S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 15, стр. 801–814 (2018) УДК 515.17
DOI 10.17377/semi.2018.15.066 MSC 28A80

TWOFOLD CANTOR SETS IN $\mathbb{R}$

K.G. KAMALUTDINOV, A.V. TETENOV

Abstract. A symmetric Cantor set K_{pq} in $[0,1]$ with double fixed points 0 and 1 and contraction ratios p and q is called *twofold Cantor set* if it satisfies special exact overlap condition. We prove that all twofold Cantor sets have isomorphic self-similar structures and do not have weak separation property and that for Lebesgue-almost all $(p,q) \in [0,1/16]^2$ the sets K_{pq} are twofold Cantor sets.

Keywords: self-similar set, weak separation property, twofold Cantor set, Hausdorff dimension.

Introduction

If a self-similar set does not possess weak separation property it can have unpredictable and surprising properties, especially if it satisfies some additional regularity conditions. For example, as it was shown in 2006 by one of the authors in [14, 15], a self-similar structure $(\gamma, \mathcal{S})$ on a Jordan arc γ in $\mathbb{R}^2$, which does not satisfy WSP, is possible only if γ is a line segment and two self-similar structures $(\gamma_1, \mathcal{S}_1)$ and $(\gamma_2, \mathcal{S}_2)$ on segments γ_1 and γ_2, which do not satisfy WSP, are isomorphic if and only if the homeomorphism $\varphi: \gamma_1 \to \gamma_2$, which induces the the isomorphism of these structures, is a linear map. So the question arises, does such rigidity phenomenon occur for self-similar sets whose dimension is smaller than 1 and which are therefore totally disconnected?

By *twofold Cantor sets* K_{pq} in $\mathbb{R}$ we mean attractors of the systems $\mathcal{S}_{pq} = \{S_1, S_2, S_3, S_4\}$ consisting of two symmetric pairs of similarities $S_1(x) = px$, $S_2(x) = qx$ and $S_3(x) = px - p + 1$, $S_4(x) = qx + q - 1$, with fixed points at 0 and 1 and real

Kamalutdinov, K.G., Tetenov, A.V., Twofold Cantor sets in $\mathbb{R}$.

Received May, 1, 2018, published July, 27, 2018.

positive coefficients p and q such that for any $m, n \in \mathbb{N}$, the similarities $S_i, S_j \in \mathcal{S}_{pq}$ having a common fixed point satisfy exact overlap condition:

$$S_i^m(K_{pq}) \cap S_j^n(K_{pq}) = S_i^m S_j^n(K_{pq}). \tag{1}$$

The condition (1) holds iff the sets $S_1^{m_1} S_2^{n_1}(A)$ and $S_1^{m_2} S_2^{n_2}(A)$, where $A = S_3(K_{pq}) \cup S_4(K_{pq})$, are disjoint for any $(m_1, n_1) \neq (m_2, n_2)$ (Proposition 3); therefore it implies that all such systems $\mathcal{S}_{pq}$ do not possess weak separation property (Proposition 10) and any two of them define isomorphic self-similar structures $(K_{pq}, \mathcal{S}_{pq})$, $(K_{p'q'}, \mathcal{S}_{p'q'})$ (Theorem 12). At the same time, each homeomorphism $f : K_{pq} \to K_{p'q'}$, which induces the isomorphism of respective self-similar structures, cannot be extended continuously to a homeomorphism of $\mathbb{R}$ to itself. Nevertheless, each f extends to a homeomorphism $\tilde{f} : \mathbb{C} \to \mathbb{C}$, but such homeomorphism $\tilde{f}$ never agrees with the systems $\mathcal{S}_{pq}, \mathcal{S}_{p'q'}$ if $(p, q) \neq (p', q')$ (Theorem 13).

Though twofold Cantor sets K_{pq} are homeomorphic to the Cantor set, they densify near 0 and in the case of positive p, q there is a topological limit $\lim\limits_{t \to +\infty} tK_{pq}$, which is $[0, +\infty)$ (Theorem 8).

If we omit the exact overlap condition the systems $\mathcal{S}_{pq}$ become a particular case of the systems

$$\mathcal{S} = \mathcal{F} \cup \mathcal{G},\ \mathcal{F} = \{F_1, ..., F_m\},\ \mathcal{G} = \{G_j : j \in J\},\ J \subset I\{1, ..., m\}, \operatorname{fix} F_i = \operatorname{fix} G_i \text{ if } i \in J$$

with non-distinct fixed points. These systems were studied by B. Barany [2, 3]. Using transversality method, which was first appeared in [11, 13], he showed that if for the subsystem $\mathcal{F} \subset \mathcal{S}$ the images $F_i([\operatorname{fix} F_1, \operatorname{fix} F_m])$, where $\operatorname{fix} F_1 < \cdots < \operatorname{fix} F_m$, are disjoint, then for Lebesgue-almost all positive vectors of parameters $(\operatorname{Lip} G_j)_{j \in J}$ in some neighbourhood of 0 the Hausdorff dimension d of the attractor K of the system $\mathcal{S}$ is a solution of equation

$$\sum_{i=1}^{m} (\operatorname{Lip} F_i)^d + \sum_{j \in J} (\operatorname{Lip} G_j)^d - \sum_{j \in J} (\operatorname{Lip} F_j \operatorname{Lip} G_j)^d = 1.$$

He also proved that $H^d(K) = 0$.
In case of twofold Cantor sets, these formulas for the dimension d and for the measure $H^d(K)$ may be proved using exact overlap property.

The properties of twofold Cantor sets, mentioned above, follow naturally from their definition. There is the most important question: do such sets exist?

The answer to this question requires different techniques.
Our problem is how to find those p, q, for which each of the intersections $S_1^m(A) \cap S_2^n(A)$ is empty, where $A = S_3(K) \cup S_4(K)$, so we analyse how large is the set of those pairs (p, q) which do not possess such property. We confine ourselves to the case when the pairs p, q lie in the set $\mathcal{V} = (0, 1/16)^2$. First we show that for any pair of non-negative $m, n \in \mathbb{Z}$ and for any $p \in (0, 1/16)$ the set $\Delta_{mn}(p)$ of those $q \in (0, 1/16)$, for which $S_1^m(A) \cap S_2^n(A) \neq \varnothing$, has dimension less than 1 (Theorem 20). Therefore for any p the set $\bigcup\limits_{m,n=0}^{\infty} \Delta_{mn}(p)$ has zero 1-dimensional Lebesgue

measure in $\{p\} \times (0, 1/16)$.

To prove that $\dim_H \Delta_{mn}(p) < 1$, we use a bunch of two statements, General Position Theorem 14 and Displacement Theorem 17 used by the authors in [16].

First theorem considers a set of pairs $(\varphi_1(\xi, x), \varphi_2(\xi, x))$ of maps $\varphi_i : D \times L_i \to \mathbb{R}^n$, $i = 1, 2$, of metric spaces L_i, depending on a parameter ξ from metric space D, and find the conditions under which the set $\{\xi \in D : \varphi_1(\xi, L_1) \cap \varphi_2(\xi, L_2) \neq \varnothing\}$ has Hausdorff dimension smaller than $\dim_H D$.
Such conditions are:
(1) Both φ_i are Hölder continuous with respect to $x \in L_i$ and
(2) The difference $\varphi_1(\xi, x_1) - \varphi_2(\xi, x_2)$ is inverse Lipschitz with respect to $\xi \in D$.

Lemma 19 allows to check the conditions of the Theorem 14 for a particular case, when we try to find those values of parameter q, for which $S_1^m(A) \cap S_2^n(A) \neq \varnothing$.

This allows us to show that the set $\mathcal{K}$ of those $(p, q) \in \mathcal{V}$ which correspond to twofold Cantor sets, has a full measure in $\mathcal{V}$ (Theorem 22), while the complement of $\mathcal{K}$ in $\mathcal{V}$ is uncountable and dense in $\mathcal{V}$.

1. Definition and basic properties of twofold Cantor sets

1.1. **Self-similar sets.** Let (X, d) be a complete metric space. A mapping $S : X \to X$ is a contraction if $\operatorname{Lip} S < 1$ and it is called a similarity if $d(S(x), S(y)) = rd(x, y)$ for all $x, y \in X$ and some fixed r.

Definition 1. *Let $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ be a system of contractions of a complete metric space (X, d). A nonempty compact set $K \subset X$ is called the attractor of the system $\mathcal{S}$, if $K = \bigcup\limits_{i=1}^{m} S_i(K)$.*

By Hutchinson's Theorem [7], the attractor K is uniquely defined by the system $\mathcal{S}$.

We also call the set K *self-similar* with respect to $\mathcal{S}$ and the pair $(K, \mathcal{S})$ is called a self-similar structure. Throughout this paper, the maps $S_i \in \mathcal{S}$ will be similarities and the space X will be $\mathbb{R}$.

Given a system $\mathcal{S} = \{S_1, ..., S_m\}$, $I = \{1, 2, ..., m\}$ is the set of indices, $I^* = \bigcup\limits_{n=1}^{\infty} I^n$ is the set of all finite I-tuples, or multiindices $\mathbf{j} = j_1 j_2 ... j_n$. By $\mathbf{ij}$ we denote the concatenation of the corresponding multiindices; we write $\mathbf{i} \sqsubset \mathbf{j}$, if $\mathbf{j} = \mathbf{ik}$ for some $\mathbf{k} \in I^*$.

We write $S_{\mathbf{j}} = S_{j_1 j_2 \ldots j_n} = S_{j_1} S_{j_2} ... S_{j_n}$, $S_i^m = S_i S_i^{m-1}$ and for the set $A \subset X$ we denote $S_{\mathbf{j}}(A)$ by $A_{\mathbf{j}}$; $I^\infty = \{\alpha = \alpha_1 \alpha_2 \ldots : \ \alpha_i \in I\}$ is the index space; $\pi : I^\infty \to K$ is the *index map*, which sends $\alpha \in \mathbf{I}^\infty$ to the point $\bigcap\limits_{n=1}^{\infty} K_{\alpha_1 \ldots \alpha_n}$.

We denote by $F_{\mathcal{S}} = \{S_{\mathbf{j}} : \mathbf{j} \in I^*\}$ (or simply by F) the semigroup, generated by $\mathcal{S}$; then $\mathcal{F} = F^{-1} \circ F$, or a set of all compositions $S_{\mathbf{j}}^{-1} S_{\mathbf{i}}$ is the *associated family of similarities*, first considered by Bandt and Graf [1]. Weak separation property (WSP) was introduced by Lau and Ngai [9]. It was shown by Zerner [17], that it is equivalent to the topological condition: system $\mathcal{S} = \{S_1, \ldots, S_m\}$ of contraction similarities satisfy the WSP iff $\mathrm{Id} \notin \overline{\mathcal{F} \setminus \mathrm{Id}}$.

Notice that throughout this paper $m = 4$, therefore $I = \{1, 2, 3, 4\}$.

1.2. **Systems $\mathcal{S}_{pq}$ and their attractors.** A system $\mathcal{S}_{pq} = \{S_1, S_2, S_3, S_4\}$ is defined by the equations $S_1(x) = px$, $S_2(x) = qx$, $S_3(x) = px + 1 - p$, $S_4(x) = qx + 1 - q$, where $p, q \in (0, 1/2)$.
Let K_{pq} be the attractor of the system $\mathcal{S}_{pq}$. We write K instead of K_{pq} if it does not cause ambiguity. We will also denote $A = S_3(K) \cup S_4(K)$ and $B = S_1(K) \cup S_2(K)$. Notice that if $q = p^n, n \in \mathbb{N}$, the attractor K_{pq} is the Cantor set K_p generated by S_1 and S_3.

There are several obvious properties of systems $\mathcal{S}_{pq}$ and their attractors K_{pq} :

Proposition 2.
(i) $S_1S_2 = S_2S_1$, $S_3S_4 = S_4S_3$;
(ii) $K_{pq} = \omega(K_{pq})$, *where* $\omega(x) = 1 - x$;
(iii) For $i = 1, 2$ *and for any natural* $m \neq n$, $S_i^m(A) \cap S_i^n(A) = \varnothing$;
(iv) For any $m, n \in \mathbb{N}$, $S_1^m S_2^n(K_{pq}) \subseteq S_1^m(K_{pq}) \cap S_2^n(K_{pq})$;
(v) $K_{pq} \setminus \{0\} = \bigcup\limits_{m,n=0}^{\infty} S_1^m S_2^n(A)$.

Proof. (i), (ii) and (iv) are obvious.
(iii) Suppose $m < n$. $S_i^m(A) \subset (S_i^m(1/2), S_i^m(1)]$, and $S_i^n(A) \subset (S_i^n(1/2), S_i^n(1)]$. Since $S_i^n(1) \le S_i^m(1/2)$, $S_i^m(A) \cap S_i^n(A) = \varnothing$.
(v) Let $\sigma = i_1 i_2 ... \in I^\infty$ and $x = \pi(\sigma)$. Notice that if $\min\{l : i_l \in \{3, 4\}\} = k \in \mathbb{N}$, then $x \in S_1^m S_2^n(A)$ for some $m, n \ge 0$ such that $m + n = k - 1$. If $x \notin \bigcup\limits_{m,n=0}^{\infty} S_1^m S_2^n(A)$, then $\sigma \in \{1, 2\}^\infty$, therefore $x = 0$. □

Notation: We use $\bigsqcup$ to denote disjoint union.

Proposition 3. *For the system $\mathcal{S}_{pq}$ the following conditions are equivalent:*
(i) For any $m, n \in \mathbb{N}$, $S_1^m(A) \cap S_2^n(A) = \varnothing$;
(ii) $K = \{0\} \cup \bigsqcup\limits_{m,n=0}^{\infty} S_1^m S_2^n(A)$;
(iii) For any $m, n \in \mathbb{N}$, $S_1^m(K) \cap S_2^n(K) = S_1^m S_2^n(K)$.

Proof. Notice that for any integers k, l, m, n,

$$S_1^{m+k} S_2^n(A) \cap S_1^m S_2^{n+l}(A) \neq \varnothing \quad \text{iff} \quad S_1^k(A) \cap S_2^l(A) \neq \varnothing.$$

Therefore (i) is equivalent to (ii). To prove (i)&(ii)⇒(iii), notice that if all the sets $S_1^m S_2^n(A)$ are disjoint, then the set $(S_1^m(K) \cap S_2^n(K)) \setminus \{0\}$ is equal to

$$\bigcup_{k,l=0}^{\infty} S_1^{m+k} S_2^l(A) \cap \bigcup_{k,l=0}^{\infty} S_1^k S_2^{n+l}(A) = \bigcup_{k,l=0}^{\infty} S_1^{m+k} S_2^{n+l}(A) = S_1^m S_2^n(K) \setminus \{0\}.$$

To prove (iii)⇒(i), notice that $S_1^m S_2^n([0,1]) \cap (S_1^m([1/2,1]) \cap S_2^n([1/2,1])) = \varnothing$ for any $m, n \in \mathbb{N}$. Then $S_1^m(A) \cap S_2^n(A) = (S_1^m(K) \cap S_2^n(K)) \cap (S_1^m(A) \cap S_2^n(A)) = S_1^m S_2^n(K) \cap (S_1^m(A) \cap S_2^n(A)) = \varnothing$. □

Definition 4. *If the system $\mathcal{S}_{pq}$ satisfies any of the conditions (i)–(iii), we call K_{pq} a* twofold Cantor set.

1.3. **Dimension theorem for twofold Cantor sets.** Next result follows from [3, Theorem 1.1], but we prove it using the approach based on exact overlap conditions:

Theorem 5. *Hausdorff dimension* $d = \dim_H K_{pq}$ *of a twofold Cantor set* K_{pq} *satisfies the equation* $\quad p^d + q^d - (pq)^d = 1/2.$

Proof. Consider the set $B = S_1(K) \cup S_2(K)$. By Proposition 2 (ii), $A = \omega(B)$, so from Proposition 3 we get

$$B = \{0\} \sqcup \left(\bigsqcup_{m+n>0} S_1^m S_2^n(A) \right) = S_1(B) \sqcup S_1\omega(B) \sqcup \left(\bigsqcup_{n=1}^{\infty} S_2^n \omega(B) \right). \tag{2}$$

Thus B is the attractor of infinite system $\mathcal{B} = \{S_1, S_1\omega, S_2\omega, S_2^2\omega, \dots, S_2^n\omega, \dots\}$ of similarities whose contraction ratios are $\{p, p, q, q^2, \dots, q^n, \dots\}$ respectively.

Then standard argument [6, Theorem 9.3] shows that if d is a solution of the equation

$$2p^d + \sum_{n=1}^{\infty} q^{nd} = 1, \tag{3}$$

then $\dim_H B \le d$.

In our case the equation (3) is equivalent to the equation $p^d + q^d - (pq)^d = 1/2$. If $p, q \in (0,1)$ it has unique positive solution which we denote by d_{pq}.

Let B_n be the attractor of a subsystem $\mathcal{B}_n = \{S_1, S_1\omega, S_2\omega, \dots, S_2^n\omega\}$ of the system $\mathcal{B}$.

For any $n \in \mathbb{N}$, $B_n \subset B_{n+1}$ and $\bigcup\limits_{n=1}^{\infty} B_n \subseteq B$, therefore $\dim_H B_n < \dim_H B \le d_{pq}$.

Since $B_n \subset B$, the equality (2) shows that the compact sets $S_1(B_n)$, $S_1\omega(B_n)$ and $S_2^k\omega(B_n)$ $(k = 1, \dots, n)$ are mutually disjoint. It means (see [7]) that the system $\mathcal{B}_n$ satisfies the open set condition.

Then Hausdorff dimension of the set B_n is the unique positive solution d_n of the equation $2p^x + \sum\limits_{k=1}^{n} q^{kx} = 1$ and the set B_n has positive finite measure in the dimension d_n.

The sequence d_n increases and $d_n < d_{pq}$, so it has a limit which satisfies the equation $p^d + q^d - (pq)^d = 1/2$. Therefore $d_{pq} = \dim_H B = \dim_H K_{pq}$. □

2. Density property and violation of WSP

For twofold Cantor sets K_{pq} the logarithms of p and q are incommensurable, which causes most of their unusual properties.

Proposition 6. *If* K_{pq} *is a twofold Cantor set, then* $\dfrac{\log p}{\log q} \notin \mathbb{Q}$.

Proof. Assume the contrary. Then for some m, n, $p^m = q^n$ implies $S_1^m(1) = S_2^n(1)$, which contradicts the condition (i) of Proposition 3. □

All the statements of this section require the condition $\dfrac{\log p}{\log q} \notin \mathbb{Q}$ only. Therefore they are valid for twofold Cantor sets.

Lemma 7. (*as an obvious consequence of* [5, Ch. 5, 1, Proposition 1]) *If* $\dfrac{\log p}{\log q} \notin \mathbb{Q}$, *then the multiplicative group G, generated by p and q is a dense subgroup in $\langle \mathbb{R}_+, \cdot \rangle$.*

Notation. Let $d_H(X_1, X_2)$ be Hausdorff distance between non-empty compacts $X_1, X_2 \subset \mathbb{R}$. Denote $\Delta_{[a,b]}(X) = d_H(X \cap [a,b], [a,b])$.

Theorem 8. *If* $\dfrac{\log p}{\log q} \notin \mathbb{Q}$, *then there is a topological limit* $\lim\limits_{t \to +\infty} tK_{pq} = [0, +\infty)$.

Proof. First we prove that $\lim\limits_{t \to +\infty} \Delta_{[0,r]}(tK_{pq}) = 0$ for any $r > 0$.

Put $G = \{p^m q^n : \ m, n \in \mathbb{Z}\}$ and $G_+ = \{p^m q^n : \ m, n \in \mathbb{N}\}$. Take some $r > 0$. Consider the sets $G_k = (pq)^{-k} G_+$. For any $k \in \mathbb{N}$, $G_k \subset G_{k+1}$ and $G = \bigcup\limits_{k=0}^{+\infty} G_k$. By Lemma 7 the set G is dense in $[0, +\infty)$, therefore $\Delta_{[0,r]}(G_k) \to 0$ as $k \to \infty$.

For any $t > 1$ and any $k \in \mathbb{N}$ we have the inequalities $\Delta_{[0,r]}(tG_+) \le \Delta_{[0,tr]}(tG_+) = t\Delta_{[0,r]}(G_+)$ and $\Delta_{[0,r]}(tG_k) \le t\Delta_{[0,r]}(G_k)$. Therefore, if $(pq)^{-k} \le t < (pq)^{-k-1}$, then $\Delta_{[0,r]}(tG_+) \le \dfrac{1}{pq} \Delta_{[0,r]}(G_k)$. Since $G_+ \subset K_{pq}$, then $\Delta_{[0,r]}(tK_{pq}) \le \Delta_{[0,r]}(tG_+) \to 0$ while $t \to +\infty$.

This means that for any $x \in [0, +\infty)$ and any $\varepsilon > 0$ there is such $M > 0$, that if $t > M$, then $d(x, tK_{pq}) < \varepsilon$. Therefore $x \in \lim\limits_{t \to +\infty} tK_{pq}$. □

The same property holds for all sets $K_{pq} - S_{\mathbf{j}}(0)$ and $S_{\mathbf{j}}(1) - K_{pq}$ with $j \in I^*$:

Corollary 9. *If* $\dfrac{\log p}{\log q} \notin \mathbb{Q}$, *then for any $r > 0$ and any $\mathbf{i} \in I^*$*

$$\lim_{t \to +\infty} t(K_{pq} - S_{\mathbf{i}}(0)) = \lim_{t \to +\infty} t(S_{\mathbf{i}}(1) - K_{pq}) = [0, +\infty).$$

The logarithmic incommensurability of p and q causes the violation of WSP for the system $\mathcal{S}_{pq}$:

Proposition 10. *If* $\dfrac{\log p}{\log q} \notin \mathbb{Q}$, *then the system $\mathcal{S}_{pq}$ does not have weak separation property (WSP).*

Proof. By Lemma 7 there is a sequence of $(m_k, n_k) \in \mathbb{N}^2$, such that $\dfrac{p^{m_k}}{q^{n_k}} \to 1$ as $k \to \infty$, therefore $S_1^{m_k}(S_2^{n_k})^{-1} \to \mathrm{Id}$. Since $\dfrac{p^{m_k}}{q^{n_k}} \ne 1$, the point Id is a limit point of $F^{-1}F$. □

3. Isomorphisms of twofold Cantor sets

Suppose $(K, \{S_1, ..., S_m\})$ and $(K', \{S'_1, ..., S'_m\})$ are self-similar structures. We say that a homeomorphism $f : \ K_{pq} \to K_{p'q'}$ realises the isomorphism of self-similar structures $(K_{pq}, \mathcal{S}_{pq})$ and $(K_{p'q'}, \mathcal{S}_{p'q'})$, if $f(S_i(x)) = S'_i(f(x))$ for any $x \in K$ and any $i \in \{1, ..., m\}$.

It was proved [15] that if the systems $\mathcal{S}$, $\mathcal{S}'$ of contraction similarities in $\mathbb{C}$ with Jordan attractors K and K' do not have WSP, and $f: K \to K'$ realises the isomorphism of self-similar structures $(K, \mathcal{S})$ and $(K', \mathcal{S}')$, then f is a linear map, and K, K' are straight line segments.

The question: to what extent such rigidity phenomenon is valid for self-similar structures having Hausdorff dimension smaller than 1, still remains open. Further we try to establish isomorphism of self-similar structures on twofold Cantor sets. First we need to represent the points $x \in K_{pq}$ in convenient form.

Notation. Let $(m,n),(m',n') \in \mathbb{Z}\times\mathbb{Z}$. We write $(m,n) < (m',n')$ if $m \le m'$, $n \le n'$ and $m+n < m'+n'$.

Let $G_+ = \{S_1^m S_2^n : (m,n) > (0,0)\}$ and $H_+ = \{S_3^m S_4^n : (m,n) > (0,0)\}$ be semigroups generated by S_1, S_2 and S_3, S_4 respectively. By Propositions 2 and 3,

$$B = \{0\} \sqcup \left(\bigsqcup_{g\in G_+} g(A) \right) \qquad A = \{1\} \sqcup \left(\bigsqcup_{h \in H_+} h(B) \right). \tag{4}$$

Moreover, the families $\{g(A) : g \in G_+\}$ and $\{h(B) : h \in H_+\}$ are locally finite in $\mathbb{C}\backslash\{0\}$ and $\mathbb{C}\backslash\{1\}$ respectively, so for any $g \in G$ and $h \in H$ the sets $B\backslash g(A)$ and $A\backslash h(B)$ are closed, so each of the sets $g(A)$, $h(B)$ is open-closed in K_{pq}.

Lemma 11. *Let K_{pq} be a twofold Cantor set. Then each $x \in K_{pq}\backslash\{0\}$ has unique alternating sum representation*

$$x = \sum_{k=0}^{N} (-1)^k p^{m_k} q^{n_k} \qquad \text{or} \qquad x = \sum_{k=0}^{\infty} (-1)^k p^{m_k} q^{n_k}, \tag{5}$$

where $(m_k, n_k) < (m_{k+1}, n_{k+1})$ for any k and $(m_0, n_0) \ge (0,0)$.
This sum is finite if $x \in \bigcup_{\mathbf{i}\in I^*} S_{\mathbf{i}}(\{0,1\})$ and infinite otherwise.

Proof. Suppose $x = x_0 \in B\backslash\{0\}$, then there is unique $g_0 \in G_+$ and $y \in A$, such that $x = g_0(y)$. If $y = 1$, then $x = g_0(1)$, otherwise there is unique $h_0 \in H_+$ and $x_1 \in B$, such that $y = h_0(x_1)$.
If $x_1 = 0$, then $x = g_0 h_0(0)$, otherwise $x = g_0 h_0(x_1)$. Proceeding by induction, we see that for any $x \in B\backslash\{0\}$ there is a unique representation having one of the forms

$$x = g_0 h_0 ... g_N h_N(0) \quad \text{or} \quad x = g_0 h_0 ... g_N(1) \quad \text{or} \quad \{x\} = \bigcap_{N=0}^{\infty} g_0 h_0 ... g_N h_N(K), \tag{6}$$

where $g_i \in G_+$, $h_i \in H_+$ $(i = 0, \dots, N)$.
We obtain respective representations for $x \in A$, putting $g_0 = \mathrm{Id}$.
Notice that $gh(x) = S_1^i S_2^j S_3^{i'} S_4^{j'}(x) = p^i q^j - p^{i+i'} q^{j+j'} + p^{i+i'} q^{j+j'} x$, therefore

$$g_0 h_0 ... g_N(x) = p^{m_{2N}} q^{n_{2N}} x + \sum_{k=0}^{N-1} (p^{m_{2k}} q^{n_{2k}} - p^{m_{2k+1}} q^{n_{2k+1}}), \tag{7}$$

$$g_0 h_0 ... g_N h_N(x) = p^{m_{2N+1}} q^{n_{2N+1}} x + \sum_{k=0}^{N} (p^{m_{2k}} q^{n_{2k}} - p^{m_{2k+1}} q^{n_{2k+1}}), \tag{8}$$

where $m_0 = i_0$, $n_0 = j_0$, $m_{2k+1} = \sum\limits_{l=0}^{k}(i_l + i'_l)$, $n_{2k+1} = \sum\limits_{l=0}^{k}(j_l + j'_l)$, $m_{2k+2} = m_{2k+1} + i_{k+1}$ and $n_{2k+2} = n_{2k+1} + j_{k+1}$.

Applying formulas (7),(8) to each of the relations in (6), we get the desired result (5). □

Theorem 12. *Let K_{pq}, $K_{p'q'}$ be twofold Cantor sets. Then:*
(i) There is a homeomorphism $f: K_{pq} \to K_{p'q'}$, which realises the isomorphism of self-similar structures $(K_{pq}, \mathcal{S}_{pq})$ and $(K_{p'q'}, \mathcal{S}_{p'q'})$.
(ii) If $(p,q) \neq (p',q')$, then f cannot be extended to a homeomorphism of $[0,1]$ to itself.

Proof. (i) It follows from Lemma 11 that each element $x \in K_{pq}$ has unique representation either as a finite sum $x = \sum\limits_{k=0}^{N}(-1)^k p^{m_k} q^{n_k}$ or as a sum of a series $\sum\limits_{k=0}^{+\infty}(-1)^k p^{m_k} q^{n_k}$. The same is true for the set $K_{p'q'}$. Therefore, one can define a bijection $f: K_{pq} \to K_{p'q'}$ by

$$f(x) = \begin{cases} \sum\limits_{k=0}^{N}(-1)^k {p'}^{m_k} {q'}^{n_k}, & \text{if} \quad x = \sum\limits_{k=0}^{N}(-1)^k p^{m_k} q^{n_k} \\ \sum\limits_{k=0}^{+\infty}(-1)^k {p'}^{m_k} {q'}^{n_k}, & \text{if} \quad x = \sum\limits_{k=0}^{+\infty}(-1)^k p^{m_k} q^{n_k}. \end{cases}$$

The sets $g(A)$ and $h(B)$ are open-closed in K_{pq}. Applying formulas (4) stepwise we conclude that each of the sets $g_0 h_0 ... g_N h_N(B)$, $g_0 h_0 ... g_N(A)$ is open-closed in K_{pq}.

The sets $U_N = \bigcup\limits_{k=0}^{N} S_1^k S_2^{N-k}(B)$ are also open-closed, because their complement is a finite union $\bigsqcup\limits_{k+l \leq N} S_1^k S_2^l(A)$. These sets form a fundamental system of open-closed neighborhoods of 0 in K_{pq}.

Therefore, an open-closed neighborhood base of every point $x = g_0 h_0 ... g_N h_N(0)$ or $x = g_0 h_0 ... g_N(1)$ is $\{g_0 h_0 ... g_N h_N(U_k), k \in \mathbb{N}\}$ or $\{g_0 h_0 ... g_N \omega(U_k), k \in \mathbb{N}\}$ respectively.

For the points having representation $x = \bigcap\limits_{N=0}^{\infty} g_0 h_0 ... g_N h_N(K)$, such open-closed neighbourhood base is $\{g_0 h_0 ... g_k h_k(B), k \in \mathbb{N}\}$.

The map f sends all these base sets to respective base sets in $K_{p'q'}$. Therefore $f: K_{pq} \to K_{p'q'}$ is a homeomorphism.

(ii) Suppose that the map $f: K_{pq} \to K_{p'q'}$ which induces the isomorphism of self-similar structures (K_{pq}, S_{pq}) and $(K_{p'q'}, S_{p'q'})$ can be extended to a homeomorphism $\tilde{f}$: $[0,1] \to [0,1]$. Then the function $\tilde{f}$ is increasing.
Put $\alpha = \dfrac{\log p'}{\log p}$, $\beta = \dfrac{\log q'}{\log q}$. There are 2 cases: $\alpha \neq \beta$ and $\alpha = \beta$.
In the first case suppose $\alpha < \beta$. Then $\dfrac{\log p'}{\log q'} < \dfrac{\log p}{\log q}$ and there is $\lambda \in \mathbb{Q}$, such that $\dfrac{\log p'}{\log q'} < \lambda < \dfrac{\log p}{\log q}$. Take $k, l, m, n \in \mathbb{N}$ such that $\dfrac{k-m}{n-l} = \lambda$. Simple computation

shows that $p^l q^k > p^n q^m$ and $p'^l q'^k < p'^n q'^m$, which violates monotonicity of $\tilde{f}$.

Let now $\alpha = \beta$. Then for any $x \in G_+ = \{p^m q^n | m, n \in \mathbb{N}\}$, $f(x) = x^\alpha$.
For any $m, n \in \mathbb{N}$ the inequalities $p^m < q^n(1-p)$ and $p'^m < q'^n(1-p')$ are equivalent because $f(p^m) = p'^m$, $q'^n(1-p') = f(q^n(1-p)) = f(S_2^n S_3(0))$, and f increases.

Consider the set $W = \left\{ \dfrac{p^m}{q^n} : \ p^m < (1-p)q^n; \ m, n \in \mathbb{N} \right\}$. The set W is dense in $(0, 1-p)$, so $\sup W = 1-p$. At the same time,

$$W' = \left\{ \frac{p'^m}{q'^n} : \ p'^m < (1-p')q'^n; \ m, n \in \mathbb{N} \right\} = \{x^\alpha, x \in W\}.$$

Therefore $\sup W' = 1 - p' = (\sup W)^\alpha$ or $1 - p^\alpha = (1-p)^\alpha$, which implies $\alpha = 1$ and $(p, q) = (p', q')$. $\square$

Theorem 13. *Let $f : K_{pq} \to K_{p'q'}$ be a homeomorphism of twofold Cantor sets which induces the isomorphism of self-similar structures $(K_{pq}, \mathcal{S}_{pq})$ and $(K_{p'q'}, \mathcal{S}_{p'q'})$. Then f has an extension to a homeomorphism $\tilde{f} : \mathbb{C} \to \mathbb{C}$. If $\tilde{f} \circ S_i(z) = S'_i \circ \tilde{f}(z)$ for any $z \in \mathbb{C}$ and for any $i \in I$ then $(p, q) = (p', q')$.*

Proof. A homeomorphism of totally disconnected compact sets in $\mathbb{C}$ has an extension to a self-homeomorphism of $\mathbb{C}$ [10, Ch. 13, Theorem 7]. Let $\tilde{f} : \ \mathbb{C} \to \mathbb{C}$ be the extension of the map f which induces the isomorphism of $(K_{pq}, \mathcal{S}_{pq})$ and $(K_{p'q'}, \mathcal{S}_{p'q'})$.

Suppose $\tilde{f} \circ S_i(z) = S'_i \circ \tilde{f}(z)$ for any z and i. Then for any $m, n \in \mathbb{Z}$, $f(p^m q^n) = p'^m q'^n$, so f sends $G = \{p^m q^n : \ m, n \in \mathbb{Z}\}$ to $G' = \{p'^m q'^n : \ m, n \in \mathbb{Z}\}$. Since K_{pq} and $K_{p'q'}$ are twofold Cantor sets, the groups G and G' are dense in $[0, 1]$ by Proposition 7, so $f([0, 1]) = [0, 1]$ which contradicts Theorem 12(ii). $\square$

4. Existence of twofold Cantor sets

4.1. **General position Theorem.** Let K be the attractor of a system $\mathcal{S} = \{S_1, ...S_m\}$ of contraction similarities of $\mathbb{R}^n$, and let $\dim_H K < n/2$. Suppose for some k, l, $S_k(K) \cap S_l(K) \neq \varnothing$. Is it possible to change the system $\mathcal{S}$ slightly to such system $\mathcal{S}' = \{S'_1, ...S'_m\}$, that its attractor K' satisfies the condition $S'_k(K) \cap S'_l(K) = \varnothing$? In the above situation such statements as [6, Theorem 8.1, Corollary 8.2] cannot help, because the transformation from $S_k(K)$ to $S'_k(K')$ is not even a homeomorphism. The following Theorem gives a way out of this situation:

Theorem 14. *Let the Cartesian products of metric spaces (D, ρ), $(L_1.\sigma_1)$, (L_2, σ_2) be supplied with canonical metrisation (see [8, §21.VI, (1)]). Let continuous maps $\varphi_1 : D \times L_1 \to \mathcal{M}$ and $\varphi_2 : D \times L_2 \to \mathcal{M}$ to the normed linear space $(\mathcal{M}, \|.\|)$ be such that:*
(a) there are $C_0 > 0$ and $\alpha > 0$ such that for any $i = 1, 2$ and for all $(\xi, x), (\xi, y)$ in $D \times L_i$ the estimate holds

$$\|\varphi_i(\xi, x) - \varphi_i(\xi, y)\| \le C_0 [\sigma_i(x, y)]^\alpha$$

(uniform α-Hölder continuity condition);
(b) there are such $M_0 > 0$ and $\beta > 0$ that for any $(x_1, x_2) \in L_1 \times L_2$ and $\xi, \xi' \in D$ the function

$$\Phi(\xi, x_1, x_2) := \varphi_1(\xi, x_1) - \varphi_2(\xi, x_2)$$

on the set $D \times L_1 \times L_2$ satisfies the condition

$$\|\Phi(\xi', x_1, x_2) - \Phi(\xi, x_1, x_2)\| \geq M_0[\rho(\xi', \xi)]^\beta \ . \tag{9}$$

Then the set $\Delta := \{\xi \in D : \varphi_1(\xi, L_1) \cap \varphi_2(\xi, L_2) \neq \varnothing\}$ is closed in D and its Hausdorff dimension satisfies

$$\dim_H \Delta \leq \min\{(\beta/\alpha) \dim_H(L_1 \times L_2), \dim_H D\} \ . \tag{10}$$

Proof. Put $\tilde{\Delta} := \{(\xi, x_1, x_2) \in D \times L_1 \times L_2 : \varphi_1(\xi, x_1) = \varphi_2(\xi, x_2)\} = \{(\xi, x_1, x_2) \in D \times L_1 \times L_2 : \Phi(\xi, x_1, x_2) = 0\}$ and notice that $\Delta = \mathrm{pr}_1 \tilde{\Delta}$, where $\mathrm{pr}_1 : D \times L_1 \times L_2 \to D$ is the canonical projection.

Applying canonical projection $\mathrm{pr}_2 : D \times (L_1 \times L_2) \to L_1 \times L_2$ we obtain a set $\Delta_L := \mathrm{pr}_2(\tilde{\Delta})$, that is,

$$\Delta_L = \{(x_1, x_2) \in L_1 \times L_2 | \ \exists \xi \in D : \varphi_1(\xi, x_1) = \varphi_2(\xi, x_2)\}.$$

The maps $\pi_D = \mathrm{pr}_1|_{\tilde{\Delta}} : \tilde{\Delta} \to \Delta$ and $\pi_L = \mathrm{pr}_2|_{\tilde{\Delta}} : \tilde{\Delta} \to \Delta_L$ are continuous open maps (by properties of canonical projections). Let us show that π_L is a bijection. Indeed, if for $(\xi', x_1', x_2') \in \tilde{\Delta}$ and $(\xi'', x_1'', x_2'') \in \tilde{\Delta}$ the equality $\pi_L(\xi', x_1', x_2') = \pi_L(\xi'', x_1'', x_2'')$ holds, then $(x_1', x_2') = (x_1'', x_2'') = (x_1, x_2)$, whereas $\Phi(\xi', x_1, x_2) = 0 = \Phi(\xi'', x_1, x_2)$. Then from (9) it follows that $0 = \|\Phi(\xi', x_1, x_2) - \Phi(\xi'', x_1, x_2)\| \geq M_0[\rho(\xi', \xi'')]^\beta$, that is, $\rho(\xi', \xi'') = 0$. This means that $\xi' = \xi''$.

Since every open bijective continuous map is a homeomorphism (see [8, §13.XIII]), the maps π_L and π_L^{-1} are homeomorphisms.

Now we find Hölder continuity estimate for a map $g = \pi_D \circ \pi_L^{-1} : \Delta_L \to \Delta$. Let $\xi' = g(x_1', x_2')$ and $\xi = g(x_1, x_2)$. Then $\Phi(\xi', x_1', x_2') = 0 = \Phi(\xi, x_1, x_2)$ and, particularly, $\varphi_1(\xi', x_1') = \varphi_2(\xi', x_2')$. The inequality (9) gives an estimate

$$M_0[\rho(\xi', \xi)]^\beta \leq \|\Phi(\xi', x_1, x_2) - \Phi(\xi, x_1, x_2)\| = \|\Phi(\xi', x_1, x_2) - 0\| =$$
$$= \|\varphi_1(\xi', x_1) - \varphi_2(\xi', x_2)\| \leq \|\varphi_1(\xi', x_1) - \varphi_1(\xi', x_1')\| + \|\varphi_1(\xi', x_1') - \varphi_2(\xi', x_2)\| =$$
$$= \|\varphi_1(\xi', x_1) - \varphi_1(\xi', x_1')\| + \|\varphi_2(\xi', x_2') - \varphi_2(\xi', x_2)\|.$$

Applying the condition (a), we come to the inequality

$$M_0[\rho(\xi', \xi)]^\beta \leq C_0[\sigma_1(x_1, x_1')]^\alpha + C_0[\sigma_2(x_2, x_2')]^\alpha \leq$$
$$\leq 2C_0 \left[\sqrt{\sigma_1(x_1, x_1')^2 + \sigma_2(x_2, x_2')^2}\right]^\alpha \ .$$

Denoting by $\tilde{\sigma}$ the metrics of Cartesian product of the spaces (L_1, σ_1) and (L_2, σ_2), we get Hölder continuity estimate of the map g:

$$\rho(g(x_1', x_2'), g(x_1, x_2)) \leq (2C_0/M_0)^{1/\beta} [\tilde{\sigma}((x_1', x_2'), (x_1, x_2))]^{\alpha/\beta}.$$

Applying [6, Proposition 2.3] and the inequality $\dim_H \Delta_L \leq \dim_H(L_1 \times L_2)$, we get the desired relation (10):

$$\dim_H \Delta = \dim_H g(\Delta_L) \leq (\beta/\alpha) \dim_H(L_1 \times L_2) \quad \text{and} \quad \dim_H \Delta \leq \dim_H D.$$

Since the maps ϕ_i are continuous, Φ is continuous too. The set $\tilde{\Delta}$ is closed in $D \times L_1 \times L_2$ as a set of zeros of Φ, so the set $\Delta = \pi_D \tilde{\Delta}$ is closed in D (by properties of canonical projections). □

Remark 15. *The proof of (10) in the Theorem does not use the condition that the functions φ_1 and φ_2 are continuous with respect to the metrisation of product spaces, so this condition may be omitted. We use it here to provide that Δ is also closed in D.*

Remark 16. *The condition (b) in the Theorem may be considered as a form of transversality condition [12], when $D \subset \mathbb{R}^n$ is an open set, $\beta = 1$ and φ_i ($i = 1, 2$) are the address maps to different copies of a self-similar set, depending of a parameter $\xi \in D$.*

4.2. **Displacement Theorem.** To evaluate the displacement $|\pi(\sigma) - \pi'(\sigma)|$ of elements $x = \pi(\sigma)$ of the set K_{pq} under the transition to the set $K_{pq'}$ we use the following Displacement Theorem, which can be considered as a modification of Barnsley's Collage Theorem [4, 3.10, Theorem 1]:

Theorem 17. *Let $\mathcal{S} = \{S_1, ..., S_m\}$ and $\mathcal{S}' = \{S_1', ..., S_m'\}$ be two systems of contractions in metric space $(\mathcal{M}, \rho)$ with attractors K and K' corresponding. Let $\pi : I^\infty \to K$ and $\pi' : I^\infty \to K'$ be the address maps with $I = \{1, ..., m\}$. Suppose $V \subseteq \mathcal{M}$ is such compact set, that for any $i \in I$, $S_i(V) \subset V$ and $S_i'(V) \subset V$.*
Then, for any $\sigma \in I^\infty$,

$$\rho(\pi(\sigma), \pi'(\sigma)) \leq \frac{\delta}{1-p}, \tag{11}$$

where $p = \max\limits_{i \in I}(\operatorname{Lip} S_i, \operatorname{Lip} S_i')$ and $\delta = \max\limits_{x \in V, i \in I} \rho(S_i'(x), S_i(x))$.

Proof. Take $\sigma = i_1 i_2 \ldots$ and denote $\sigma_k = i_k i_{k+1} \ldots$. Since $\pi(\sigma_k) = S_{i_k}\pi(\sigma_{k+1})$, $\rho(\pi(\sigma_k), \pi'(\sigma_k)) \leq \rho(S_{i_k}\pi(\sigma_{k+1}), S_{i_k}\pi'(\sigma_{k+1})) + \rho(S_{i_k}\pi'(\sigma_{k+1}), S_{i_k}'\pi'(\sigma_{k+1}))$, so $\rho(\pi(\sigma_k), \pi'(\sigma_k)) \leq p \cdot \rho(\pi(\sigma_{k+1}), \pi'(\sigma_{k+1})) + \delta$ for any $k \in \mathbb{N}$.
Therefore $\rho(\pi(\sigma), \pi'(\sigma)) \leq p^{n+1}\rho(\pi(\sigma_{n+1}), \pi'(\sigma_{n+1})) + \delta \sum\limits_{k=0}^{n} p^k$, which becomes (11) as k tends to ∞. □

Notation: The space I_a^∞. Let $0 < a < 1$ and I_a^∞ be the space I^∞ supplied with the metrics $\rho_a(\sigma, \tau) = a^{s(\sigma,\tau)}$, where $s(\sigma, \tau) = \min\{k : \sigma_k \neq \tau_k\} - 1$.

This metrics turns I^∞ to a self-similar set having Hausdorff dimension $\dim_H I_a^\infty = -\dfrac{\log 4}{\log a}$. Particularly, if $0 < a < \dfrac{1}{16}$, then $\dim_H I_a^\infty < 1/2$.

We use the space I_a^∞ to parametrise the sets K_{pq} and further apply this in Theorem 14:

Lemma 18. *Let $p, q \in (0, a)$ and $a \in (0, 1/16)$. Then $\pi_{pq} : I_a^\infty \to K_{pq}$ is a 1-Lipschitz map.*

Proof. Take $\sigma, \tau \in I^\infty$ and let $s(\sigma, \tau) = k$. Then $\rho_a(\sigma, \tau) = a^k$. There is $\mathbf{i} \in I^k$ such that $\mathbf{i} \sqsubset \sigma$ and $\mathbf{i} \sqsubset \tau$, so both $\pi_{pq}(\sigma)$ and $\pi_{pq}(\tau)$ are contained in $S_{\mathbf{j}}(K)$, whose diameter is equal to $\operatorname{Lip}(S_{\mathbf{j}}) < a^k$. Therefore $\dfrac{|\pi_{pq}(\sigma) - \pi_{pq}(\tau)|}{\rho_a(\sigma, \tau)} < 1$. □

Lemma 19. *Let $p \in (0, 1/16)$, $D_{mn}(p) = \left\{ q \in (0, 1/16) : \dfrac{15}{16} \leq \dfrac{q^n}{p^m} \leq \dfrac{16}{15} \right\}$.*
Let $\varphi_1(q, \sigma) = S_1^m S_i \pi_{pq}(\sigma)$ and $\varphi_2(q, \tau) = S_2^n S_j \pi_{pq}(\tau)$, where $i, j \in \{3, 4\}$.
Then for any $\sigma, \tau \in I^\infty$ and for any $q, q' \in D_{mn}(p)$:

$$|\varphi_1(q, \sigma) - \varphi_2(q, \tau) - \varphi_1(q', \sigma) + \varphi_2(q', \tau)| > 11p^m |q' - q| \tag{12}$$

Proof. Take $\mathcal{S}_{pq} = \{S_1, S_2, S_3, S_4\}$ and $\mathcal{S}_{pq'} = \{S'_1, S'_2, S'_3, S'_4\}$, where $q, q' \in D_{mn}(p)$.

Notice that $S'_2(x) = q'x$ and $S'_4(x) = 1 - q' + q'x$, while $S'_1 = S_1$ and $S'_3 = S_3$. Let $x = \pi_{pq}(\sigma)$, $x' = \pi_{pq'}(\sigma)$, $y = \pi_{pq}(\tau)$, $y' = \pi_{pq'}(\tau)$ be the images of σ, τ in K_{pq} and $K_{pq'}$.

Denote $\delta = |q - q'|$, $\delta_{1i} = S_1^m S'_i x' - S_1^m S_i x$ and $\delta_{2j} = {S'}_2^n S'_j y' - S_2^n S_j y$.

Thus, we have to find a lower bound for $|\delta_{2j} - \delta_{1i}|$ valid for any $i, j \in \{3, 4\}$.

It follows from Theorem 17 that $|x - x'|$ and $|y - y'|$ do not exceed $\dfrac{16\delta}{15}$.
Without loss of generality we can take $q < q'$. By Lagrange theorem for any $k \in \mathbb{N}$,

$$kq^{k-1}\delta \le |q'^k - q^k| \le kq'^{k-1}\delta \tag{13}$$

Let us evaluate $|S'_i(x') - S_i(x)|$ for $i = 3, 4$.

$|S'_3(x') - S_3(x)| = |p(x' - x)| \le \dfrac{p \cdot 16\delta}{15} < \dfrac{\delta}{15}$.

$|S'_4(x') - S_4(x)| = |q'(1-x') - q(1-x)| \le |\delta(1-x')| + |q(x'-x)| < \delta + \dfrac{q \cdot 16\delta}{15} < \dfrac{16\delta}{15}$.

Writing $\delta_{2j} = ({S'_2}^n S'_j(y') - {S_2}^n S'_j(y')) + ({S_2}^n S'_j(y') - {S_2}^n S_j(y))$, we notice that by (13),

$$|{S'_2}^n S'_j(y') - {S_2}^n S'_j(y')| \ge \frac{15}{16} n q^{n-1}\delta$$

So we have $|\delta_{2j}| > \left(\dfrac{15n}{16q} - \dfrac{16}{15}\right) q^n \delta$.

Taking $n \ge 1$ and $q^n \ge \dfrac{15}{16}p^m$, we get $|\delta_{2j}| > (14n - 1)p^m\delta \ge 13p^m\delta$.

For $\delta_{1i} = {S_1}^m S'_i(x') - {S_1}^m S_i(x)$ we see that $|\delta_{1i}| < \dfrac{16p^m\delta}{15} < 2p^m\delta$.

Therefore $|\delta_{2j} - \delta_{1i}| > 11p^m\delta$. □

4.3. **Almost all K_{pq} are twofold Cantor sets.**

Theorem 20. *Let $p \in (0, 1/16)$. Then for any $m, n \in \mathbb{N}$ the set $\Delta_{mn}(p) = \{q \in (0, 1/16) : S_1^m(A) \cap S_2^n(A) \ne \varnothing\}$ is closed and nowhere dense in $(0, 1/16)$.*

Proof. Take some $a \in (0, 1/16)$ and $p \in (0, a)$.

Note that $S_1^m(A) \cap S_2^n(A) \ne \varnothing$ implies $\dfrac{15}{16} \le \dfrac{q^n}{p^m} \le \dfrac{16}{15}$.

Consider the set $D_{mn}(p) = \left\{q \in (0, 1/16) : \dfrac{15}{16} \le \dfrac{q^n}{p^m} \le \dfrac{16}{15}\right\}$.

Consider the functions $\varphi_1(q, \sigma) = S_1^m S_i \pi_{pq}(\sigma)$ and $\varphi_2(q, \sigma) = S_2^n S_j \pi_{pq}(\sigma)$, where $i, j \in \{3, 4\}$ as maps from I_a^∞ to K_{pq}. It follows from Lemma 18 that they are 1-Lipschitz with respect to σ, and from Lemma 19 it follows that if $q, q' \in D_{mn}(p) \cap (0, a)$ and $\Phi(q, \sigma, \tau) = \varphi_1(q, \sigma) - \varphi_2(q, \tau)$ then:

$$|\Phi(q', \sigma, \tau) - \Phi(q, \sigma, \tau)| \ge 11p^m|q' - q| \tag{14}$$

Applying Theorem 14 to the set $D_{mn}(p) \cap (0, a)$ we get that the set $\Delta_{mn}(p) \cap (0, a)$ is closed in $(0, a)$ and its dimension is not greater than $2 \dim_H I_a^\infty < 1$. Therefore $\Delta_{mn}(p)$ is closed in $(0, 1/16)$, its dimension is less or equal to 1, it has zero H^1-measure and is nowhere dense in $(0, 1/16)$. □

Corollary 21. *The set* $\widetilde{\Delta}_{mn} = \{(p,q) :\ p,q \in (0,1/16),\ S_1^m(A) \cap S_2^n(A) \neq \varnothing\}$ *is a null-measure closed subset in* $(0,1/16)^2$*.*

Proof. Define a function $\Psi : (0,1/16)^2 \times (I^\infty)^2 \to \mathbb{R}$ by $\Psi(p,q,\sigma,\tau) = |S_1^m S_i \pi_{pq}(\sigma) - S_2^n S_j \pi_{pq}(\tau)|$. It is continuous, therefore the set $\Psi^{-1}(\{0\})$ is closed in $(0,1/16)^2 \times (I^\infty)^2$. The projection of this set to $(0,1/16)^2$ is $\widetilde{\Delta}_{mn}$. Since $(I^\infty)^2$ is compact, $\widetilde{\Delta}_{mn}$ is closed in $(0,1/16)^2$.

By Fubini's Theorem, 2-dimensional Lebesgue measure of the set $\widetilde{\Delta}_{mn}$ is equal to

$$\iint\limits_{(0,1/16)^2} \chi(p,q)dp\ dq = \int\limits_0^{1/16} dp \int\limits_0^{1/16} \chi(p,q)dq,$$

where $\chi(p,q)$ is a characteristic function of the set $\widetilde{\Delta}_{mn}$.

By Theorem 20, $\int\limits_0^{1/16} \chi(p,q)dq = 0$ for any $p \in (0,1/16)$. Therefore the set $\widetilde{\Delta}_{mn}$ has zero measure in $(0,1/16)^2$. $\square$

Theorem 22. *The set* $\mathcal{K}$ *of those* $(p,q) \in \mathcal{V} = (0,1/16)^2$*, for which* K_{pq} *is a twofold Cantor set, has full measure in* $\mathcal{V}$*, and its complement is uncountable and dense in* $\mathcal{V}$*.*

Proof. By Corollary 21, $\widetilde{\Delta}_{mn}$ has zero Lebesgue 2-measure in $\mathcal{V}$ for any $m,n \in \mathbb{N}$. Since $\Delta = \bigcup\limits_{m,n=1}^{\infty} \widetilde{\Delta}_{mn}$, the same is true for Δ and its complement $\mathcal{K}$ has full measure in $\mathcal{V}$.

From the other side, the set $\Delta_0 = \{(p,q) \in (0,1) : \dfrac{\log p}{\log q} \in \mathbb{Q}\} \cap \mathcal{V} \subseteq \Delta$ is a countable union of curves $q = p^s$ in $\mathcal{V}$. The set $\{p^s :\ s \in \mathbb{Q}\}$ is dense in $(0,1/16)$ for any $p \in (0,1/16)$, therefore Δ is uncountable and dense in $\mathcal{V}$. $\square$

Acknowledgments

The authors would like to thank Vladislav Aseev and Christoph Bandt for useful discussions at different stages of writing the paper.

The research of authors was supported by Russian Foundation of Basic Research grants No. 16-01-00414, No. 18-501-51021.

References

[1] Ch. Bandt, S. Graf, *Self-similar sets 7. A characterization of self-similar fractals with positive Hausdorff measure,* Proc. Amer. Math. Soc. **114**:4 (1992), 995–1001. MR1100644

[2] B. Barany, *On the Hausdorff dimension of a family of self-similar sets with complicated overlaps,* Fund. Math. **206** (2009), 49–59. MR2576260

[3] B. Barany, *Iterated function systems with non-distinct fixed points,* J. Appl. Math. Anal. Appl. **383**:1 (2011), 244–258. MR2812735

[4] M.F. Barnsley, *Fractals Everywhere,* Academic Press, Orlando, Florida, (1988). MR0977274

[5] N. Bourbaki, *General Topology: Chapters 5–10*, Springer-Verlag Berlin Heidelberg, (1989). MR979295

[6] K. J. Falconer, *Fractal geometry: mathematical foundations and applications,* J. Wiley and Sons, New York, 1990. MR1102677

[7] J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30**:5 (1981), 713–747. MR0625600

[8] K. Kuratowski, *Topology*, vol. 1, PWN and Acad. Press, (1966). MR0217751

[9] K. S. Lau and S. M. Ngai, *Multifractal measures and a weak separation condition,* Adv. Math. **141** (1999), 45–96. MR1667146

[10] E. Moise, *Geometric Topology in Dimensions 2 and 3.* Springer-Verlag, New York, (1977). MR0488059

[11] M. Pollicott, K. Simon, *The Hausdorff dimension of λ-expansions with deleted digits,* Trans. Am. Math. Soc. **347**:3 (1995), 967–983. MR1290729

[12] K. Simon, B. Solomyak, M. Urbański, *Hausdorff dimension of limit sets for parabolic IFS with overlaps,* Pacific J. Math. **201**:2 (2001), 441–478. MR1875903

[13] B. Solomyak, *On the random series $\sum \pm\lambda^i$ (an Erdos problem),* Annals of Math **142** (1995), 611–625. MR1356783

[14] A. V. Tetenov, *Self-similar Jordan arcs and graph-oriented IFS,* Sib. Math. J. **47**:5 (2006), 1147–1153.

[15] A. V. Tetenov, *On the rigidity of one-dimensional systems of contraction similitudes,* Sib. Elektron. Mat. Izv. **3** (2006), 342–345. MR2276040

[16] A. Tetenov, K. Kamalutdinov, D. Vaulin, *Self-similar Jordan arcs which do not satisfy OSC*, arXiv:1512.00290

[17] M. P. W. Zerner, *Weak separation properties for self-similar sets,* Proc. Amer. Math. Soc. **124**:11 (1996), 3529–3539. MR1343732

Kirill Kamalutdinov
Novosibirsk State University,
Novosibirsk, Russia
E-mail address: kirdan15@mail.ru

Andrei Tetenov
Novosibirsk State University &
Gorno-Altaisk State University
E-mail address: atet@mail.ru