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MANY–VALUED MULTI–MODAL LOGICS, SATISFIABILITY PROBLEM

M.A. MOOR, V.V. RYBAKOV

Abstract. This paper investigates many–valuated multi–modal logics. The suggested semantics consists of relational Kripke–Hintikka models which have various accessibility relations and distinct valuations for propositional statements (letters). So we study a multi–agent approach when each agent has its own accessibility relation and also its own valuation for propositional letters. We suggest the rules for computation of truth values of formulas, illustrate our approach, and study the satisfiability problem.

Using a modification of the filtration technique, we obtain a solution for satisfiability problem in basic but most important wide classes of multi–valued multi–modal models. We comment on possible applications and describe open problems.

Keywords: many–valued logic, multi–agent logic, multi–modal logic, computability, satisfiability, decidability, deciding algorithms

1. Introduction

Looking at information sciences and non-classical logic, one can find different fruitful approaches to verification of the correctness of models and for constructing mathematical models investigating human reasoning by using multi–valued logics. Initially multi–valued logics were used to represent truth relations for kernel logical language — Boolean logic. That may be dated to Łukasiewicz (1917) and his three–valued and many–valued propositional calculi, as well as to Gödel (1932) who refuted the finite–validness of intuitionistic logic. One of the first appearances of multi–modal logics may be dated to Arthur Prior's tense logic, temporal logic, with

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two modalities, F and P, corresponding to *sometime in the future* and *sometime in the past*. For modal and temporal logics, in the pioneering works by A. Tarski (1951) on topological Boolean algebras and by S. Kripke (1960x), the relational models (Kripke–Hintikka models) were suggested as semantical objects — algebras and models — which are multi–valued by nature.

Applications of logic in AI and Information Sciences use various tools to automate logical inferences. The logics in the approach are primarily non–classical, such as modal, temporal logics, description logics (which all contribute to research in semantic web). Nowadays, e.g. the temporal logic is a very popular, highly technical and fruitful area (cf. e.g. Gabbay and Hodkinson [8, 9, 10]) with various particular areas of applications in CS and AI. At present time, rather the most complete monograph that collected and summarized many results and techniques of multi–modal logic is the book by Gabbay et al. [11].

The approach via symbolic logic works in CS for verification of computational processes, for verification of correctness in the representation of information and knowledge, etc. (cf. for example Wooldridge et al. [25, 26, 27], Lemniscio et al. [12, 2], Bambini and Vagarious [3], Vagarious [24]).

The technique of multi–modal logics was in particular translated to description logics which found many applications in the study of ontology, cf. e.g. F. Baader et al. [1], F. Wolter, [28], F. Wolter et al. [13]. Earlier we have also studied multi–agents' logic with distances, the satisfiability problem for it (Rybakov et al. [19]), models for the conception of Chance Discovery in multi–agents' environment (Rybakov [20, 22]). A logic modeling uncertainty via agents' views also was investigated (cf. McLean et al [14]); the study of the conception of knowledge from the viewpoint of multi agency based at temporal logic may be found in works by Rybakov [15, 17, 18, 21, 16, 17]. From technical viewpoint, perhaps the first approach to the multi–valued modal logics (when different valuations are taken on algebraic lattices) may be found in works by M. Fitting [5, 6]; multi–valued approaches were also used for such a popular area as model checking (cf. e.g. G. Bruns, P. Godefroid [4]).

In this paper we study many–valuated multi–modal logics. The innovating point here is that we base our approach on relational Kripke–Hintikka models which have several accessibility relations (which reflects the presence of several distinct modalities) and simultaneously several distinct valuations for propositional statements (letters). This may be looked at as modeling a multi–agent approach when each agent has its own accessibility relation and also its own valuation for propositional letters. Thus, looking at this logic we see that it is a multi–modal logic; but the rules of computation of the truth values for composed formulas (statements) may use simultaneously several modal operations and several truth valuations V_i for initial propositional letters (upon which the formulas are built). That looks, in our view, as a more precise modeling of the multi–agency when we use the relational models as a base. After some short motivating part and fixing computational rules, we turn to the satisfiability problem in the classes of multi–modal and many–valued models (the satisfiability problem in corresponding logics). Using an appropriate modification of the filtration technique, we obtain a solution of the satisfiability problem in basic but most wide and important classes of multi–valued multi–modal models. We conclude the paper with comments on possible application areas and point out some open problems in the suggested direction.

2. Preliminaries, Definitions, Syntax and Semantics

Generally speaking, multi–modality means a usage of a collection of modal operations $\Box_i$ instead of a single one $\Box$ (which usually means 'necessary'). Nowadays multi–modal logic is applied in the formalization of knowledge representation (for example, epistemic logic allows several agents and managing the belief or knowledge of each agent). In the possible worlds semantics (Kripke–Hintikka semantics), a multi–modal extension of Kripke semantics uses the introduction of distinct accessibility relations instead of a single *common* accessibility relation. Many new results and modern technique in this field may be found in Gabbay et al. [11]. In this section we just briefly recall the notation and the necessary definitions.

The language of multi–modal logic consists of a (potentially infinite) set of propositional letters (propositional variables), standard Boolean logical operations $\wedge, \vee, \rightarrow, \neg$ and a finite set of unary logical operations $\Box_i$ ('necessary' from the viewpoint of the agent i). The formation rules for formulas are as usual; in particular, if φ is a formula, then for any $\Box_i$, $\Box_i\varphi$ is a formula. The operation $\Diamond_i$ ('possible' from the viewpoint of the agent i) may be introduced as follows: $\Diamond_i := \neg\Box_i\neg$. It may be applied in the opposite definition — to take modal operations $\Diamond_i$ as basic and express $\Box_i$ as $\Box_i := \neg\Diamond_i\neg$. The semantics for multi–modal logic, which we will offer here, differs from the standard relational semantics of multi-modal logics. Concerning the part of our semantics related the frames themselves, it is defined as usual.

Definition 1. *A k–multi–modal frame is a tuple $F_k = \langle W, R_1, \ldots, R_k, \rangle$, where W is a set (of worlds/states) and all R_i are binary relations on W.*

Definition 2. *A k–multi–modal frame with the objective accessibility relation R_0 is a $k+1$-multi-modal frame $F_k = \langle W, R_0, R_1, \ldots, R_k, \rangle$ such that $\forall i\ R_i \subseteq R_0$.*

We mean here that the relation R_0 is the objective one, which does not depend on peseption/knowledge of the agents (objects) i. Therefore we suggest $R_i \subseteq R_0$; the whole arbitrary R_i may be not complete — if we consider R_i as modeling the accessibility relation of the agents (because R_i may mean the accessibility, possible individual independent computational runs, etc.). Though R_0 absorbs all the possible accessibility and besides it may include some parts which are not visible for all agents themselves.

Turning to relational models with multi–valuations, the definition differs from the standard one, since we consider various (in particular, possibly different) valuations for subjects (agents) i.

Definition 3. *A k–multi-valued, model M_k is a pair $\langle F_k, V_1, \ldots, V_k\rangle$, where (i) F_k is a k–multi–modal frame; (ii) Any V_l is a valuation of a fixed for M_k set P of propositional letters in this frame, that is, for any letter $p \in P$, $V_l(p) \subseteq |F_k|$, we will use notation $(M_k, a) \Vdash_{V_l} p$ iff $a \in V_l(p)$.*

We may extend the valuations from propositional letters to all formulas as follows:

Definition 4. For any $a \in M_k$ and any V_j:

$$(M_k, a) \Vdash_{V_j} \neg\varphi \quad \Leftrightarrow \quad (M_k, a) \nVdash_{V_j} \varphi;$$

$$(M_k, a) \Vdash_{V_j} (\varphi\ \wedge\ \psi) \quad \Leftrightarrow \quad ((M_k, a) \Vdash_{V_j} \varphi) \wedge ((M_k, a) \Vdash_{V_j} \psi);$$

$$(M_k, a) \Vdash_{V_j} (\varphi \ \vee \ \psi) \quad \Leftrightarrow \quad ((M_k, a) \Vdash_{V_j} \varphi) \vee ((M_k, a) \Vdash_{V_j} \psi);$$
$$(M_k, a) \Vdash_{V_j} (\varphi \ \to \ \psi) \quad \Leftrightarrow \quad ((M_k, a) \Vdash_{V_j} \psi) \vee ((M_k, a) \nVdash_{V_j} \varphi);$$

The part above is a standard one, but for the formulas of kind $\Box_i\psi$, $\Diamond_i\psi$, and any V_j the rules are the following:

$$(M_k, a) \Vdash_{V_j} \Box_i\varphi \quad \Leftrightarrow \quad (\forall b,\ aR_ib \Rightarrow (M_k, b) \Vdash_{V_j} \varphi);$$

$$(M_k, a) \Vdash_{V_j} \Diamond_i\varphi \quad \Leftrightarrow \quad (\exists b,\ aR_ib \Rightarrow (M_k, b) \Vdash_{V_j} \varphi).$$

For example, consider the definition

$$(M_k, a) \Vdash_{V_j} \Diamond_i\varphi \quad \Leftrightarrow \quad \exists b,\ aR_ib \Rightarrow (M_k, b) \Vdash_{V_j} \varphi;$$

Here we need to know the truth values with respect to the valuation V_j; therefore we consider the truth values only w.r.t. the valuation V_j in the accessible sates. However, we evaluate the modality $\Diamond_i$, therefore we use R_i as the accessibility relation.

3. Main Technical Results, Satisfiability

Let K be any class of k–multi–modal frames. We may, following the classical scheme, define the multi–modal logic $L(K)$ of this class as the class of all formulas which are true w.r.t. any valuation at all worlds of any multi–valued model M based at any multi–modal frame from K (Though, it seems, in multi–agent environment, the general logic itself would not be of main point of attraction — the satisfiability looks to be more important, but nonetheless, we may model in logics the desirable laws about dependencies and hierarchy of the accessibility relations, etc., we will comment on it later.). The standard well known argument verifies the truth of the following

Lemma 1. *The following hold*
(i) All substitutional examples of all classical tautologies belong to $L(K)$;
(ii) $\forall i,\ \Box_i(p \to q) \to (\Box_i p \to \Box_i q) \in L(K)$;
(iii) $L(K)$ is closed w.r.t. model ponens, the rules of generalization: $x/\Box_i x$ for all i, and the rule of substitution of arbitrary formulas instead of letters.

(This technique is well known since 1960x, for modern layout you may refer to e.g. [11].) So, in our modification, logics $L(K)$ look as usual multi-modal logics and the presence of various valuations does not break basic logical properties. Besides if K will consist of (i) all reflexive (w.r.t. all accessibility relations R_i), (ii) transitive, or (iii) equivalence relations, the logics $L(K)$ will contain the corresponding well known modal logical laws:

(a) for (i): $\forall i \ \ \Box_i p \to p \in L(K)$;
(b) for (ii): $\forall i \ \ \Box_i p \to \Box_i\Box_i p \in L(K)$;
(c) for (ii): $\forall i \ \ \Diamond_i p \to \Box_i\Diamond_i p \in L(K)$.

To extend these standard properties, we may express the interference between various accessibility relations. For example, if we allow an objective accessibility relation R_0, which is the strongest one among others, we have

$$\forall i,\ \Box_0 p \to \Box_i p \in L(K), \quad \Diamond_i p \to \Diamond_0 p \in L(K).$$

Notice that if R_0 itself is transitive but the others maybe not, then nonetheless

$$\Box_0 p \to \Box_{i_1}\Box_{i_2}\ldots\Box_{i_m} p \in L(K),\ \Diamond_{i_1}\Diamond_{i_2}\ldots\Diamond_{i_m} p\ \to\ \Diamond_0 p \in L(K).$$

If at least one R_i is reflexive, this implies that R_0 is reflexive. But if some R_i is transitive, it does not imply that R_0 is transitive, because R_0 may have among accessible worlds those which are not accessible by that R_i.

The satisfiability problem is very popular in logic and applications to CS. It is a general well known problem which, in particular, implies, as a rule, the decidability of the corresponding logic (because φ is a theorem iff $\neg\varphi$ is not satisfiable). Turning to our case, let a class of frames K be given. A formula φ is said to be j–satisfiable in a model based at a frame F_k from K if there is a state a of this frame where φ is true at a w.r.t. the valuation V_j.

Turning to the satisfiability, let us start from the class K_f of all frames (with R_0 or without R_0). For a formula φ, $Sub(\varphi)$ denotes the set of all subformulas of the formula φ and all formulas $\Diamond_i\psi$, $i \neq 0$, when $\Diamond_0\psi$ is a subformula of ψ.

It will be convenient for us now to consider $\Diamond_i$ as the basic modal operations considering the operations $\Box_i$ as abbreviations for $\neg\Diamond_i\neg$ and think then that all the formulas in our considerations contain modal operations $\Diamond_i$ only.

Lemma 2. *If a formula φ is j–satisfied in a model M based on a frame from K_f by a valuation V_j, that is, for some a, $(M_k, a) \Vdash_{V_j} \varphi$ then φ may be satisfied in a finite model M_{fin} based at a frame from K_f by a valuation V_j, where $||M_{fin}|| \leq 2^{2^{||Sub\varphi|| \times k}}$.*

Proof. We will use a modified filtration technique. Let us define the following equivalence relations on states (worlds) from M:

$$\forall a, b \in |M|,\ a \equiv b\ \Leftrightarrow$$

$$[\forall\psi \in Sub(\varphi),\ \forall V_j, (M, a) \Vdash_{V_j} \psi \Leftrightarrow (M, b) \Vdash_{V_j} \psi].$$

If $\forall a, b \in |M|$,

$$a \equiv_j b\ \Leftrightarrow [\forall\psi \in Sub(\varphi),\ (M, a) \Vdash_{V_j} \psi \Leftrightarrow (M, b) \Vdash_{V_j} \psi].$$

Let, for any a from M, $[a]_\equiv$ be the set of all states from M equivalent to a w.r.t. the relation $\equiv$. For any j, the sets $[a]_{\equiv_j}$ have similar meaning. Then it is easy to see that

$$[a]_\equiv = \bigcap_j [a]_{\equiv_j}. \tag{1}$$

We define the valuations V_j on the equivalence classes as follows:

$$[a]_\equiv \Vdash_{V_j p}\ \Leftrightarrow a \Vdash_{V_j} p. \tag{2}$$

It is easy to see that the definition is correct and does not depend on the choice of the state a generating $[a]_\equiv$. Now define the accessibility relations R_i on classes $[a]_\equiv$ as follows:

$$\forall a, b \in |M|, \forall i,\ [a]_\equiv R_i [b]_\equiv \Leftrightarrow \forall\Diamond_i\psi \in Sub(\varphi), \tag{3}$$
$$\forall j,\ (M, b) \Vdash_{V_j} \psi \Rightarrow (M, a) \Vdash_{V_j} \Diamond_i\psi.$$

Since (1), this definition is correct as well and does not depend on the choice of states generating the equivalence classes. Now it only remains to apply a light modification of the filtration technique. Let $M_{\equiv}$ denote the obtained model on the equivalence classes. We need to show that

$$(4) \qquad \forall a \in |M|, \forall \psi \in Sub(\varphi),\ \forall j,\ [(M_{\equiv}, [a]_{\equiv}) \Vdash_{V_j} \psi \Leftrightarrow (M, a) \Vdash_{V_j} \psi].$$

Proof of (4) follows by induction on the length of ψ. For ψ to be the propositional letter it follows from (2). For formulas with main logical operations being the Boolean operations the proof of (4) is a standard routine computation.

Let now $\Diamond_i\psi \in Sub(\varphi), a \in |M|$, and $(M, a) \Vdash_{V_j} \Diamond_i\psi$. Then there exists a $b \in |M|$ such that $[(aR_ib)\&(M, b) \Vdash_{V_j} \psi)]$. It is easy to see that then $[a]_{\equiv}R_i[b]_{\equiv}$, which immediately follows from (3). By inductive assumption we have that $(M_{\equiv}, [b]_{\equiv}) \Vdash_{V_j} \psi$ and then, consequently, we obtain $(M_{\equiv}, [a]_{\equiv}) \Vdash_{V_j} \Diamond_i\psi$. Conversely, let now for some a, $(M_{\equiv}, [a]_{\equiv}) \Vdash_{V_j} \Diamond_i\psi$. Then

$$\exists b \in |M|,\ [([a]_{\equiv}R_i[b]_{\equiv})\&((M_{\equiv}, [b]_{\equiv}) \Vdash_{V_j} \psi)].$$

Then by the inductive assumption we obtain $(M, b) \Vdash_{V_j} \psi$ and by the definition of the relation R_i on $M_{\equiv}$ — cf. (3) we obtain that $(M, a) \Vdash_{V_j} \Diamond_i\psi$. This completes the proof of (4). Thus the model $M_{\equiv}$ satisfies the formula φ and the size of $M_{\equiv}$ is at most $2^{2^{||Sub\varphi|| \times k}}$.

Lemma 3. *If the model M has objective accessibility relation then the relation R_0 in the filtrated model $M_{\equiv}$ is also objective.*

Proof. Let $i \neq 0$, $[a]_{\equiv}R_i[b]_{\equiv}$, $\Diamond_0\psi \in Sub(\varphi)$, and $(M, b) \Vdash_{V_0} \psi$. Then also $\Diamond_i\psi \in Sub(\varphi)$ and by definition of R_i on $M_{\equiv}$, we obtain $(M, a) \Vdash_{V_0} \Diamond_i\psi$. Then for some c, aR_ic, where $(M, c) \Vdash_{V_0} \psi$. Since R_0 is stronger than R_i we get aR_0c and $(M, c) \Vdash_{V_0} \psi$, so $(M, a) \Vdash_{V_0} \Diamond_0\psi$. By definition of R_0 on on $M_{\equiv}$ it follows that $[a]_{\equiv}R_0[b]_{\equiv}$. □

Using Lemmas 2 and 3 we get

Theorem 1. *The satisfiability problem for the class K_f is decidable.*

Now we would like to extend this result to other classes of frames but not only to the class K_f of all frames itself. It is reasonable here to go along the list of classes of frames with popular restrictions for accessibility relations.

Theorem 2. *Let K_r, K_t, $K_{r,t}$ be respectively the sets of all reflexive, transitive, and reflexive and transitive frames without R_0. Then the satisfiability problem for any of these classes is decidable.*

Proof. For the class K_r, in order to prove the statement of Theorem 2, it is sufficient just to extend the proof of Lemma 2 — that is, to show that the relations R_i defined in the proof of Lemma 2 are reflexive; so, cf. (3)

$$\forall a, b \in |M|, \forall i, \quad [a]_{\equiv}R_i[b]_{\equiv} \quad \Leftrightarrow \quad \forall \Diamond_i\psi \in Sub(\varphi),$$
$$\forall j,\ [(M, b) \Vdash_{V_j} \psi \Rightarrow (M, a) \Vdash_{V_j} \Diamond_i\psi].$$

Let $\psi \in Sub(\varphi)$ and $(M, a) \Vdash_{V_j} \psi$. Since M is reflexive, we get aR_ia and hence $(M, a) \Vdash_{V_j} \Diamond_i\psi$, consequently $[a]_{\equiv}R_i[a]_{\equiv}$. Thus any R_i in the filtrated model is reflexive. And the statement of our theorem holds for K_r. Notice that if the model M would have R_0 then the relation R_0 in the filtrated model $M_{\equiv}$ would also be objective – cf. Lemma 3. That will be already not a case for the remaining classes of models. So, recall that in the sequel we do not have the relation R_0 and the corresponding modal operations.

For K_t such a simple argument as above does not work and we need to redefine the relations R_i on the classes $[a]_\equiv$. First, we need to consider the modal operations $\Box_i$ instead of $\Diamond_i$, and so, we accept that all formulas φ may contain only modal operations $\Box_i$ (since operations $\Box_i$ and $\Diamond_i$ are mutually expressible). We take the filtrated model as earlier but now redefine the accessibility relations as follows:

$$(5) \qquad \forall a, b \in |M|, \forall i, \quad [a]_\equiv R_i [b]_\equiv \;\Leftrightarrow\; \forall \Box_i\psi \in Sub(\varphi)$$
$$\forall j,\ (M, a) \Vdash_{V_j} \Box_i\psi \Rightarrow (M, b) \Vdash_{V_j} \Box_i\psi \wedge \psi.$$

It is clear that this definition is correct and does not depend on the choice of the representatives in the equivalence classes. Besides, it immediately follows from the definition that any R_i is transitive. It remains only to prove that

$$(6) \qquad \forall a \in |M|, \forall\psi \in Sub(\varphi),\ \forall j,\ (M, [a]_\equiv) \Vdash_{V_j} \psi \;\Leftrightarrow\; (M, a) \Vdash_{V_j} \psi.$$

Again, the proof of (6) follows by induction on the length of ψ. For ψ to be a propositional letter it follows from definition of the valuations — cf. (2). For formulas with main logical operations being the Boolean operations the proof of (6) is a standard routine computation.

Let now $\Box_i\psi \in Sub(\varphi), a \in |M|$ and $(M, a) \Vdash_{V_j} \Box_i\psi$. Let $[a]_\equiv R_i [b]_\equiv$. Then by definition of R_i on the equivalence classes, we have $(M, b) \Vdash_{V_j} \Box_i\psi \wedge \psi$; in particular, $(M, b) \Vdash_{V_j} \psi$ and by the inductive assumption we get $[b]_\equiv \Vdash_{V_j} \psi$. So, $[a]_\equiv \Vdash_{V_j} \Box_j\psi$.

Vice versa, let now $[a]_\equiv \Vdash_{V_j} \Box_j\psi$. Assume that aR_ib. Then by the definition of R_i on the equivalence classes we have $[a]_\equiv R_i [b]_\equiv$ because K_t consists of transitive frames only. Therefore $[b]_\equiv \Vdash_{V_j} \psi$, and by inductive assumption $(M, b) \Vdash_{V_j} \psi$. So, (6) holds, hence the filtrated model is transitive and will satisfy φ.

Consider now the class $K_{r,t}$. Now we use the same filtration as for K_t and we need only to show that in this case the filtrated models will also be reflexive. Let $\Box_i\psi \in Sub(\varphi)$ and $a \in M \in K_{r,t}$ and $(M, a) \Vdash_{V_j} \Box_i\psi$. Since M is reflexive we get $a \Vdash_{V_j} \Box\psi \wedge \psi$ and consequently $[a]_\equiv R_i [a]_\equiv$. So, these filtrated models are reflexive and transitive, which concludes the proof of our theorem. □

Notice that we cannot yet prove the property of the objective accessibility relations R_0 (if we will allow it) for classes K_t and $K_{r,t}$ the same way as we did for the class K_f in Lemma 3, so the question is still open.

4. Areas of Applications, Problems

First we would like to briefly illustrate a possible usage of multi–valued multi–modal (multi–agent) logics in AI and CS. Fist we comment on modeling of the multi–agency. We used the modeling of multi–agency via different accessibility relations. This looks meaningful and it well correlates with our intuition. For example:

(1) Consider a discussion, reasoning, and obtaining a common decision in a large group of experts. Here the accessibility relations R_i may model an access to personal databases of experts, a possibility to get consultancy with other experts,

etc. These databases may have a non–empty intersection, a large common part, or to be totally independent. Evaluating the information, reasoning about its correctness and consistency may include operations similar to $\Diamond_i$ and to their applications to various formulas and various Boolean combinations of such formulas.

(2) Another example is multi–thread computations and common computational runs. Relations R_i are the accessibility relations of agents (for example, appropriate software) to time check points of the computation. They (checkpoints) may be located in independent intervals of multi–thread computations or in some common parts; and the verification of a checkpoint for correctness (truth) can be undertaken by any agent (or, responsible, a program code). This altogether increases the correctness of decisions about the expected behavior of a computation.

(3) One can say some things similar to (2) about the internet communication and the work of agents (as humans or using software) looking for security and efficiency of networks. Similarly, the accessibility relations R_i may be viewed as approved rules for agents to control individual web sites for content and to verify the correctness.

Now we would like to comment on the presence of multi–valuations in our approach and on our decision to consider many valuations in the relations models, for each individual agent.

(4) It looks very well justified, since any agent has its own knowledge on the facts (coded by propositional letters) to be true or false. Usually in multi–modal logics the valuation in single — so to say — it is objective; it is the same for all agents for all coding propositional letters. Though a clearly more general and important case of modeling multi–agency is the situation when the distinct agents have their own valuations; they may have a good agreed common part or may differ much; everything depends on the particular case of the modelling.

(5) A more important part of our work is the consideration of truth values for different truth valuations V_i of compound complicated formulas which may contain as subformulas formulas of kind $\Diamond_j\varphi$ for various $\Diamond_j$. So, here V_j is the valuation of the agent j; but all $\Diamond_j$ while the computation of truth values of formulas $\Diamond_j\varphi$ w.r.t. V_j nonetheless uses distinct accessibility relations R_l. Fortunately, rather standard technique works well and we can find computational algorithm for satisfiability problem. We have completed it for most standard versions of multi-modal logics only, and for many ones it is an open problem.

(6) If we work with the multi–agency as in (1) — (3) above, then all we commented there also holds when we consider distinct individual valuations V_i for distinct agents i. This looks as a more precise modeling, when we cannot lay upon restrictions for valuations to be the same, and consider principally distinct valuations reflecting knowledge of the agents instead.

It is relevant to note that we may consider the case when the number of accessibility relations R_i (and corresponding modal operations) does not coincide with the number of valuations V_j. All the proofs and results remain to be the same as earlier. They can make sense, when we consider the agents responsible for accessibility relations and agents responsible for the valuations to be different, totally independent or coinciding in some parts.

This research area has many open questions. For example, as we commented earlier, we cannot prove the property of the objective accessibility relations R_0 for classes K_t and $K_{r,t}$, and the question is open. Another interesting open case is to extend our results to multi–valued versions of the other popular modal systems, as for example $S4.1$, or e.g. versions of the Gabbay do Jongh logics with bounded branching (cf. [7]). Similar may be said about multi–valuated versions of temporal logics. Another open question is to study multi–valuated relational models with distinct (for agents) base sets of possible worlds, which can have non–empty intersections (common knowledge for agents, so to say — objective parts).

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Michael Alexandrovich Moor
Institute of Mathematics and Fundamental Informatics,
Siberian Federal University
Siberian Federal University,
79 Svobodny pr.,
660041 Krasnoyarsk, Russia

Vladimir Vladimirovich Rybakov
Institute of Mathematics and Fundamental Informatics,
Siberian Federal University,
79 Svobodny pr.,
660041, Krasnoyarsk, Russia
A.P. Ershov Institute of informatics systems SB RAS,
Acad. Lavrentjev pr., 6,
Novosibirsk 630090, Russia
E-mail address: vadimir_rybakov@mail.ru