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ON COORDINATE VECTOR-FUNCTIONS OF QUASIREGULAR MAPPINGS

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Abstract. Let $f : R^n \to R^n = R^k \times R^{n-k}$ $(1 \le k \le n-1)$ be a K-quasiregular mapping and $\pi : R^n \to R^k$ denotes the canonical projection. Then we obtain a lower estimate for the distortion of the values of generalized angles in R^k under the multy-valued function $F = f^{-1} \circ \pi^{-1} : R^k \to R^n$. This estimate is Möbius invariant and depends only on K and n.

Keywords: quasiregular map, conformal capacity of condenser, Teichmüller's ring, generalized angle, mapping of bounded angular distortion.

1. Basic notions and the main result

The *generalized angle* in R^n is a four of sets $\Psi = (A_1, A_2; B_1, B_2)$ where $B_1 \neq \emptyset \neq B_2$, and $B_1 \cup B_2$ contains at least two distinct points. It's value is

$$\alpha(\Psi) := \inf_{a_1 \in A_1, a_2 \in A_2} \sup_{b_1 \in B_1, b_2 \in B_2} \frac{|a_1 - a_2| \cdot |b_1 - b_2|}{|a_1 - b_1| \cdot |a_2 - b_2| + |a_1 - b_2| \cdot |a_2 - b_1|},$$

where infimum through emtyset is constituted to be equal 1. Note that $\alpha(\Psi) \le 1$ for any generalized angle Ψ in R^n.

Set-valued mapping $F : X \to 2^Y$ is called *hyperinjective* if $x_1 \neq x_2$ is equivalent to $F(x_1) \cap F(x_2) = \emptyset$. The hyperinjective mapping $F : X \to 2^Y$ transforms generalized angle Ψ in X into generalized angle $F(\Psi)$ in Y, so in the case where $X \subset R^n$, $Y = R^n$ one can compare the values $\alpha(F(\Psi))$ and $\alpha(\Psi)$. Given an increasing real function $\omega : [0, 1] \to [0, 1]$ the hyperinjective mapping $F : X \to 2^Y$ is

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said to have ω-BAD property (Bounded Angular Distortion) if $\alpha(F(\Psi)) \geq \omega(\alpha(\Psi))$ for every generalized angle Ψ in X. The notion of generalized angle and it's value was introduced in [1]. Some basic properties of generalized angles and mappings with BAD property in Ptolemaic spaces may be found in [1]-[2].

The definition and general properties of *quasiregular mappings* in R^n may be found, for example, in [3]-[5]. The advanced theory of quasiregular mappings was developed in well-known books [6]-[8], and for the information concerning with *conformal capacity of a condenser* and *the capacity of Teichmüller's extremal ring* the reader is referred to those monographs.

In order to formulate the main result we need the following:

Definition 1 (6, 3.2)**.** *Compact proper subset $E \subset \bar{R}^n$ is said to be a set of null capacity, if for any compact set $B \subset \bar{R}^n \setminus E$ the conformal capacity $Cap(E, B)$ of the condenser (E, B) is equal to zero.*

Definition 2 (8, 7.18)**.** *Given $e \in R^n$ with $|e| = 1$ the Teichmüller function $T_n(s)$ with parameter $s > 0$ is the conformal capacity of the condenser $(\{te : t \geq s\}, \{te : -1 \leq t \leq 0\})$ in R^n which is known as extremal Teichmüller's ring.*

This real function strictly decreases from $+\infty$ to 0 as s runs from 0 to $+\infty$.

Theorem 1 (4, 2.3)**.** *Given a K-quasiregular mapping $f : D \to R^n$ of a domain $D \subset R^n$ with it's complementary $\bar{R}^n \setminus D$ being a set of null capacity, for any open subset $A \subset D$ and a compact set $E \subset A$ the following inequality for conformal capacities holds*

$$Cap(f(E)), \bar{R}^n \setminus f(A)) \leq K \cdot Cap(E, \bar{R}^n \setminus A) \ . \tag{1.1}$$

It follows that *if $\bar{R}^n \setminus D$ is a set of null capacity then $\bar{R}^n \setminus f(D)$ is also a set of null capacity.* Since a set of null capacity has zero topological dimension, the image $f(D) \subset R^n = R^k \times R^{n-k}$ under quasiregular mapping $f : D \to R^n$ mensioned in Theorem 1 has the canonical projection $\pi(f(D)) = R^k$ onto the whole coordinate k-plane R^k, $1 \leq k \leq n-1$. So we can consider the set-valued hyperinjective mapping $F : R^k \to 2^D$, $F(x) = f^{-1}(\pi^{-1}(x))$, $x \in R^k$.

The $(n-1)$-dimensional Hausdorff measure of $\partial B(0,1) \subset R^n$ will be denoted as Ω_{n-1}

Our main result is the following

Theorem 2. *Let a domain $D \subset R^n$ be such that its complement $\bar{R}^n \setminus D$ has null capacity and let $f : D \to R^n$ be a K-quasiregular mapping. Let $a \leq k \leq n-1$ and $\pi : R^n = R^k \times R^{n-k} \to R^k$ be the canonical projection of R^n onto R^k. Then the set-valued mapping $F = f^{-1} \circ \pi^{-1} : R^k \to 2^D$ has the ω-BAD property with the control function*

$$\omega(t) = (3 - \sqrt{2}) \min \left\{ t, exp \left[- \left(\frac{K \cdot \Omega_{n-1}}{T_n(1/t)} \right)^{\frac{1}{n-1}} \right] \right\} \ . \tag{1.2.1}$$

That is, the inequality

$$\alpha(F(\Psi)) \geq \omega(\alpha(\Psi)) \tag{1.2.2}$$

is valid for every generalized angle Ψ in R^k.

The similar result for the case $k = 1$ has been obtained in [2]. The counterexample in [2, §10] shows that in the case $k = n$ Theorem 2 is not valid with any control function ω that depends only on K and n.

2. Auxiliary lemmas

Lemma 1. *Let a domain $D \subset R^n$ be such that it's complement $\bar{R}^n \setminus D$ has null capacity. Let $\Phi = (\{z_1\}, \{z_2\}; P_1, P_2)$ be a generalized angle in D such that P_1 and P_2 are closed in D, z_1, z_2 are distinct points in $D \setminus (P_1 \cup P_2)$, and*

$$\alpha(\Phi) = t_0 < 3 - 2\sqrt{2}. \tag{2.1}$$

Then there exists a continuum $\gamma \subset D \setminus (P_1 \cup P_2)$ such that $z_1, z_2 \in \gamma$ and

$$Cap(\gamma, \bar{R}^n \setminus (D \setminus (P_1 \cup P_2))) \leq \frac{\Omega_{n-1}}{[\log((1-t_0)^2/(4t_0))]^{n-1}} . \tag{2.2}$$

Proof. The conformal capacity as well as the value of a generalized angle does not change under Möbius transformations in $\bar{R}^n$. Thus we may assume without any loss of generality that $\infty \in P_1$, $z_1 = 0$, and $z_2 = e$ where $|e| = 1$. Thus the estimate

$$t_0 = \alpha(\Phi) = \sup_{b_1 \in P_1, b_2 \in P_2} \frac{|b_1 - b_2|}{|b_1| \cdot |b_2 - e| + |b_2| \cdot |b_1 - e|} \geq \frac{1}{|b| + |b - e|}$$

holds for all points $p \in P_2$. Then for every $b \in P_2$

$$2|b| + 1 \geq |b| + |b - e| \geq \frac{1}{t_0} \ ; \ |b| \geq \frac{1 - t_0}{t_0}$$

and it means that $P_2 \subset \{x \in \bar{R}^n : |x| \geq (1 - t_0)/t_0\}$.

Choosing some point $q \in P_2$ we get the estimate

$$t_0 = \alpha(\Phi) \geq \frac{|b_1 - q|}{|q| \cdot |b_1 - e| + |b_1| \cdot |q - e|}$$

for all points $b_1 \in P_1$. Consequently

$$|q| - |b_1| \leq |b_1 - q| \leq t_0|q| \cdot |b_1 - e| + t_0|b_1| \cdot |q - e| \leq 2t_0|q| \cdot |b_1| + t_0|q| + t_0|b_1|$$

so that

$$|b_1| \geq \frac{(1 - t_0)|q|}{2t_0|q| + t_0 + 1} \geq \frac{(1 - t_0)^2}{4t_0} .$$

It means that $P_1 \subset \{x \in \bar{R}^n : |x| \geq (1 - t_0)^2/(4t_0)\}$.

Thus $P_1 \cup P_2 \subset \{x : |x| \geq (1 - t_0)^2/(4t_0)\}$. The condition (2.1) gives the estimate $(1 - t_0)^2/(4t_0) > 1$, so that the sets $\bar{B}(0, 1)$ and $\bar{P}_1 \cup \bar{P}_2$ are separated by ring $\{x : 1 < |x| < (1 - t_0)^2/(4t_0)\}$.

Since $\bar{R}^n \setminus D$ has zero topological dimension it does not break the connectivity of $\bar{B}(0, 1)$, so the points $z_1 = 0$ and $z_2 = e$ may be connected with a continuum $\gamma \subset D \cap \bar{B}(0, 1)$. The inequality

$$\mathrm{Cap}(\gamma, \bar{P}_1 \cup \bar{P}_2) \leq \mathrm{Cap}(\gamma, (\bar{P}_1 \cup \bar{P}_2) \cup (\bar{R}^n \setminus D)) \leq$$
$$\mathrm{Cap}(\gamma, \bar{P}_1 \cup \bar{P}_2)) + \mathrm{Cap}(\gamma, \bar{R}^n \setminus D) = \mathrm{Cap}(\gamma, \bar{P}_1 \cup \bar{P}_2))$$

means that

$$\mathrm{Cap}(\gamma, \bar{R}^n \setminus (D \setminus (P_1 \cup P_2))) = \mathrm{Cap}(\gamma, (\bar{R}^n \setminus D) \cup (P_1 \cup P_2)) = \mathrm{Cap}(\gamma, \bar{P}_1 \cup \bar{P}_2) . \tag{2.3}$$

Since the ring $\{x : 1 < |x| < (1 - t_0)^2/(4t_0)\}$ separates γ from $\bar{P}_1 \cup \bar{P}_2$ the following estimate holds

$$\mathrm{Cap}(\gamma, \bar{P}_1 \cup \bar{P}_2) \leq \frac{\Omega_{n-1}}{[\log((1 - t_0)^2/(4t_0))]^{n-1}} . \tag{2.4}$$

Now (2.3) and (2.4) give the desired estimate (2.2). □

Lemma 2. *Let $\Psi = (\{a_1\}, \{a_2\}; \{b_1\}, \{b_2\})$ be a generalized angle in R^k ($1 \le k \le n-1$). In the euclidean space $R^n = R^k \times R^{n-k}$, consider the sets*

$$A_1 = \{a_1\} \times R^{n-k}; A_2 = \{a_2\} \times R^{n-k}; B_1 = \{b_1\} \times R^{n-k}; B_2 = \{b_2\} \times R^{n-k} \ ,$$

and a continuum $\Gamma \subset R^n \setminus (B_1 \cup B_2)$. If $\Gamma \cap A_1 \neq \emptyset \neq \Gamma \cap A_2$, then

$$Cap(\Gamma, B_1 \cup B_2) \ge T_n \left(\frac{1}{\alpha(\Psi)} \right) \tag{2.5}$$

Proof. Let $d = \min\{|a_1 - b_1|, |a_1 - b_2|\} = |a_1 - b_j|$ and $w_1 \in A_1 \cap \Gamma$, $w_2 \in A_2 \cap \Gamma$. There is a point $w_3 \in B_j$ with $|w_3 - w_1| = |a_1 - b_j| = d$. We use the well-known estimate (see [8, Lemma 7.34])

$$\mathrm{Cap}(\Gamma, B_j) \ge T_n \left(\frac{|w_1 - w_3|}{|w_1 - w_2|} \right) \ge T_n \left(\frac{d}{|a_1 - a_2|} \right) \ . \tag{2.6}$$

Since

$$\alpha(\Psi) = \frac{|a_1 - a_2| \cdot |b_1 - b_2|}{|a_1 - b_1| \cdot |a_2 - b_2| + |a_1 - b_2| \cdot |a_2 - b_1|} \le$$
$$\frac{|a_1 - a_2|}{d} \cdot \frac{|b_1 - b_2|}{|b_2 - a_2| + |a_2 - b_1|} \le \frac{|a_1 - a_2|}{d} \ ,$$

we obtain from (2.6) the desired estimate (recall that T_n is a strictly decreasing function)

$$\mathrm{Cap}(\Gamma, B_1 \cup B_2) \ge \mathrm{Cap}(\Gamma, B_j) \ge T_n \left(\frac{d}{|a_1 - a_2|} \right) \ge T_n \left(\frac{1}{\alpha(\Psi)} \right) \ .$$

□

3. Proof of the main theorem

It suffices to prove (1.2.2) for generalized angles $\Psi = (\{a_1\}, \{a_2\}; \{b_1\}, \{b_2\})$ in R^k (see [2, Lemma 4.2]). Consider the sets in $R^n = R^k \times R^{n-k}$

$$A_1 = \{a_1\} \times R^{n-k}; A_2 = \{a_2\} \times R^{n-k}; B_1 = \{b_1\} \times R^{n-k}; B_2 = \{b_2\} \times R^{n-k}.$$

Since $\bar{R}^n \setminus f(D)$ is a set of null capacity and has zero topological dimension, the sets $A'_j = A_j \cap f(D)$, $B'_j = B_j \cap f(D)$ $(j = 1, 2)$ are nonempty and closed in $f(D)$. Their preimages $Z_j = f^{-1}(A'_j) = F(a_j)$, $P_j = f^{-1}(B'_j) = F(b_j)$ are nonempty mutually disjoint closed sets in D.

Since $\alpha(\Psi) \le 1$ the inequality (1.2.2) holds in the case where $\alpha(F(\Phi)) \ge (3-\sqrt{2})$. Thus we have to consider only the case where $\alpha(F(\Phi)) < (3 - \sqrt{2})$. In that case the inequality

$$\alpha(F(\Psi)) = \alpha(Z_1, Z_2; P_1, P_2) < 3 - \sqrt{2}$$

means that given any sufficiently small $\varepsilon > 0$ there exist points $z_1 \in Z_1$, $z_2 \in Z_2$ such that

$$t_0 := \alpha(\{z_1\}, \{z_2\}; P_1, P_2) < \alpha(F(\Psi)) + \varepsilon < 3 - \sqrt{2} \ .$$

Thus we have the situation described in Lemma 1 by which we have a continuum $\gamma \in D \setminus (P_1 \cup P_2)$ connecting z_1 to z_2 and satisfying the inequality (2.2). Since $B_1 \cup B_2 \subset \bar{R}^n \setminus f(D \setminus (P_1 \cup P_2))$ Theorem 1 may be applied to the open set $A = D \setminus (P_1 \cup P_2)$ and the compact set $E = \gamma$. Thus for the continuum $\Gamma = f(\gamma)$ we obtain the inequality

$$\mathrm{Cap}(\Gamma, B_1 \cup B_2) \le \mathrm{Cap}(\Gamma, \bar{R}^n \setminus f(D \setminus (P_1 \cup P_2))) \le K \cdot \mathrm{Cap}(\gamma, \bar{R}^n \setminus (D \setminus (P_1 \cup P_2))) \ .$$

Then Lemma 1 gives the estimate

$$\mathrm{Cap}(\Gamma, B_1 \cup B_2) \le \frac{K \cdot \Omega_{n-1}}{[\log((1-t_0)^2/(4t_0))]^{n-1}} \ ,$$

where $t_0 = \alpha(\{z_1\}, \{z_2\}; P_1, P_2)$. Now by Lemma 2 we have

$$T_n\left(\frac{1}{\alpha(\Psi)}\right) \le \frac{K \cdot \Omega_{n-1}}{[\log((1-t_0)^2/(4t_0))]^{n-1}} \ . \tag{3.1}$$

Since $t_0 < 3 - \sqrt{2}$ it follows from (3.1) that

$$\frac{(2-2\sqrt{2})^2}{4t_0} \le \exp\left[\left(\frac{K \cdot \Omega_{n-1}}{T_n(1/\alpha(\Psi))}\right)^{\frac{1}{n-1}}\right] \ ,$$

so that

$$\alpha(F(\Psi)) + \varepsilon > t_0 \ge (3-2\sqrt{2}) \exp\left[-\left(\frac{K \cdot \Omega_{n-1}}{T_n(1/\alpha(\Psi))}\right)^{\frac{1}{n-1}}\right] \ . \tag{3.2}$$

Since $\varepsilon > 0$ is arbitrarily close to 0 the desired estimate (1.2.1) follows immediately from (3.2). The Theorem 2 is proved.

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