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THE CAUCHY PROBLEM FOR THE DEGENERATED PARTIAL DIFFERENTIAL EQUATION OF THE HIGH EVEN ORDER

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Abstract. In this paper we develop a method for investigating the Cauchy problem for a degenerate differential equation of high even order. Applying the generalized Erdélyi-Kober operator, the formulated problem reduces to a problem for an equation without degeneracy. Further, necessary and sufficient conditions for reducing the order of the equation are proved. Two examples demonstrate the application of the developed method.

Keywords: Fractional integrals and derivatives, generalized Erdélyi-Kober operator, Bessel operator, degenerate differential equations.

1. Introduction

Degenerate equations simulate processes occurring near the boundaries of different domains. The influence of the boundaries leads to the fact that near the boundary the type or order of the equation describing the process changes. In this case we say that the equation degenerates.

Degenerate equations are encountered in the theory of boundary layers, in the theory of shells (if the loading does not comprise moments), in the theory of diffusion processes, in particular in the theory of Brownian motion, and in many other problems in physics and mechanics.

The degenerate second-order equations and boundary value problems for them are well studied. Second-order degenerate equations of elliptic and parabolic types have been most extensively studied. Besides, most of the studies of degenerate hyperbolic equations concern second-order equations with two independent variables which degenerate at the boundary of the domain. These studies were inspired, in the first place, by the study of equations of mixed type and related problems in

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gas dynamics. More detailed information about this direction can be found in the monographs of A.V. Bitsadze [1], M.M. Smirnov [2], R. Carroll and R. Showalter [3].

In the last years a lot of papers involving degenerate and singular partial differential equations have been published in important journals of the world. In particular, these are the works [4, 5, 6, 7, 8] and others. An overview of some of the works can be found in [9].

In this paper, in contrast to the cited sources, in investigating the Cauchy problem for a degenerate higher-order equation, we apply a different approach. Namely, by a change of variables we reduce the degenerate equations to an equation with singular coefficients, and then, taking into account the specific nature of the singular equations, we use the generalized Erdélyi - Kober operator [10]. The use of this operator allows us to reduce equations with the lowest term of $\lambda^2 u$ and with a singular Bessel operator that acts on one or several variables to non-singular equations without the minor term of $\lambda^2 u$. We demonstrate this approach by the example of a solution of the Cauchy problem for a degenerate differential equation of high even order.

Let R^n $(n \geq 1)$ be a Euclidean n -dimensional space, $x = (x_1, x_2, \ldots, x_n) \in R^n$ a point of this space, m be a natural number. In the domain $\Omega = \{(x,t) : x \in R^n,\ t \in R,\ t > 0\}$ we consider the degenerate equation

$$u_{tt} + t^p L^{2m}(u) + \lambda^2 t^p u = f(x,t),\ \ (x,t) \in \Omega, \tag{1}$$

where L is a linear differential operator of any order in the variable $x \in R^n$, independent of the variable t; the $L^{2m} = L^{2m-1}L$ is the $2m$ -th composition of the L operator; $f(x,t)$ is a given smooth function; λ, $p \in R$ and $p \geq 0$.

Suppose that the natural number of the $k > 1$ is the order of the L operator. We denote by M the class of functions that are continuously differentiate once by t and mk times in x in the closed domain of $\bar{\Omega} = \{(x,t) : x \in R^n,\ t \in R, t \geq 0\}$ and are continuously differentiable twice by t and $2mk$ times by x in the domain of an Ω.

The Cauchy problem. Find the solution $u(x,t) \in M$ of equation (1) satisfying the initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),\ x \in R^n, \tag{2}$$

where $\varphi(x)$, $\psi(x)$ are given smooth functions.

In the problem (1), (2) we make the change of variables of $y = [2/(p+2)]t^{(p+2)/2}$. Then equation (1) and initial conditions (2) will accordingly take the form

$$A_\beta(u) \equiv u_{yy} + \frac{2\beta}{y} u_y + L^{2m}(u) + \lambda^2 u = f_0(x,y), \tag{3}$$

$$u(x,0) = \varphi(x),\ \lim_{y \to +0} y^{2\beta} u_y(x,y) = \psi_0(x),\ x \in R^n, \tag{4}$$

where $f_0(x,y) = (1-2\beta)^{4\beta} y^{-4\beta} f[x,\ (1-2\beta)^{2\beta-1} y^{1-2\beta}]$, $\psi_0(x) = (1-2\beta)^{2\beta}\psi(x)$, $2\beta = p/(p+2)$ and at $p > 0$, we have $0 < 2\beta < 1$.

First, we construct a solution of (3) satisfying the semi-homogeneous initial conditions

$$u(x,0) = \varphi(x), \quad u_y(x,0) = 0,\ x \in R^n. \tag{5}$$

In this paper, to construct a solution of the Cauchy problem (3), (5), we shall use the generalized Erdélyi - Kober operator of fractional order [10]. Therefore, we give some properties of this operator.

2. The generalized Erdélyi - Kober operator

In the theory and applications various modifications and generalizations of classical operators of integration and differentiation of fractional order are widely used. Such modifications include in particular Erdélyi - Kober operators [10, 11, 12].

Lowndes [11] introduced and studied the generalized Erdélyi - Kober operator with Bessel function in the kernel:

$$J_\lambda(\eta,\,\alpha)f(x) = 2^\alpha\lambda^{1-\alpha}x^{-2\alpha-2\eta}\int\limits_0^x t^{2\eta+1}(x^2-t^2)^{(\alpha-1)/2}J_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)f(t)dt, \tag{6}$$

where $\alpha,\eta,\,\lambda\in R,\ \alpha>0,\ \eta\geq-(1/2)$, $J_\nu(z)$ is the Bessel function of the first kind of order ν. The operator (6) for $\lambda\to0$ coincides with the usual Erdélyi - Kober operator [10]

$$I_{\eta,\alpha}f(x)=\frac{2x^{-2(\eta+\alpha)}}{\Gamma(\alpha)}\int\limits_0^x(x^2-t^2)^{\alpha-1}t^{2\eta+1}f(t)dt, \tag{7}$$

where $\Gamma(\alpha)$ is the Euler gamma function [13].

An inverse operator of the operator (6) with $0<\alpha<1$ has the form [10]

$$J_\lambda^{-1}(\eta,\alpha)f(x)=\frac{x^{-2\eta-1}}{\Gamma(1-\alpha)}\frac{d}{dx}\int\limits_0^x\frac{\bar{I}_{-\alpha}\left(\lambda\sqrt{x^2-s^2}\right)}{(x^2-s^2)^\alpha}s^{2(\eta+\alpha)+1}f(s)ds, \tag{8}$$

where $\bar{I}_\nu(z)=\bar{J}_\nu(iz)=\Gamma(\nu+1)(z/2)^{-\nu}I_\nu(z)$, $I_\nu(z)$ is the Bessel function of the imaginary argument. Hence, taking $\bar{I}_\nu(0)=1$ into account, for $\lambda=0$ we obtain from (8) the inverse operator of the operator (7):

$$I_{\eta,\alpha}^{-1}g(x)=\frac{x^{-2\eta-1}}{\Gamma(1-\alpha)}\frac{d}{dx}\int\limits_0^x(x^2-s^2)^{-\alpha}s^{2(\eta+\alpha)+1}g(s)ds \tag{9}$$

The basic properties of these operators can be found in [10].

Some properties of the operator (6) were generalized in [14, 15], in particular, the following theorem is proved.

Let $l\in N\cup\{0\}$, $[B_\eta^x]^0=E$, E be the unit operator and $[B_\eta^x]^l=[B_\eta^x]^{l-1}[B_\eta^x]$ be the l-th power of the Bessel operator $B_\eta^x=x^{-2\eta-1}\dfrac{d}{dx}x^{2\eta+1}\dfrac{d}{dx}=\dfrac{d^2}{dx^2}+\dfrac{2\eta+1}{x}\dfrac{d}{dx}$.

Theorem 1. *Let* $\alpha>0,\eta\geq-(1/2)$; $f(x)\in C^{2l}(0,b)$, $b>0$; *the functions* $x^{2\eta+1}[B_\eta^x]^kf(x)$ *be integrable at zero and* $\lim\limits_{x\to0}x^{2\eta+1}\dfrac{d}{dx}[B_\eta^x]^kf(x)=0,\ k=\overline{0,l-1}$. *Then*

$$[B_{\eta+\alpha}^x+\lambda^2]^lJ_\lambda(\eta,\alpha)f(x)=J_\lambda(\eta,\alpha)[B_\eta^x]^lf(x),$$

in particular, if $\lambda=0$, *then*

$$[B_{\eta+\alpha}^x]^lI_{\eta,\alpha}f(x)=I_{\eta,\alpha}[B_\eta^x]^lf(x).$$

Theorem 1 allows us to apply the operator (6) as a transmutation operator (for the transmutation operator, see [16, 17, 18]), which allows one to transform equations with singular coefficients into equations without singular coefficients. This fact is applicable to the solution of the Cauchy problem (3), (5). We note that in [19, 20] the Erdélyi - Kober operator was applied to the solution of the Cauchy problem for partial differential equations of hyperbolic type.

3. Application of the Erdélyi - Kober operator

Suppose that the solution of the problem (3), (5) exists. We seek this solution in the form of a generalized Erdélyi - Kober operator (6):

$$u(x,y) = J_{\lambda}^{(y)}(-(1/2),\,\beta)\;V(x,y) \tag{10}$$

where $V(x,y)$ is an unknown twice continuously differentiable function.

We substitute (10) into the equation (3) and the initial conditions (5). Then, using Theorem 1 for $l=1, \alpha=\beta, \eta=-1/2$ and the inverse operator (8) (for $\alpha=\beta, \eta=-1/2$), we obtain the following problem: find the solution of $V(x,y)$ of equation

$$\frac{\partial^2 V}{\partial y^2} + L^{2m}(V) = F_0(x,y), \tag{11}$$

satisfying the initial conditions

$$V(x,0) = k_0\varphi(x), \quad V_y(x,0) = 0, \quad x \in R^n, \tag{12}$$

where $k_0 = \Gamma(\beta+1/2)/\sqrt{\pi}$,

$$F_0(x,y) = \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial y}\int\limits_0^y (y^2-s^2)^{-\beta}\bar{I}_{-\beta}(\lambda\sqrt{y^2-s^2})s^{2\beta}f_0(x,s)ds. \tag{13}$$

The following theorem holds.

Theorem 2. *In order that the $V(x,y) \in M$ function be a solution of the problem* (11), (12) *it is necessary and sufficient that the function*

$$U(x,y) = V(x,y) + i\int\limits_0^y L^m(V(x,\tau))d\tau, \tag{14}$$

was a solution of equation

$$U_y - iL^m(U) = F(x,y), \quad (x,y)\in\Omega, \tag{15}$$

satisfying the initial condition

$$U(x,0) = k_0\varphi(x), \quad x \in R^n, \tag{16}$$

where $F(x,y) = \int\limits_0^y F_0(x,\tau)d\tau$, *$i$ is the imaginary unit.*

Proof. 1. Necessity. Let the $V(x,y)\in M$ solution of the problem (11), (12). Let us show that the function $U(x,y)$, defined by (14), is a solution of the problem (15), (16).

Differentiating equation (14) with respect to y, then applying to it the operator iL^m and taking into account the initial condition $V_y(x,0)=0$, we obtain

$$U_y - iL^m(U) - F(x,y) = \int_0^y \left[V_{\tau\tau} + L^{2m}(V) - F_0(x,\tau)\right] d\tau. \tag{17}$$

Thus, as a function of $V(x,y)$ is a solution of equation (11), it follows from (17) that the function $U(x,y)$ is a solution of equation (15).

Since $V(x,y)\in M$ from equation (14) with $y=0$, we have

$$U(x,0) = V(x,0). \tag{18}$$

This implies the satisfaction of the initial condition (16).

2. Sufficiency. Suppose that the function $U(x,y)$ defined by (14) is a solution of the problem (15), (16). Let us show that the function $V(x,y)\in M$ is a solution of the problem (11), (12).

First, we show that the initial conditions (12) are satisfied. From (18) follows the feasibility of the initial condition $V(x,0)=k_0\varphi(x)$. From equation (14) and equation (15), we obtain $V_y = iL^m(U-V)+F(x,y)$. Hence for $y\to 0$, in view of (18) and $F(x,0)=0$, we have $V_y(x,0)=0$. Then equality (17) holds. By virtue of the fact that the function $U(x,y)$ is a solution of equation (15), from (17) follows the satisfaction of equation (11). This completes the proof of Theorem 2. □

The following corollary is true.

Corollary 1. *Suppose that the function $U(x,y)\in M$ is a solution of the problem* (15), (16), *where $\varphi(x)$, $F(x,y)$ are real functions. Then the function $V(x,y)=\operatorname{Re}U(x,y)$ is a solution of the problem* (11), (12), *where* $\operatorname{Re}U$ *denotes the real part of the function $U(x,y)$.*

To construct a solution of equation (3), satisfying the initial conditions

$$u(x,0)=0,\ \lim_{y\to+0} y^{2\beta}u_y(x,y)=\psi_0(x),\ x\in R^n, \tag{19}$$

we apply the following property of this equation:

Lemma 1. *If $u(x,y;1-\beta)$ is a solution of the equation $A_{1-\beta}(u)=0$ satisfying conditions* (5), *then the function $w(x,y;\beta)=y^{1-2\beta}u(x,y;1-\beta)$ is a solution of the equation $A_\beta(w)=0$ satisfying conditions*

$$w(x,0)=0,\ \lim_{y\to+0} y^{2\beta}w_y(x,y)=(1-2\beta)\varphi(x),\ x\in R^n.$$

This lemma is proved by direct computation. Taking into account Lemma 1 and replacing $(1-2\beta)\varphi(x)$ by $\psi_0(x)$ on the basis of the solution of the equation $A_\beta(u)=0$, satisfying conditions (5), we can construct a solution of the equation $A_\beta(u)=0$, satisfying conditions (19).

4. Examples

4.1. **An example of applying the method to a fourth-order equation.** As an example, we consider the equation

$$u_{tt} + t^p u_{xxxx} + \lambda^2 t^p u = 0.$$

This equation of the fourth order coincides with equation (1) for $n = 1,\ m = 1$, $L \equiv \partial^2/\partial x^2$ and $f(x, y) = 0$. The Cauchy problem for this equation has not been investigated previously.

In this case, equation (3) takes the form

$$u_{yy} + \frac{2\beta}{y} u_y + \frac{\partial^4 u}{\partial x^4} + \lambda^2 u = 0, \tag{20}$$

and equation (11) goes into equation

$$\frac{\partial^2 V}{\partial y^2} + \frac{\partial^4 V}{\partial x^4} = 0. \tag{21}$$

The last equation occurs in problems of oscillation of an elastic rod [21].

To solve the problem (11), (12) we apply the corollary to Theorem 2. In this case, equation (15) becomes a one-dimensional Schrëdinger equation

$$\frac{\partial U}{\partial y} - i \frac{\partial^2 U}{\partial x^2} = 0.$$

The solution of the problem (15), (16) in this case takes the form [21]

$$U(x, y) = \int_{-\infty}^{+\infty} \varphi(\xi) G(x, \xi, y) d\xi,$$

where

$$G(x, \xi, y) = \frac{1}{2\sqrt{\pi y}} \exp\left[i \left(\frac{(x-\xi)^2}{4y} - \frac{\pi}{4} \right) \right].$$

Then, by virtue of the corollary to Theorem 2, the solution of the problem (11), (12) has the form

$$V(x, y) = k_0 \int_{-\infty}^{+\infty} \varphi(\xi) G_1(x, y, \xi) d\xi, \tag{22}$$

where

$$G_1(x, y, \xi) = \frac{1}{2\sqrt{\pi y}} \cos\left[\frac{(\xi - x)^2}{4y} - \frac{\pi}{4} \right].$$

We substitute (22) into (10). Then we change the order of integration and calculate the inner integral. As a result, we get

$$u(x, y) = \frac{k_0}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(x + 2\xi\sqrt{y}) G_2(y, \xi; \beta) d\xi, \tag{23}$$

where

$$G_2(y, \xi; \beta) = \frac{\Gamma(1/4)}{\Gamma(\beta + (1/4))} K_1 \left(\beta + \frac{1}{4};\ \frac{3}{4},\ \frac{1}{2};\ -\frac{\xi^4}{4},\ -\frac{1}{4}\lambda^2 y^2 \right)$$
$$+ \frac{\Gamma(-1/4)}{\Gamma(\beta - (1/4))} \xi^2 K_1 \left(\beta - \frac{1}{4};\ \frac{5}{4},\ \frac{3}{2};\ -\frac{\xi^4}{4},\ -\frac{1}{4}\lambda^2 y^2 \right),$$

$$K_1(a, b, c;\ x, y) = \sum_{m=0}^{\infty} \frac{y^m}{(a)_m m!} {}_1F_2 \left(1 - a - m;\ b,\ c;\ x \right),$$

${}_1F_2\,(a;\ b,\ c;\ z)$ is a generalized hypergeometric function [13].

Now, to construct a solution of (20) satisfying the initial conditions (19), we use Lemma 1. Replacing $(1-2\beta)\varphi(x)$ by $\psi_0(x)$, on the basis of solution (23), we have

$$w(x,y) = k_1 y^{1-2\beta} \int_{-\infty}^{+\infty} \psi_0(x+2\xi\sqrt{y}) G_2(y,\xi;1-\beta)d\xi,$$

where $k_1 = \Gamma(1/2-\beta)/(2\pi\sqrt{2})$.

4.2. **An example of a method application to a fourth-order equation containing the square of the Bessel operator.** The above method can be applied also in the case when in equation (1) the L operator has a singular coefficient. As an example, in the region $\Omega^+ = \{(x,y) : 0 < x < +\infty, 0 < y < +\infty\}$ for a fourth-order equation

$$\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2}{\partial x^2} + \frac{2\alpha}{x}\frac{\partial}{\partial x}\right)^2 u = 0, \tag{24}$$

one can put and study the problem with initial

$$u(x,0) = \varphi(x),\ 0 \le x < +\infty,\ u_y(x,0) = 0,\ 0 < x < +\infty \tag{25}$$

and boundary conditions

$$u_x(0,y) = 0, \quad u_{xxx}(0,y) = 0, \quad 0 < y < +\infty, \tag{26}$$

where $\alpha \in R$, and $0 < \alpha < (1/2)$, $\varphi(x)$ is a given function.

Equation (24), in particular, arises in the study of the equation for the multi - dimensional free transverse vibration of a thin elastic plate $u_{yy} + \Delta^2 u = 0$ for axial symmetry in a spherical coordinate system, where $\Delta^2 = \Delta\Delta$ is a biharmonic operator and Δ is a multidimensional Laplace operator.

As in the previous example, we seek the solution of equation (24) in the form

$$u(x,y) = I^{(x)}_{-1/2,\,\alpha} V(x,y) \tag{27}$$

where $I^{(x)}_{-1/2,\,\alpha}$ is the Erdélyi - Kober operator (7), and $V(x,y)$ is an unknown function.

We substitute (27) into equation (24) and the initial conditions (25). Then, using Theorem 1 for $\lambda = 0,\ l = 2$ and taking into account the boundary conditions (26), we obtain the following problem: find the solution of the $V(x,y)$ of equation (21) satisfying the initial conditions

$$V(x,0) = \Phi(x),\ 0 \le x < +\infty,\ V_y(x,0) = 0,\ 0 < x < +\infty, \tag{28}$$

and homogeneous boundary conditions

$$V_x(0,y) = 0, \quad V_{xxx}(0,y) = 0, \quad 0 < y < +\infty. \tag{29}$$

where

$$\Phi(x) = \left(I^{(x)}_{-1/2,\,\alpha}\right)^{-1} \varphi(x).$$

To solve the problem (21), (28), (29), we can not directly use formula (22), since for negative values of the argument the function $\Phi(x)$ is not defined. We extend the function $\Phi(x)$ evenly to the negative part of the Ox -axis and denote by $\Phi_0(x)$ the extended function. Thus, we ensure the fulfillment of the homogeneous boundary condition (29) and the right to use formula (22), which has the form

$$V(x,y) = \frac{1}{2\sqrt{\pi y}} \int\limits_{-\infty}^{+\infty} \Phi_0(\xi) \cos\left[\frac{(\xi - x)^2}{4y} - \frac{\pi}{4}\right] d\xi. \tag{30}$$

Taking into account the parity of the function $\Phi_0(x)$, we represent the equality (30) in the form

$$V(x,y) = \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} \Phi(\xi) G_1(x,y,\xi) d\xi, \tag{31}$$

where

$$G_1(x,y,\xi) = \frac{1}{2\sqrt{y}}\left[\cos\frac{(x+\xi)^2}{4y} + \sin\frac{(x+\xi)^2}{4y} + \cos\frac{(x-\xi)^2}{4y} + \sin\frac{(x-\xi)^2}{4y}\right]$$

$$= \frac{4}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} \cos(y\eta^2)\cos(x\eta)\cos(\xi\eta)d\eta.$$

By (9), we rewrite $\Phi(x) = \left(I^{(x)}_{-1/2,\,\alpha}\right)^{-1} \varphi(x)$ in the form $\Phi(x) = \Phi_1'(x)$, where

$$\Phi_1(x) = \frac{1}{\Gamma(1-\alpha)} \int\limits_{0}^{x} (x^2 - s^2)^{-\alpha} s^{2\alpha} \varphi(s) ds.$$

We substitute the expression $\Phi(x) = \Phi_1'(x)$ into (31). We apply the formula of integration by parts. Then, we change the order of integration and calculate the inner integral, we find

$$V(x,y) = \frac{2^{1-\alpha}}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} \varphi(s) s^{(1/2)+\alpha} G_2(x,y,s) ds, \tag{32}$$

where

$$G_2(x,y,s) = \int\limits_{0}^{+\infty} \xi^{(1/2)+\alpha} \cos(y\xi^2)\cos(x\xi) J_{\alpha-(1/2)}(s\xi) d\xi.$$

We substitute (32) in (27) and change the order of integration. Then, calculating the inner integral, we find the solution of the problem (24) – (26) in the form

$$u(x,y) = \tag{33}$$

$$\frac{x^{(1/2)-\alpha}}{2y} \int\limits_{0}^{+\infty} \varphi(\xi)\xi^{(1/2)+\alpha} J_{\alpha-(1/2)}\left(\frac{x\xi}{2y}\right) \sin\left[\frac{x^2+\xi^2}{4y} + \frac{\pi}{2}\left(\frac{1}{2} - \alpha\right)\right] d\xi.$$

Remark *For $\alpha = 1/2$, the equation (24) goes over into the equation*

$$\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x}\right)^2 u = 0,$$

and formula (33) *into the well-known formula* [21, p. 629]

$$u(x,y) = \frac{1}{2y}\int\limits_0^{+\infty} \varphi(\xi)\xi J_0\left(\frac{x\xi}{2y}\right)\sin\left(\frac{x^2+\xi^2}{4y}\right)d\xi.$$

Similarly, we can study the Cauchy problem for other values of the parameters n, m and the operator L.

5. Conclusions

In this paper we applied the generalized Erdélyi - Kober operator of fractional order to investigate the Cauchy problem for a degenerate differential equation of high even order. In this case, the problem posed is reduced to the problem for the equation without degeneracy. We also proved necessary and sufficient conditions for the reduction of the order of the equation. The proposed approach is very effective and allows us to construct an exact solution of the formulated problem. These exact solutions can be used as test cases for asymptotic, approximate and numerical methods.

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