S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 15, стр. 759–767 (2018) УДК 519.633
DOI 10.17377/semi.2018.15.061 MSC 65M06

ON REFLECTED WAVES IN THE SOLUTIONS OF DIFFERENCE PROBLEMS FOR THE WAVE EQUATION ON NON-UNIFORM MESHES

A.S. ANISIMOVA, YU.M. LAEVSKY

Abstract. The paper discusses the problem of numerical reflected waves when using difference schemes on strongly nonuniform grids for solution to the wave equation. The relationship between the amplitude of the reflected wave and the order of approximation on the interface of the transition from a coarse grid to a fine grid is shown. A simple modification of the difference scheme on the interface is proposed, which increases the order of approximation, and, as a consequence, reduces the amplitude of the reflected wave.

Keywords: wave equation, reflected wave, difference scheme, nonuniform mesh, step jump, homogeneous scheme, compound scheme, computational experiment

Introduction

In this paper, using the example of the simplest difference schemes for the wave equation, the question of the effect of a jump of the spatial step to the amplitude of the resulting computational artifacts, having the form of waves reflected from the interface, is numerically investigated. The existence of such reflected waves has long been known (see, for example, [1]), and there is an extensive literature devoted to algorithms that minimize the impact of such artifacts. Let us note that the schemes with jumps of spatial steps are equivalent to the difference problems with discontinuous coefficients on the "physical interface" (unlike the "computational interface") [2], [3], [4]. For one-dimensional problems, the problem of reflected waves

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Anisimova, A.S., Laevsky, Yu.M., On numerical artifacts in the solutions of difference problems for the wave equation on non-uniform meshes.

The work is supported by RFBR (grant 16-29-15122).
Received May, 7, 2018, published July, 5, 2018.

caused by the jump of the spatial step is studied in detail in the work [5]. The article [6] considers the method for arbitrarily large jump. Questions of the stability of methods in the presence of computational artifacts are discussed in the review [7]. We note that in all the cited works the process of wave propagation is described as a system of first-order differential equations. Without claiming for completeness of the list of investigations of the computational reflected waves, we have cited only a small part of the articles in which, in our opinion, the problem is most clearly described.

Let us note, one more direction of the researches connected with refinement of a spatial mesh, and, as consequence, with possibility of occurrence of computational artifacts. We are talking about the refinement of the space-time mesh in explicit difference schemes in such a way that the restriction on the local Courant number ensures stability. The refinement of the space-time mesh was simultaneously, but independently, presented in the works [8], [9], and almost immediately the method from [9] was theoretically investigated in the article [10]. The method from [9] is based on implicit schemes for parabolic and hyperbolic equations. In [11], an analogous technique was theoretically investigated in the norm of the space of continuous functions, and applied to the solution of the problem of the propagation of a laminar flame wave. In the works [12], [13] such a multilevel approach was developed on the basis of explicit schemes, and an exhaustive analysis of stability is given. Let us note that in the algorithms from [8], [12], based on explicit schemes, auxiliary values on the interface are calculated by linear interpolation in time. For hyperbolic equations this way of conjugation can lead to instability. To eliminate this drawback, the article [14] is proposed to provide energy conservation during the transition through the interface. Earlier, in the article [13], the conjugation is carried out on the basis of Neumann conditions, which actually means energy conservation in the interface. Recently approach using space-time mesh refinement was developed for simulation of elastic wave propagation [15]. Let us note that among articles, dealing with the refinement of time-space mesh, computational reflected waves are considered only in [15].

The work is organized as follows. In Section 1 we consider the grid Cauchy problem approximating the problem for the wave equation on a non-uniform grid, and the results of computational experiments that establish the characteristic values of the amplitudes of the reflected waves appearing at the junction of the grids with different steps are presented. In this case, a simple method of increasing the accuracy on such an interface, which essentialy weakens the effect of the reflected wave, is considered. In the second section we present analogous results for the two-dimensional problem. The main attention is paid to the construction of the compound scheme, which increases the order of approximation on the interface and reduces the amplitude of the reflected waves. The results of computational experiments demonstrating such a decrease are presented. In the third section, we give a short resume to the results of numerical experiments.

1. 1D REFLECTED WAVES

In this section we consider the Cauchy problem for a one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \ -\infty < x < \infty\,, \tag{1}$$

with initial data

$$u(0,x) = \varphi(x), \quad \frac{\partial u}{\partial t}(0,x) = 0, \quad -\infty < x < \infty\,. \tag{2}$$

For this problem we have exact d'Alembert's solution:

$$U(t,x) = \frac{1}{2}(\varphi(x-t) + \varphi(x+t))\,.$$

Further experiments will be carried out for sufficiently smooth finite initial data with support in the interval $I = (-h_0, h_0)$. In particular, let

$$\varphi(x) = \begin{cases} \cos^4(\pi x/2h_0)\,, & -h_0 < x < h_0\,, \\ 0\,, & -\infty < x \le -h_0\,,\ \ h_0 \le x < \infty\,. \end{cases} \tag{3}$$

This function has three continuous derivatives and a bounded fourth derivative that is discontinuous at the points $x = \pm h_0$. In this case, instead of the problem (1), (2) we consider the boundary value problem on the interval $(-1,1)$, on the temporal interval during which the wave is within the spatial interval of consideration. Those, we use the boundary conditions

$$u(t,-1) = u(t,1) = 0\,. \tag{4}$$

For numerical solution of the problem (1)–(4) we will use the simplest explicit difference scheme on the non-uniform spatial mesh (see, [16], p.499):

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2} = (\Lambda^{(3)} u^n)_i\,, \quad n = 1,2,\ldots\,, \quad i = -L+1,\ldots,L-1\,, \tag{5}$$

where $\Lambda^{(3)}$ is a standard three-point mesh operator:

$$(\Lambda^{(3)} v)_i = \frac{2}{x_{i+1} - x_{i-1}} \left(\frac{v_{i+1} - v_i}{x_{i+1} - x_i} - \frac{v_i - v_{i-1}}{x_i - x_{i-1}} \right)\,, \tag{6}$$

u^n and v are mesh functions with values u_i^n and v_i, respectively. According to (2) and (4) let the initial data and the boundary conditions be given by the following equalities

$$u_i^0 = \varphi(x_i), \quad u_i^1 = u_i^0 + \frac{\tau^2}{2}(\Lambda^{(3)} u^0)_i\,, \qquad i = -L+1,\ldots,L-1\,, \tag{7}$$
$$u_{-L}^n = u_L^n = 0\,, \quad n = 1,2,\ldots$$

We comment on the second of the conditions (7). Integration of the equation (1) in time on the interval $(0,\tau)$ with the second of the conditions (2) gives:

$$\frac{\partial u}{\partial t}(\tau,x) = \int\limits_0^\tau \frac{\partial^2 u}{\partial x^2}(t,x)\,dt \approx \tau \frac{\partial^2 u}{\partial x^2}(\tau/2,x).$$

Then the second order of time approximation provides the following difference relation:

$$\frac{u_i^2 - u_i^0}{2\tau} = \tau \left(\Lambda^{(3)} \frac{u^0 + u^1}{2} \right)_i .$$

Substitution u_i^2 from this equality into (5) at $n = 1$ lesds to the second condition in (7).

Let us introduce the following non-uniform spatial mesh. Let $l < L$ be the integer and h_1, h_2 be the positive real numbers that $x_i = ih_1$ at $-l \le i \le l$, and $x_i = x_l + (i-l)h_2$ at $i > l$ and $x_i = x_{-l} + (i+l)h$ at $i < -l$. Taking into account the expression of initial data (3) we assume $h_1 \le h_0$. In the experiments, we will consider the transition of a wave from the domain with a coarse grid with the step h_1 to the domain with a fine grid with the step $h_2 \leq h_1$. At the points of transition from a coarse grid to a fine grid, the difference equations have the form:

$$\frac{u_l^{n+1} - 2u_l^n + u_l^{n-1}}{\tau^2} = (\Lambda^{(3)} u^n)_l, \quad \frac{u_{-l}^{n+1} - 2u_{-l}^n + u_{-l}^{n-1}}{\tau^2} = (\Lambda^{(3)} u^n)_{-l}. \tag{8}$$

The order of local approximation of the scheme (5) is $O(\tau^2 + h_1^2)$ in subdomains with constant step, and the order decreases to $O(\tau^2 + h_1)$ in the points of step jumps. A stability is provided by the Courant condition cour $= \tau/h_2 \leq 1$.

Presented below computational results are obtained at $h_0 = 0.1$ (the half-width of the carrier of the initial pulse) and $x_l = 0.5$. Figure 1 illustrates the dynamics of the initial and reflected waves for the mesh parameters $\tau = 1/20480$, $h_1 = 1/80$, $h_2 = 1/10240$ (cour $= 1/2$, $\sigma = h_1/h_2 = 128$). For clarity, the amplitude of the reflected wave is increased by 50 times.

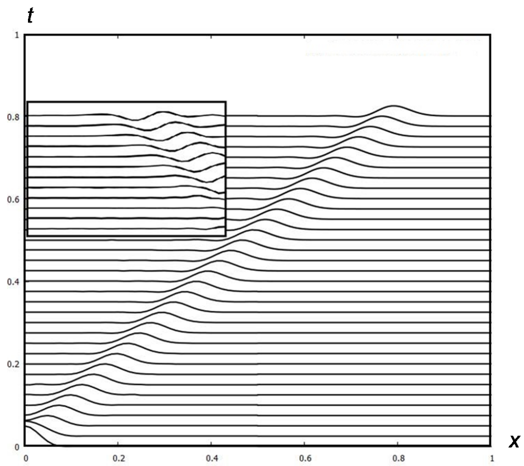

Fig. 1. The dynamics of the initial and reflected waves

We give a simple 5-point modification of the equations (8), which provides the order of approximation of $O(\tau^2 + h_1^2)$ at the points of step jumps, and, as a consequence, essentially which reduces the amplitude of the reflected wave. In what follows we will use the notation:

$$(\Lambda^{(5)} v)_k = \frac{1}{2(h_1 + h_2)} \left(\frac{-v_{k+2} + 8v_{k+1} - 7v_k}{h_p} - \frac{7v_k - 8v_{k-1} + v_{k-2}}{h_q} \right), \tag{9}$$

where $p = 2$, $q = 1$ at $k = l$, and $p = 1$, $q = 2$ at $k = -l$. Let us consuder the following equations at the points x_l and x_{-l}:

$$\frac{u_k^{n+1} - 2u_k^n + u_k^{n-1}}{\tau^2} = (\Lambda^{(5)} u^n)_k, \quad k = l, \ -l. \tag{10}$$

Let us note that in the examples considered it is not necessary to modify the second of equalities (7) for the case $n = 0$ because at the initial time instant the solution at the points of step jumps is zero. As a result, a compound scheme is obtained in which the calculations are carried out according to equalities (10) at $k = l$ and $k = -l$, respectively, and in other cases according to the equation (5). Figure 2 shows the errors for the 3-point and compound schemes at different time instants.

At time $t = 0.4$, when the wave does not reach the points of the step jump, the

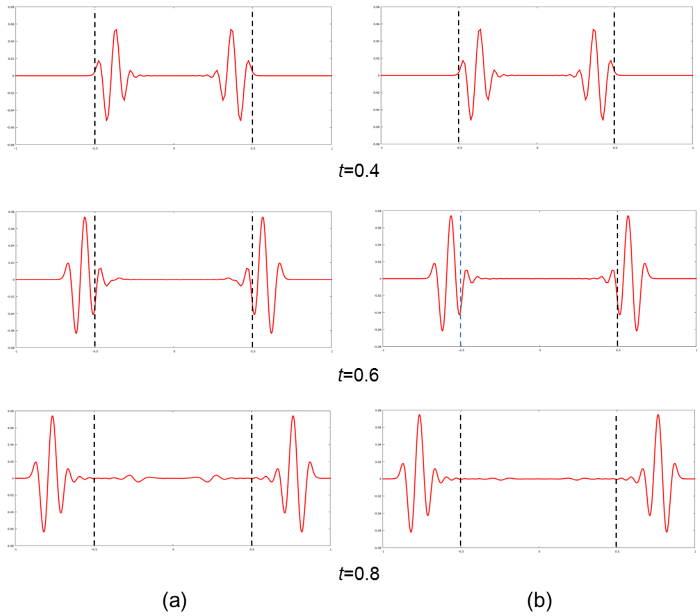

Fig. 2. Error of solution for 3-point (a) and compound (b) 1D schemes

solutions obtained for both schemes do not differ. At time $t = 0.6$, the wave passes through the points of the step jump, and one can observe the difference associated with the begining of the formation of reflection waves. At $t = 0.8$, reflection waves were completely formed, and their amplitude for the compound scheme is much smaller than the amplitude in the 3-point scheme. Figure 3 shows the dependences of the reflected waves amplitudes on the magnitude of the mesh step jump for the 3-point scheme (red line) and for the compound scheme (green line). In this case,

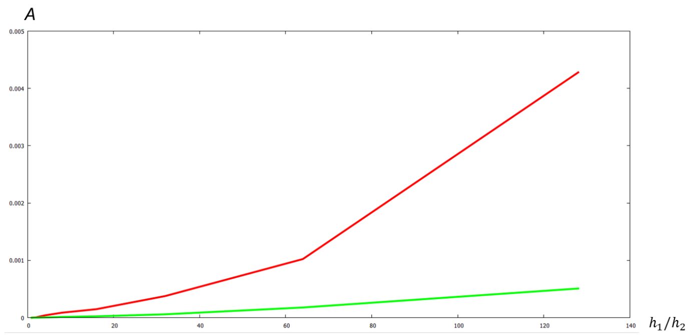

Fig. 3. Amplitudes of 1D reflected waves for 3-point (red) and compound (green) schemes

the ratio h_1/h_2 was calculated for a fixed $h_2 = 1/10240$, and $h_1 = 2^k h_2$, $k = 2, ..., 7$. Let us note that as the ratio h_1/h_2 decreases, the amplitudes of the reflected waves decrease, since the calculations are realized on a more detailed grid. However, the amplitude ratio does not decreases. The corresponding data are given in Table 1, where $A^{(3)}$ and $A^{(3,5)}$ denote the amplitudes of the reflected waves for the 3-point and compound schemes, respectively.

Tab. 1

h_1	$A^{(3)}$	$A^{(3,5)}$	$A^{(3)}/A^{(3,5)}$
1/80	4.28e-3	5.09e-4	8.41
1/160	1.02e-3	1.78e-4	5.73
1/320	3.77e-4	5.89e-5	6.40
1/640	1.48e-4	2.38e-5	6.22
1/1280	8.71e-5	1.32e-5	6.60
1/2560	3.95e-5	4.30e-6	9.18
1/5120	6.34e-6	1.57e-7	40.4

2. 2D reflected waves

In this section the following two-dimensional problem is considered:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (t,x,y) \in (0,1.5) \times (-2,2)^2, \tag{11}$$

with initial data

$$u(0,x,y) = \varphi(x,y), \quad \frac{\partial u}{\partial t}(0,x,y) = 0, \quad (x,y) \in [-2,2]^2, \tag{12}$$

where

$$\varphi(x,-2) = \varphi(x,2) = 0, \ \ x \in [-2,2], \quad \varphi(-2,y) = \varphi(2,y) = 0, \ \ y \in [-2,2],$$

and boundary conditions

$$\begin{aligned} u(t,x,-2) = u(t,x,2) = 0, \ \ (t,x) \in (0,1.5] \times [-2,2], \\ u(t,-2,y) = u(t,2,y) = 0, \ \ (t,y) \in (0,1.5] \times [-2,2]. \end{aligned} \tag{13}$$

The numerical solution of this problem is carried out by a difference scheme using a standard 5-point operator and a compound scheme using a combination of a 5-point operator and a 9-point operator at nodes where the mesh steps jump. Let us define the operators $\Lambda_x^{(\alpha)}$ and $\Lambda_y^{(\alpha)}$ where α takes the values 3 or 5 by formulars (6) or (9), respectively, and I_x, I_y are identity operators. Then

$$\Lambda^{(\alpha,\beta)} = \Lambda_x^{(\alpha)} \otimes I_y + I_x \otimes \Lambda_y^{(\beta)} \tag{14}$$

The homogeneous scheme is defined by the оператор $\Lambda^{(3,3)}$. For the compound scheme the operators $\Lambda^{(5,3)}$ or $\Lambda^{(3,5)}$ are used in points of jumps in x or y directions, respectively, and the operator $\Lambda^{(5,5)}$ is used in the vertices where the jump of steps occurs in both directions. These operators are defined on 5-point, 7-point (different in directions) and 9-point mesh templates (see Figure 4). Thus, as a homogeneous difference scheme we mean the equation

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\tau^2} = (\Lambda^{(3,3)} u^n)_{i,j}\,, \tag{15}$$

which holds at all points of the computational domain. As a compound difference scheme we mean the equations

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\tau^2} = (\Lambda^{(\alpha,\beta)} u^n)_{i,j}\,, \tag{16}$$

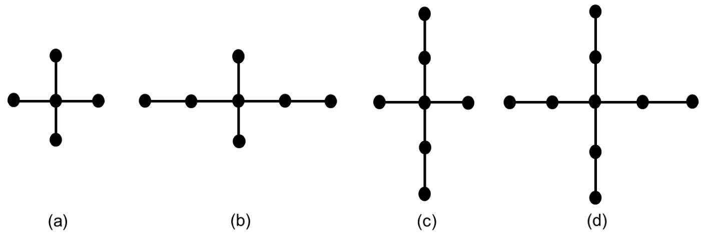

Fig. 4. Mesh templates: (a) $\Lambda^{(3,3)}$, (b) $\Lambda^{(5,3)}$, (c) $\Lambda^{(3,5)}$, (d) $\Lambda^{(5,5)}$

where $\alpha = \beta = 3$ (scheme (15)) at points where steps in both directions do not change, $\alpha = 5$, $\beta = 3$ at points where only step h_x changes, $\alpha = 3$, $\beta = 5$ at points where only step h_y changes, and $\alpha = \beta = 5$ at points where steps in both directions change. For the problem with nonuniform mesh the order of local approximation of the scheme (15) is $O(\tau^2 + h)$, where $h = \max h_x, h_y$, but for compound scheme the order of local approximation is $O(\tau^2 + h^2)$ in all mesh nodes.

In presented bellow computational experiments the initial function which is used in inital data has the form

$$\varphi(x,y) = \begin{cases} \cos^4(\pi x/2h_0)\cos^4(\pi y/2h_0)\,, & (x,y) \in [-h_0, h_0]^2\,, \\ 0\,, & (x,y) \in [-2,2]^2 \setminus [-h_0,h_0]^2. \end{cases}$$

All computational results are obtained at $h_0 = 0.1$. Let us consider the following grid in the domain $[-2,2]^2$: $h_x = h_y = 1/80$ in subdomain $[-1,1]^2$ (coarse grid), and $h_x = h_y = 1/640$ in subdomain $[-2,2]^2 \setminus [-1,1]^2$ (fine grid). Time step $\tau = 1/1280$ provides a stability of homogeneous and compound difference schemes.

Figure 5 shows the solutions obtained by schemes (15) and (16) at different time instants. At time $t = 1.1$, when the reflected wave begins to form, the figures (a) and (b) are practically indistinguishable. At moments $t = 1.3$ and $t = 1.5$, the difference between the reflected waves is clearly visible. The wave amplitude for the compound scheme is much smaller than when using a homogeneous scheme. Figure 6 shows the dependences of the reflected waves amplitudes on the magnitude of the mesh step jump for the homogeneous scheme and for the compound scheme. Magnitude of the mesh step jump is defyned as relation h_1/h_2, where $h_1 = h_x = h_y$ in subdomayn $[-1,1]^2$, and $h_2 = h_x = h_y$ in subdomayn $[-2,2]^2 \setminus [-1,1]^2$.

3. Conclusion

The main result of this study is the obtaining of computational values of amplitudes of reflected waves when the wave equation is approximated on a strongly nonuniform grid, and establishing the connection of these values with the order of approximation on the interface between the coarse and fine grids. Let us note that the amplitudes of the reflected waves in the experiments performed are no more than 1% of the value of the amplitude of the original wave. The modified scheme reduces this value by an order of magnitude, and the problem of computational reflected waves becomes practically not actual, although the urgency of the problem largely depends on the applications under consideration.

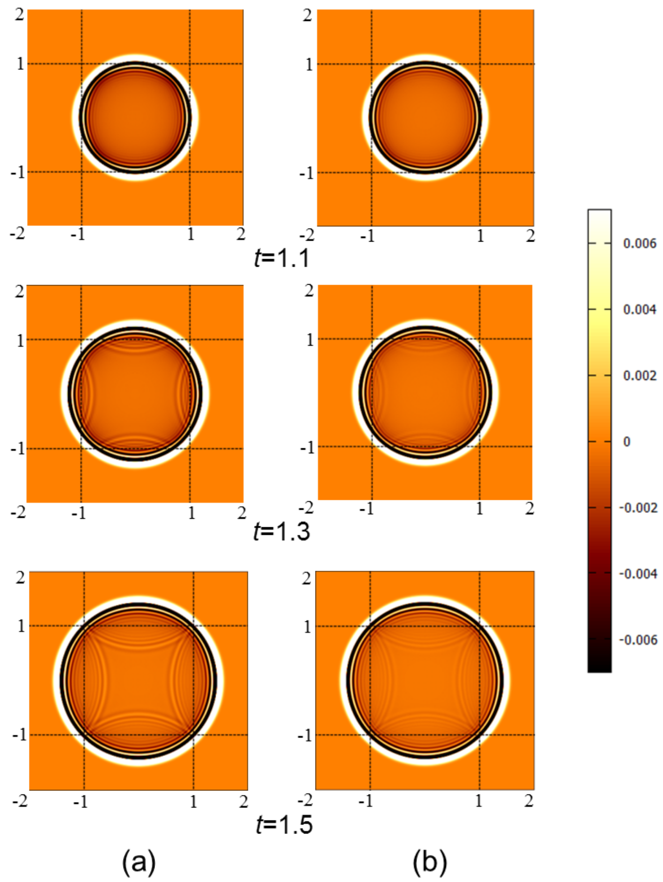

Fig. 5. Solution for homogeneous (a) and compound (b) 2D schemes

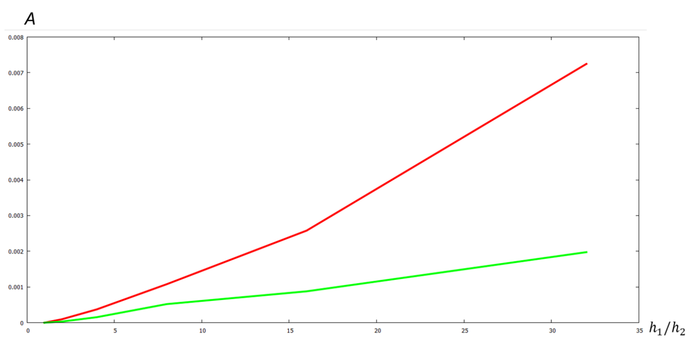

Fig. 6. Amplitudes of 2D reflected waves for homogeneous (red) and compound (green) schemes

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Anastasiya S. Anisimova
Novosibirsk State University,
ul. Pirogova, 2,
630090, Novosibirsk, Russia

Yuri M. Laevsky
Institute of Computational Mathematics and Mathematical Geophysics SB RAS,
pr. Akad. Lavrent'eva, 6,
630090, Novosibirsk, Russia,
Novosibirsk State University
E-mail address: laev@labchem.sscc.ru