x}) \in S(A)$, then $I(T \cup tp(\bar{a}/\emptyset), \omega) \geq \omega$, where $\bar{a}$ is a tuple enumerating the set A.*

Proof. Since p is eventually extremely trivial, there are $m_1 \geq 1$ and m_1 realizations $\bar{\beta}_1$, $\bar{\beta}_1$, ... $\bar{\beta}_{m_1}$ of p which are the only realizations of p in the prime model $\mathfrak{M}_1 := \mathfrak{M}(\bar{\beta}_1, \bar{\beta}_2, ..., \bar{\beta}_{m_1}, \bar{a})$. We have $|p(M_1)| = m_1$. By the same consideration, for every $i \geq 1$ there is a model $\mathfrak{M}_i$ (prime over a finite set) with $|p(M_i)| = m_i \geq m_{i-1}+1$. All those models are non-isomorphic, so we have at least countable number of models of $I(T \cup tp(\bar{a}/\emptyset))$. □

Definition 3.2. [5, 6, 7] *Let M be a linearly ordered structure, $A \subseteq M$, M be $|A|^+$-saturated, and $p \in S_1(A)$ be non-algebraic.*

*1) An A-definable formula $\varphi(x, y)$ is said to be p–***stable** *if there exist α, γ_1, $\gamma_2 \in p(M)$ such that $p(M) \cap [\varphi(\alpha, M) \setminus \{\alpha\}] \neq \emptyset$ and $\gamma_1 < \varphi(\alpha, M) < \gamma_2$.*

2) A p-stable formula $\varphi(x, y)$ is said to be **convex to the right (left)** *if there exists $\alpha \in p(M)$ such that $p(M) \cap \varphi(\alpha, M)$ is convex, α is the left (right) endpoint of the set $\varphi(\alpha, M)$, and $\alpha \in \varphi(\alpha, M)$.*

3) A p-stable convex to the right (left) formula $\varphi(x, y)$ is a **quasi-successor** *on p if for any $\alpha \in p(M)$ there exists $\beta \in \varphi(\alpha, M) \cap p(M)$ such that*

$$p(M) \cap [\varphi(\beta, M) \setminus \varphi(\alpha, M)] \neq \emptyset.$$

We use the following result from [7].

Theorem 3.3. [7] *Let T be a theory of (an expansion of) linear order, A be a finite subset of a model of T, and $p(x) \in S_1(A)$. If there exists an A-definable quasi-successor on p, then T has $2^{\aleph_0}$ countable models.*

Lemma 3.4. *Let T be a small complete theory of (an expansion of) linear order, which has less than $2^{\aleph_0}$ countable non-isomorphic models. Let A be a finite subset of a model of T, and $p(x) \in S_1(A)$ be a non-principal 1-type over A. Then for every pair of realizations of p, $\beta \models p$ the set of formulas*

$$\{\alpha < x < \beta\} \cup p(x)$$

is consistent.

Proof. Let us assume the contrary. Then there exists a finite subset $\Phi \subset p(x)$ which is inconsistent with the formula $\{\alpha < x < \beta\}$ in T. Denote $\theta(x, \bar{a}) := \bigwedge_{\varphi \in \Phi} \varphi(x)$.

Take a countable saturated model $\mathfrak{M} \models T$ with $\alpha, \beta \in M$, and $A \subset M$. By our assumption we have $\mathfrak{M} \models \neg\exists x \, \big(\alpha < x < \beta \wedge \theta(x, \bar{a})\big)$.

Now take an elementary monomorphism which maps α to β. This monomorphism can be extended to an automorphism $f \in Aut_A(\mathfrak{M})$. Since $\alpha < \beta$, $\beta = f(\alpha) < f(\beta)$, and so on: $f^n(\beta) < f^{n+1}(\beta)$, $n \in \mathbb{Z}$. By this we obtain that $\theta(M, \bar{a})$ contains an infinite discretely ordered chain

On the set $\theta(M, \bar{a})$ we introduce a binary relation $<^*$, defined by the following formula: $x <^* y := x < y \wedge \theta(x, \bar{a}) \wedge \theta(y, \bar{a}) \wedge \neg\exists z\big(\theta(z, \bar{a}) \wedge x < z < y\big)$.

Consider the following set of formulas:

$$p(x) \cup p(y) \cup \Big\{x < y \wedge \forall z\Big(\big(x < z < y \wedge \theta(z, \bar{a})\big) \rightarrow$$
$$\exists u_1 \exists u_2 \big(\theta(u_1, \bar{a}) \wedge \theta(u_2, \bar{a}) \wedge x < u_1 <^* z <^* u_2 < y\big)\Big)\Big\} \cup$$
$$\Big\{\exists u_1 \exists u_2 ... \exists u_n \Big(\bigwedge_{1 \le i \le n} \theta(u_i, \bar{a}) \wedge x < u_1 <^* u_2 <^* ... <^* u_n < y\Big)\Big\}.$$

This set is consistent, therefore, it can be completed to a 2-type over A. Fix some realization, γ_1, γ_2, of the obtained type in the model $\mathfrak{M}$.

Let $r(x)$ be a completion of the formula $\gamma_1 < x < \gamma_2$ to a type over $A \cup \{\gamma_1, \gamma_2\}$.

Then the formula

$$\varphi(x, y, \bar{a}) := x = y \vee x <^* y$$

is a quasi-successor on the type r.

Therefore by Theorem 3.3 the theory $T \cup tp(\alpha, \beta, \gamma_1, \gamma_2, \bar{a})$ has $2^{\aleph_0}$ countable models. Any model of the theory T has only ω countable non-isomorphic models of

$T \cup tp(\alpha, \beta, \gamma_1, \gamma_2, \bar{a})$, consequently, $I(T, \omega) = 2^{\aleph_0}$, which contradicts the hypothesis of the theorem. $\square$

Lemma 3.5. *Let M be a model of a small countable complete theory T, where A and D be finite subsets of M, and B is a countable subset of M. For each $(A \cup B \cup D)$-formula, $\varphi(x, \bar{a}, \bar{b}, \bar{d})$, where $\bar{a}$ enumerates the set A, $\bar{b} \in B$, and $\bar{d} \in D$, there exists a type $q_\varphi = q \in S_1(A \cup B \cup D)$ such that*

1) $\varphi(x, \bar{a}, \bar{b}, \bar{d}) \in q$;

2) B can be written as union of finite subsets B_n such that for every n, $q \restriction B_n$ is principal.

Proof. Enumerate B as $\{b_1, b_2, ..., b_i, ...\}$. For $i < \omega$ denote $\bar{b}_i := \langle b_1, b_2, ...b_i \rangle$, and let $\bar{d}'$ be a tuple enumerating the set D. Because the theory T is small, there exists a formula $\varphi_0(x, \bar{a}, \bar{b}_n, \bar{d}')$ that implies $\varphi(x, \bar{a}, \bar{b}_n, \bar{d})$ and generates a principal type over $(A \cup \{\bar{b}_n\} \cup D)$. In turn there is a principal subformula over $(A \cup \{\bar{b}_{n+1}\} \cup D)$ that implies $\varphi_0(x, \bar{a}, \bar{b}_n, \bar{d}')$. Repeating this procedure, we obtain a consistent infinite decreasing chain of principal over parameters formulas $\varphi_i(x, \bar{a}, \bar{b}_{n+i}, \bar{d}')$: ... $\subseteq \varphi_{i+1}(N, \bar{a}, \bar{b}_{n+i+1}, \bar{d}') \subseteq \varphi_i(N, \bar{a}, \bar{b}_{n+i}, \bar{d}') \subseteq ... \subseteq \varphi_0(N, \bar{a}, \bar{b}_n, \bar{d}') \subseteq \varphi(N, \bar{a}, \bar{b}_n, \bar{d})$, where $\mathfrak{N}$ is an arbitrary model of T with $(A \cup B \cup D) \subseteq N$. Let $\bar{b}_n$ enumerate B_n, we have defined the desired complete type over $(A \cup B \cup D)$. $\square$

Theorem 3.6. *Let T be a countable complete theory of (an expansion of) linear order. If there exists a finite subset A of a model $\mathfrak{M} \models T$ and a non-principal extremely trivial type $p(x) \in S_1(A)$, then T has $2^{\aleph_0}$ countable non-isomorphic models.*

Proof. Since every theory which is not small has $2^{\aleph_0}$ countable non-isomorphic models, it remains to prove the case, when the theory T is small.

Denote by $\mathfrak{N}$ an $\aleph_1$-saturated elementary extension of $\mathfrak{M}$.

During the proof, for an arbitrary infinite sequence of zeros and ones, $\tau := \langle \tau(1), \tau(2), ..., \tau(i), ... \rangle_{i<\omega}$, $\tau(i) \in \{0, 1\}$, we will construct a countable model $\mathfrak{M}_\tau \prec \mathfrak{N}$ such that for any $\tau_1 \neq \tau_2$, $\mathfrak{M}_{\tau_1} \not\cong \mathfrak{M}_{\tau_2}$.

Until the end of the proof fix such a sequence, τ.

Denote by $\mathbb{Q}_\tau$ the following subset of rational numbers: $\mathbb{Q}_\tau := \bigcup_{n \geq 0} (2n, 2n + 1) \cup \bigcup_{\substack{n \geq 1, \\ \tau(n)=0}} \{2n - \frac{1}{3}, 2n - \frac{2}{3}\} \cup \bigcup_{\substack{n \geq 1, \\ \tau(n)=1}} \{2n - \frac{1}{5}, 2n - \frac{2}{5}, 2n - \frac{3}{5}\}$.

Now, pick from the set $q(N)$ a subset, ordered by the type of $\mathbb{Q}_\tau$. If such a subset does not exist, then by Lemma 3.4 T has $2^{\aleph_0}$ countable models, and the theorem is proved. Denote this subset by $B := \{b_1, b_2, ..., b_i, ...\}_{i<\omega}$. Also, for each $n < \omega$ let $\bar{b}_n$ denote $\langle b_1, b_2, ..., b_n \rangle$. For the constructed model $\mathfrak{M}_\tau$ we will have $p(\mathfrak{M}_\tau) = B$.

We will use Tarski-Vaught criterion in order to show that the set M_τ is a universe of an elementary substructure of $\mathfrak{N}$. On each step of the construction we will be fixing a set of parameters and promising to realize each satisfiable 1-formula over it. We must keep coming back to the same set of parameters and deal with another formula. So the different sets of parameters are being attacked in parallel. We will choose the realizations in a certain way, which, together with extreme triviality of the type p, will imply that the only realizations of this type will be the elements of the set B.

Step 1. Denote by Φ_1 the set of all A-definable unary formulas, $\Phi_1 := \{\varphi_i^1(x, \bar{a}) | i < \omega\}$, where $\bar{a}$ is a tuple enumerating the set A. Choose the least i such that $\mathfrak{N} \models$

$\exists x\varphi_i^1(x,\bar{a})$. To satisfy the Tarski-Vaught property, we must find a witness for $\varphi_i^1(x,\bar{a})$. Since the sets A, B and the formula φ_i^1 are as in Lemma 3.5 (consider the set D to be empty), there exists an $A\cup B$-type $q_{\varphi_i^1}$ satisfying conditions 1) and 2) from the lemma. And since the model $\mathfrak{N}$ is $\aleph_1$-saturated, this type is realized in $\mathfrak{N}$ by some element, denote it by d_1. Thus, d_1 is principal over A.

Step 2. Choose the least j such that the formula $\varphi_j^1(x,\bar{a})\in\Phi_1$ was not considered before and $\mathfrak{N}\models\exists x\varphi_j^1(x,\bar{a})$. We find a special witness for $\varphi_j^1(x,\bar{a})$, which will satisfy the Tarski-Vaught condition but not realize p. Apply Lemma 3.5 to the sets A, B and $\{d_1\}$, and the formula $\varphi_j^1(x,\bar{a})$, to find a realization d_2 of the type $q_{\varphi_j^1}$, which exists by the lemma. We can arrange that d_2 is principal over Ab_1d_1.

Now take b_1 and consider the set of all $(A\cup\{b_1\}\cup\{d_1\})$-definable 1-formulas $\Phi_2 := \{\varphi_i^2(x,\bar{a},b_1,d_1)|i<\omega\}$. Choose the least index i such that the formula $\varphi_i^2(x,\bar{a},b_1,d_1)\in\Phi_2$ was not considered previously, and $\mathfrak{N}\models\exists x\varphi_i^2(x,\bar{a},b_1,d_1)$, and find a realization d_3 by applying Lemma 3.5 to A, B, $\{d_1,d_2\}$, and φ_i^2.

By the end of step k we will have the following sets:

- Nested sets $D_1=\{d_1\}$, $D_2=\{d_1,d_2,d_3\}$, $D_3=\{d_1,d_2,...,d_6\}$, ..., $D_k=\{d_1,d_2,...,d_{\frac{(k+1)k}{2}}\}$, where D_i is constructed on step i by adding i new realizations to the set D_{i-1}. It is possible that $d_i=d_j$ for some i and j with $1\le i<j\le\frac{(k+1)k}{2}$.
- The family of all A-definable 1-formulas Φ_1, and for every m, $2\le m\le k$, a family of $\left(A\cup\{\bar{b}_{m-1}\}\cup D_{m-1}\right)$-definable 1-formulas, Φ_m.

Further we will use the usual notation $\bar{d}_i=\langle d_1,d_2,...,d_i\rangle$, $i<\omega$.

Step $k+1$. Firstly we realize one formula from each of the families we defined earlier. To do this, for each m, $1\le m\le k$, find smallest index i_m such that the formula $\varphi_{i_m}^m\in\Phi_m$ was not considered before, and definable set of which in the model $\mathfrak{N}$ is not empty. Apply Lemma 3.5 to the sets A, B and $\{\bar{d}_{\frac{(k+1)k}{2}+m-1}\}$, and the formula $\varphi_{i_m}^m$, to find realization $d_{\frac{(k+1)k}{2}+m}$ of the type $q_{\varphi_{i_m}^m}$.

Now denote by Φ_{k+1} the set of all $(A\cup\{\bar{b}_k\}\cup D_k)$-definable 1-formulas, find a smallest index i such that $\mathfrak{N}\models\exists x\varphi_i^{k+1}(x,\bar{a},\bar{b}_k,\bar{d}_{\frac{(k+2)(k+1)}{2}})$. And choose $d_{\frac{(k+1)k}{2}+k+1}$ as before, as a realization of a type $q_{\varphi_i^{k+1}}$, which exists by Lemma 3.5 applied to the sets A, B, $\{\bar{d}_{\frac{(k+1)k}{2}+k}\}$, and formula φ_i^{k+1}. Let D_{k+1} be the set $\{d_1,d_2,...,d_{\frac{(k+1)k}{2}+k+1}\}$. We can arrange that each new d_i is principal over $A\bar{b}_n$ and the d_j's for $j<i$.

Denote $M_\tau := A\cup B\cup\bigcup\limits_{i<\omega}D_i$.

Suppose that there exists a realization $\delta\in p(N)\backslash B$. Since the type p is not principal, $\delta\notin A$, then for some $k<\omega$, $\delta=d_k$. For every $n<\omega$ the type $tp(d_k/\bar{a}\bar{b}_n)$ is non-principal. Otherwise, it should be realized in $\mathfrak{M}(\bar{a},\bar{b}_n)$ by some element not from $\bar{b}_n$, which is impossible since the type p is extremely trivial. Also, for every $i<\omega$, by choosing d_i to be as in Lemma 3.5, the type $tp(d_i/\bar{a},\bar{b}_n,\bar{d}_{i-1})$ is principal. From the last statement it easily follows by induction that the type $tp(\bar{d}_k/\bar{a}\bar{b}_n)$ is principal, and therefore $tp(d_k/\bar{a}\bar{b}_n)$ is also principal. This is a contradiction, and we have $p(\mathfrak{M}_\tau)=B$.

The Tarski-Vaught criterion implies that the obtained structure $\mathfrak{M}\tau$ is an elementary substrucutre of $\mathfrak{N}$. Since the number of different infinite sequences τ of zeros and ones equals to $2^{\aleph_0}$, $I(T\cup tp(\bar{a}),\omega)=2^{\aleph_0}$. As the theory T is small it has at most countably many distinct complete extensions by realizing an n-type, $T \cup tp(\bar{a})$; consequently, $I(T, \omega) = 2^{\aleph_0}$. □

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