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LARGE DEVIATIONS FOR PROCESSES ON HALF-LINE: RANDOM WALK AND COMPOUND POISSON

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ABSTRACT. We establish, under the Cramer exponential moment condition in a neighbourhood of zero, the Extended Large Deviation Principle for the Random Walk and the Compound Poisson processes in the metric space \mathbb{V} of functions of finite variation on $[0, \infty)$ with the modified Borovkov metric.

Keywords: Large Deviations, Random Walk, Compound Poisson Process, Cramer's condition, rate function, Extended Large Deviation Principle.

1. Introduction

The theory of Large Deviations for trajectories of processes seen as elements of the appropriate function space is well developed. However, for functions defined on infinite intervals, such as \mathbb{R}^+ , the typical metric used for LDP is that of uniform convergence on compacts, for example, for LDP for continuous processes the space of continuous functions \mathbb{C} is used with the metric (e.g. [13], [7], [9])

$$(1.1) \quad \rho^{(P)}(f, g) := \sum_{k=1}^{\infty} 2^{-k} \min \left\{ \sup_{0 \leq t \leq k} |f(t) - g(t)|, 1 \right\}.$$

Convergence in metric $\rho^{(P)}$ is equivalent to convergence in $\mathbb{C}[0, T]$ with uniform metric for any $T \geq 0$, e.g. [14]. Hence a drawback of this metric is that it is «not sensitive» to the behaviour of functions at infinity. In [12] the LDP in the space \mathcal{C} with metric $\rho(f, g) = \sup_{t \geq 0} \frac{|f(t) - g(t)|}{1+t}$, on \mathbb{R}^+ is obtained for Diffusions and

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Random Walk, and is shown to be «more precise» than in the space $(\mathbb{C}, \rho^{(P)})$. For discontinuous processes, Dobrushin and Pecherskij [8] give the LDP for Compound Poisson processes on the half-line using a metric based on the uniform metric. Here we work under less stringent conditions, assuming exponential moments in the neighborhood of zero ($[\mathbf{C}_0]$), rather than on the whole line ($[\mathbf{C}_\infty]$), and generalize their results by using a different metric. LDP for Compound Poisson processes on $[0, 1]$ in the space of functions of bounded variation on $[0, 1]$ is given in [10] and recently generalized in [11] by using Borovkov's metric $\rho_{\mathbb{B}}$ instead of uniform. Here we extend results of [11] to the half-line yielding a generalization of [8]. Since here we work under less stringent moment conditions $[\mathbf{C}_0]$, we also generalize results of [12], that give the classical LDP on the half-line for Random Walk under $[\mathbf{C}_\infty]$. The proofs are different to those in [12].

For $k \geq 0$, denote by $S_k = \sum_{i=1}^k \xi_{(i)}$ the partial sums of i.i.d. r.v.'s $\{\xi_{(i)}\}_{i \geq 1}$ distributed as ξ , and $S_0 = 0$. Let $s = s(t)$ be the continuous piecewise linear function on $[0, \infty)$ going through the points $(0, S_0), (1, S_1), \dots, (k, S_k), \dots$. Define the process s_n by

$$s_n = s_n(t) := \frac{1}{x} s(tn), \quad t \geq 0, \quad n = 1, 2, \dots,$$

where $x = x(n) \sim n$ as $n \rightarrow \infty$.

Similarly the process ξ_T for a real $T \geq 1$ is defined. Consider a Compound Poisson process

$$\xi = \xi(t), \quad t \geq 0,$$

and let

$$(1.2) \quad \xi_T = \xi_T(t) := \frac{1}{x} \xi(tT), \quad t \geq 0, \quad T \geq 1,$$

where $x = x(T) \sim T$ as $T \rightarrow \infty$.

Two families s_n and ξ_T (processes $s(t)$ and $\xi(t)$) have much in common. $\xi(t)$ has independent increments, and so does $s(t)$, when taken at integer times $t = n$. If the r.v. ξ in the definition of $s(t)$ is taken to be $\xi(1)$, then $s(t)$ is the linear interpolation of $\xi(t)$, going through the points

$$(0, \xi(0)), (1, \xi(1)), \dots, (k, \xi(k)), \dots$$

Therefore it is not surprising, that under mild assumptions, the families s_n and ξ_T satisfy Large Deviation Principle with the common Rate Function, determined by the r.v. ξ . We assume throughout that r.v. ξ in definition (1.3) satisfies Cramer condition $[\mathbf{C}_0]$.

$[\mathbf{C}_0]$. For some $\delta > 0$

$$\mathbf{E} e^{\delta|\xi|} < \infty.$$

We establish the *Extended Large Deviation Principle* for two families

$$(1.3) \quad \{s_n = s_n(t); t \geq 0\}_{n \geq 1}, \quad \text{and} \quad \{\xi_T = \xi_T(t); t \geq 0\}_{T \geq 1}$$

defined on the half-line, $t \in [0, \infty)$.

The precise definition of ELDP is given in Section 3, (see also [4] or [5], ch. 4). ELDP holds under less stringent requirements on the rate function than the classical LDP, in particular the space is not required to be complete, and the rate function, while lower semi-continuous, is not required to be compact. However, if ELDP holds with a good rate function, then LDP follows.

ELDP for processes in (1.3) defined on $[0, 1]$ under the assumption $[\mathbf{C}_0]$ in the space of functions on $[0, 1]$ was established earlier in [6] and [11] (see also [5], ch. 4). The main contribution of this work is to extend the results of [6] and [11] to processes defined on $[0, \infty)$ and establish ELDP in the space of functions defined on the half line.

An extension of the classical LDP to the half-line was recently done in [12], in particular for Random Walk s_n , but under a stronger Cramer condition $[\mathbf{C}_\infty]$.

$[\mathbf{C}_\infty]$. For all $\lambda \in \mathbb{R}$

$$\mathbf{E}e^{\lambda\xi} < \infty.$$

Note here that $[\mathbf{C}_\infty]$ is a necessary and sufficient condition for the classical LDP for s_n in the metric space $\mathbb{C}[0, 1]$ with the uniform metric (by Puhalski's Theorem LDP is equivalent to exponential tightness which is equivalent to $[\mathbf{C}_\infty]$, eg. Lemma 4.4.5 in [5]).

The paper is organised as follows. In Section 2 we introduce the metric space of functions of bounded variation \mathbb{V} defined on $[0, \infty)$, with the metric ρ , based on the Borovkov's metric. Section 3 contains main definitions and results. Sections 4-9 contain proofs.

2. The space (\mathbb{V}, ρ) .

We look at processes $s_n = s_n(t)$ and $\xi_T = \xi_T(t)$ as random elements of the space \mathbb{V} of functions $f = f(t)$, defined for $t \in \mathbb{R}$, *having bounded variation on any interval $[0, T]$, without discontinuities of the second kind, such that $f(t) = 0$ for $t \leq 0$; at a point of discontinuity t_0 the function f can take any value $f(t_0)$, in the interval $[f(t_0-), f(t_0+)]$* . Define $\xi_T(t)$ for $t \leq 0$ accordingly,

$$\xi_T = \xi_T(t) := \begin{cases} 0, & t \leq 0; \\ \frac{1}{x}\xi(tT), & t \geq 0. \end{cases}$$

Similarly define $s_n(t) = 0$ for $t \leq 0$. It is well known, (eg. [1]), that it is possible to define the process $\xi = \xi(t)$ to be left-continuous, so that

$$\mathbf{P}(\xi_T \in \mathbb{V}) = 1 \quad \text{for all } T > 0.$$

For every $f \in \mathbb{V}$ consider its *graph* Γ_f , a simply connected set in \mathbb{R}^2 , determined by its sections

$$\Gamma_f|_u := \Gamma_f \cap \{(t, \alpha) \in \mathbb{R}^2 : t = u\} = \{[(u, f(u-)), (u, f(u+))]\},$$

where $[[\alpha, \beta]]$ denotes the line connecting two points $\alpha, \beta \in \mathbb{R}^2$. If f is continuous at $t = u$ then

$$\Gamma_f|_u = (u, f(u))$$

consists of the single point $(u, f(u))$. If $t = u$ is a point of discontinuity of f , then the section of the graph at this point is a vertical line segment

$$\Gamma_f|_u = \{(u, \alpha) : \alpha \in [\alpha_-, \alpha_+]\}, \quad \text{where } \alpha_- := f(u-), \quad \alpha_+ := f(u+).$$

It is convenient to use the «square» norm in \mathbb{R}^2 , in which the graphs of $f \in \mathbb{V}$ lie, $[(t, \alpha)] := \max\{|t|, |\alpha|\}$ (ε -neighborhood $(\gamma)_\varepsilon$ of any point $\gamma = (t, \alpha) \in \mathbb{R}^2$, in this norm is a square with the centre at γ , with sides parallel to the coordinates and length 2ε). For a set $A \subset \mathbb{R}^2$ denote by $(A)_\varepsilon$ its ε -neighborhood in this «square» norm.

We use the Borovkov's metric in the space \mathbb{V} , $\rho_{\mathbb{B}} = \rho_{\mathbb{B}}(f, g)$, defined as follows $\rho_{\mathbb{B}}(f, g) < \varepsilon$ if and only if both relations hold

$$\Gamma_f \subset (\Gamma_g)_{\varepsilon} \quad \text{and} \quad \Gamma_g \subset (\Gamma_f)_{\varepsilon}.$$

The metric $\rho_{\mathbb{B}}$ was introduced by Borovkov A.A. in [2] (for the space \mathbb{F} , wider than \mathbb{V} , see also [3]); the topology generated by $\rho_{\mathbb{B}}$, is the same as Skorohod M_2 topology, described in [15]. The metric $\rho_{\mathbb{B}}$ was effectively used in [6], [11]. Note that $\rho_{\mathbb{B}}$ is weaker than the uniform metric $\rho_{\mathbb{U}}$, $\rho_{\mathbb{U}}(f, g) := \sup_t |f(t) - g(t)|$, i.e.

$$(2.1) \quad \rho_{\mathbb{B}}(f, g) \leq \rho_{\mathbb{U}}(f, g) \quad \text{for all } f, g \in \mathbb{V}.$$

Indeed, for any $t \in \mathbb{R}$ the point (t, α) in Γ_f is given by

$$(t, \alpha) = (t, pf(t-) + qf(t+)), \quad \text{where } p \geq 0, \quad q \geq 0, \quad p + q = 1,$$

moreover $|f(t\pm) - g(t\pm)| \leq \rho_{\mathbb{U}}(f, g)$, so that (2.1) holds.

The main metric for our analysis $\rho = \rho(f, g)$ is obtained from the Borovkov's metric $\rho_{\mathbb{B}}$ by weighing functions f and g

$$\hat{f}(t) := \frac{f(t)}{1 + |t|}, \quad \hat{g}(t) := \frac{g(t)}{1 + |t|}, \quad t \in \mathbb{R},$$

$$\rho(f, g) := \rho_{\mathbb{B}}(\hat{f}, \hat{g}).$$

LDP for the family s_n on the half-line under condition $[\mathbf{C}_{\infty}]$ is given in [12] where another metric $\hat{\rho}$ was used, obtained by weighing functions in the uniform metric

$$(2.2) \quad \hat{\rho}(f, g) = \rho_{\mathbb{U}}(\hat{f}, \hat{g}).$$

ρ is weaker than $\hat{\rho}$, since by (2.1)

$$(2.3) \quad \rho(f, g) \leq \hat{\rho}(f, g) \quad \text{for any } f, g \in \mathbb{V},$$

Denote by $\mathbb{V}^0 \subset \mathbb{V}$ the class of functions $f \in \mathbb{V}$, such that

$$\lim_{t \rightarrow \infty} \frac{|f(t)|}{1 + t} = \lim_{t \rightarrow \infty} |\hat{f}(t)| = 0.$$

To summarise, the families of processes $\{s_n\}$, $\{\xi_T\}$ have trajectories in \mathbb{V} . The distribution of s_n (ξ_T) is determined by the norming sequence $x = x(n) \sim n$ ($x = x(T) \sim T$) and the distribution of the rv. ξ , that denotes the jump in the random walk (the increment of $\xi(t)$ on the unit interval $\xi(1)$). In this way ξ denotes two different random variables from the two families. The main moment conditions and the rate function are given in terms of ξ (Section 3).

Without loss of generality, by changing the drift if necessary, we can assume $\mathbf{E}\xi = 0$. In this case the trajectories of s_n and ξ_T , belong to $\mathbb{V}^0 \subset \mathbb{V}$ with probability one. As it will be seen in Lemma 3.1, any $f \in \mathbb{V}$ with $J(f) < \infty$ (see (3.2)) belongs to \mathbb{V}^0 . However, the main Theorem 3.1 uses the space \mathbb{V} (while we could have used \mathbb{V}^0).

3. Statements of main results. The rate function $J = J(f)$.

Let ξ be a non-degenerate rv. with $\mathbf{E}\xi = 0$, satisfying the Cramer condition $[\mathbf{C}_0]$. Let $\psi(\lambda)$ be the Laplace transform of ξ

$$\psi(\lambda) := \mathbf{E}e^{\lambda\xi}, \quad \lambda \in \mathbb{R},$$

and denote by (λ_-, λ_+) the largest interval for which $\psi(\lambda)$ is finite. Due to condition $[\mathbf{C}_0]$, this interval is not empty and contains the point $\lambda = 0$. Denote the Legendre transform of $\log \psi(\lambda)$ (the deviation function of ξ) by

$$\Lambda(\alpha) := \sup_{\lambda} \{\lambda\alpha - \log \psi(\lambda)\}, \quad \alpha \in \mathbb{R}.$$

The properties of Λ are well known, it is non-negative, convex, lower semi-continuous, equals to zero at a single point $a = \mathbf{E}\xi$ (in our case $a = 0$), (e.g. [7], [1] or [5], ch 2).

Recall the decomposition of any $f \in \mathbb{V}$ into absolutely continuous and singular components f_a and f_s ,

$$f = f_a + f_s = f_a + f_{s+} - f_{s-}, \quad f_a \in \mathbb{V}, \quad f_{s\pm} \in \mathbb{V},$$

where f_{s+} and $-f_{s-}$ are non-decreasing and non-increasing parts of f_s . Using this representation define the functional (cf. [5], ch.4) for any $U \in (0, \infty)$

$$(3.1) \quad J_0^U(f) := \int_0^U \Lambda(f'_a(t))dt + \lambda_+ f_{s+}(U) + |\lambda_-| f_{s-}(U).$$

It is clear that $J_0^U(f)$ is non-decreasing in U , therefore there is a limit as $U \rightarrow \infty$, which defines the rate function

$$(3.2) \quad J(f) := \lim_{U \rightarrow \infty} J_0^U(f), \quad f \in \mathbb{V}.$$

The properties of $J(f)$ are summarised in the following Lemma 3.1.

Lemma 3.1.

I. $J(f)$ is lower semi-continuous in the space (\mathbb{V}, ρ) :

$$(3.3) \quad \liminf_{\rho(f_n, f) \rightarrow 0} J(f_n) \geq J(f).$$

II. For some $C < \infty$ and all $f \in \mathbb{V}$, if $J(f) \leq N$ then

$$|f(U)| \leq C\sqrt{U}(N + \sqrt{N}), \quad U \geq 1.$$

III. For any $f \in \mathbb{V}$ there exists a sequence of absolutely continuous functions $f_n \in \mathbb{V}$ such that

$$\rho(f_n, f) \rightarrow 0, \quad \text{and} \quad J(f_n) \rightarrow J(f), \quad \text{as } n \rightarrow \infty.$$

The proof of Lemma 3.1 is given in Section 5, and now we turn to the main result. Denote, as usual, for a measurable non empty set $B \subset \mathbb{V}$

$$J(B) := \inf_{f \in B} J(f), \quad J(\emptyset) = \infty.$$

Denote by (B) , $[B]$ the interior and the closure of B respectively, and by $(B)_\varepsilon$ the ε -neighbourhood of B with respect to our metric ρ . Finally, let

$$J(B+) := \lim_{\varepsilon \downarrow 0} J((B)_\varepsilon).$$

Since for any $\varepsilon > 0$, the following inclusions hold

$$(B) \subset B \subset [B] \subset (B)_\varepsilon,$$

we have that

$$J((B)) \geq J(B) \geq J([B]) \geq J(B+).$$

Theorem 3.1. *I. The family s_n satisfies Extended Large Deviation Principle in the space (\mathbb{V}, ρ) with the rate function J , namely for any measurable set $B \subset \mathbb{V}$*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(s_n \in B) \leq -J(B+),$$

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(s_n \in B) \geq -J((B)).$$

II. The family ξ_T satisfies Extended Large Deviation Principle in the space (\mathbb{V}, ρ) with the rate function J , namely for any measurable set $B \subset \mathbb{V}$

$$(3.4) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in B) \leq -J(B+),$$

$$(3.5) \quad \underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in B) \geq -J((B)).$$

The proof of Theorem 3.1 is given in Section 4.

Now let us compare this result with previously known results. Large Deviation Principles for Compound Poisson processes on $[0, 1]$ were established in [10] and [11] under Cramer condition $[\mathbf{C}_0]$ in the space $\mathbb{V}[0, 1]$ of functions of bounded variation on $[0, 1]$ with different metrics. In [10] uniform metric $\rho_{\mathbb{U}}$ is used, while in [11] Borovkov's metric $\rho_{\mathbb{B}}$ is used. The main result of [11] strengthens, in particular, the main result of [10] for an important class of boundary crossing sets

$$B_c := \{f \in \mathbb{V} : \sup_{0 \leq t \leq 1} f(t) \geq c\}, \quad c > 0.$$

Indeed, since the sequence

$$f_n(t) := \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} - \frac{1}{n}; \\ 1, & \frac{1}{2} - \frac{1}{n} < t \leq \frac{1}{2}; \\ 0, & \frac{1}{2} < t \leq 1, \end{cases}$$

converges weakly to $f_0 := f_0(t) \equiv 0$, f_0 belongs to the closure $[B_1]$ of B_1 in the topology of weak convergence. $J(f_0) = 0$ (when the mean of the underlying process is zero) giving a non informative trivial upper bound in [10], different to the lower bound. It can be seen (Lemma 10.1 of Appendix) that

$$J((B_1)) = J(B_1+) = \inf_{0 \leq v \leq 1} J(g_v),$$

where g_v is a continuous piece-wise linear function given by $g_v(t) = \frac{t}{v}$ for $t \in [0, v]$, and $g'_v(t) = a$ with $a = \mathbf{E}\xi(1)$ for $t \in (v, 1]$. Therefore ELDP in [11], Theorem 1.1 allows to obtain the «correct» logarithmic asymptotic for probability of B_1 .

The paper [8] generalizes the LDP for Compound Poisson processes in [10] from the interval to the half-line using a metric based on the uniform metric. Here we generalize ELDP of [11] from the interval to the half-line using a metric based on the Borovkov's metric. The above illustration of LDP's with different metrics shows advantages of our result as compared to that in [8].

If the underlying random variable ξ satisfies a stronger Cramer condition $[\mathbf{C}_\infty]$ instead of $[\mathbf{C}_0]$, then our result implies the classical LDP on the half-line. Indeed, in this case $\lambda_+ = |\lambda_-| = \infty$, and the rate function becomes

$$J(f) = I(f) := \begin{cases} \int_0^\infty \Lambda(f'(t))dt, & \text{if } f \text{ is absolutely continuous;} \\ \infty, & \text{otherwise.} \end{cases}$$

Further, as shown in [12], the rate function $I(f)$ is a «good» rate function in the space of continuous functions on the half-line $(\mathbb{C}, \hat{\rho})$ with the metric $\hat{\rho}$ in (2.2), constructed by using the uniform metric $\rho_{\mathbb{U}}$. This means that for any $v \geq 0$ the set $\{f \in \mathbb{C} : I(f) \leq v\}$ is a compact in $(\mathbb{C}, \hat{\rho})$. Since ρ is weaker than $\hat{\rho}$, see (2.3), it is clear that I remains a «good» rate function in the space (\mathbb{V}, ρ) . If for any v the set $\{f \in \mathbb{V} : I(f) \leq v\}$ is a compact, then

$$I(B+) = I([B]),$$

where $[B]$ is the closure of B in (\mathbb{V}, ρ) , this is shown in [4] (see also Lemma 4.1.1 in [5]). Therefore Theorem 3.1 implies LDP on half-line for ξ_T as well as recovers the LDP for s_n in $(\mathbb{C}, \hat{\rho})$, given recently in [12].

Corollary 3.1. *I. Let r.v. ξ satisfy $[\mathbf{C}_\infty]$, $\mathbf{E}\xi = 0$. Then the family s_n satisfies LDP in the space (\mathbb{V}, ρ) with rate function I : for any measurable set $B \subset \mathbb{V}$*

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(s_n \in B) \leq -I([B]),$$

$$\underline{\lim}_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(s_n \in B) \geq -I((B)).$$

II. Let r.v. ξ satisfy $[\mathbf{C}_\infty]$, $\mathbf{E}\xi = 0$. Then the family ξ_T satisfies LDP in the space (\mathbb{V}, ρ) with rate function I : for any measurable set $B \subset \mathbb{V}$

$$\overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in B) \leq -I([B]),$$

$$\underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in B) \geq -I((B)).$$

4. Proof of Theorem 3.1.

We prove only the second statement for Compound Poisson processes. Part *I*, for Random Walk, has a similar proof, being simpler in places. Since the proof uses results for ELDP on compacts, one needs to replace references to results in [11], where ELDP is established for ξ_T on $[0, 1]$, by reference to [6] ([5]), where ELDP is established for s_n on $[0, 1]$.

The proof is based on Lemmas 4.1, 4.2 and 4.3 below.

Lemma 4.1. *For any $\varepsilon > 0$ and $f \in \mathbb{V}$*

$$(4.1) \quad \underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in (f)_\varepsilon) \geq -J(f).$$

Lemma 4.2. *For any $\varepsilon \in (0, \frac{1}{10})$ and $f \in \mathbb{V}$*

$$(4.2) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in (f)_\varepsilon) \leq -J((f)_{59\varepsilon}).$$

Lemma 4.3. *For any $\varepsilon > 0$ and $N < \infty$ there are $M < \infty$ and a collection of functions $\{g_1, \dots, g_M\}$, $g_i \in \mathbb{V}$ so that*

$$(4.3) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \notin \cup_{k=1}^M (g_k)_\varepsilon) \leq -N.$$

We prove these Lemmas later, and now give the proof of the theorem.

Proof. of Theorem 3.1.

II. (i). Upper bound. For any $\varepsilon > 0$ and $N < \infty$ by Lemma 4.3 there are functions $\{g_1, \dots, g_M\}$ in \mathbb{V} , such that (4.3) holds. Denote $\mathcal{M} := \{i \in \{1, \dots, M\} : B \cap (g_i)_\varepsilon \neq \emptyset\}$. The following bound clearly holds

$$(4.4) \quad \mathbf{P}(\xi_T \in B) \leq \mathbf{P}(\xi_T \notin \cup_{i=1}^M (g_i)_\varepsilon) + \sum_{i \in \mathcal{M}} \mathbf{P}(\xi_T \in (g_i)_\varepsilon).$$

Further, by Lemma 4.2, for $i \in \mathcal{M}$

$$(4.5) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in (g_i)_\varepsilon) \leq -J((g_i)_{59\varepsilon}).$$

By using (4.3), (4.5), we obtain from (4.4) the bound

$$(4.6) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in B) \leq -\min\{N, \min_{i \in \mathcal{M}} J((g_i)_{59\varepsilon})\}.$$

Since for any $i \in \mathcal{M}$ there is $f_i \in B$ such that $\rho(f_i, g_i) < \varepsilon$, we have

$$(g_i)_{59\varepsilon} \subset (f_i)_{60\varepsilon} \subset (B)_{60\varepsilon}.$$

Therefore

$$\min_{i \in \mathcal{M}} J((g_i)_{59\varepsilon}) \geq J((B)_{60\varepsilon}),$$

and now it follows from (4.6)

$$(4.7) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in B) \leq -\min\{N, J((B)_{60\varepsilon})\}.$$

Taking $N \rightarrow \infty$ and then $\varepsilon \rightarrow 0$ in (4.7), we obtain (3.4).

(ii). Lower bound (3.5) follows directly from Lemma 4.1. Theorem 3.1 is proved. \square

5. Proof Lemma 3.1.

Proof. I. Lower-semicontinuity of J . From the definition of $J(f)$ it follows that for any $N < \infty$ and $\delta > 0$ there is $U = U_{N, \delta} < \infty$ so that

$$J_0^U(f) \geq \min\{J(f) - \delta, N\}.$$

(N is used in case $J(f) = \infty$). We can see by using lower semi-continuity of $J_0^U(f)$ in space (\mathbb{V}, ρ) that for $V > U$ and $\rho(f_n, f) \rightarrow 0$

$$(5.1) \quad \underline{\lim}_{n \rightarrow \infty} J_0^V(f_n) \geq J_0^U(f).$$

Now we have

$$\underline{\lim}_{n \rightarrow \infty} J(f_n) \geq \underline{\lim}_{n \rightarrow \infty} J_0^V(f_n) \geq J_0^U(f) \geq \min\{J(f) - \delta, N\}.$$

Since $N < \infty$ and $\delta > 0$ are arbitrary, (3.3) follows.

It remains to show (5.1). The proof is similar to that of Theorem 4.2.2 part (ii) in [5] for lower semi-continuity of J when f belongs to a space of functions defined on $[0, 1]$. Since we work on \mathbb{R}^+ there are differences, and we give them here. Denote by $\mathbf{t}_K = (t_0, \dots, t_K)$, $0 = t_0 < t_1 < \dots < t_K = U$ a partition of $[0, U]$ into K parts. For a function $g \in \mathbb{V}$, $g^{\mathbf{t}_K}$ denotes the continuous pice-wise linear function on $[0, U]$ going through the points

$$(t_k, g(t_k)), \quad k = 0, \dots, K.$$

Then, by definition

$$J_0^U(f^{\mathbf{t}_K}) = \int_0^U \Lambda((f^{\mathbf{t}_K})'(t)) dt.$$

Theorem 4.2.1 of [5] states that

$$(5.2) \quad J(f) = \sup_{\mathbf{t}_K} I(f^{\mathbf{t}_K}),$$

where sup is over all partitions of $[0, U]$. Therefore for any $N < \infty$, $\delta > 0$ there is a partition \mathbf{t}_K such that

$$(5.3) \quad J_0^U(f^{\mathbf{t}_K}) \geq \min\{J_0^U(f) - \delta, N\}.$$

For this partition we have, due to $\rho(f_n, f) \rightarrow 0$ as $n \rightarrow \infty$, that for any $k \in \{0, 1, \dots, K\}$ there is $(t_k^{(n)}, \hat{\alpha}_k^{(n)}) \in \Gamma_{\hat{f}_n}$ such that

$$(5.4) \quad |t_k - t_k^{(n)}| \rightarrow 0, \quad \left| \frac{f(t_k)}{1+t_k} - \hat{\alpha}_k^{(n)} \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Construct now the function g_n from f_n by replacing its values at $t_k^{(n)}$, $f(t_k^{(n)})$ by $g(t_k^{(n)}) := (1+t_k^{(n)})\hat{\alpha}_k^{(n)}$. Clearly, the graph of f_n does not change, therefore by (5.2) we have for $V \geq V_N := t_K^{(n)}$

$$(5.5) \quad J_0^V(f_n) = J_0^V(g_n) \geq J_0^{V_n}(g_n) \geq J_0^{V_n}\left(g_n^{\mathbf{t}_K^{(n)}}\right).$$

From (5.4) it follows that

$$\max_{0 \leq k \leq t_K} \{|t_k - t_k^{(n)}|\} \rightarrow 0, \quad \max_{0 \leq k \leq t_K} \{|f(t_k) - g_n(t_k^{(n)})|\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence by lower semi-continuity of $\Lambda(\alpha)$ we have

$$(5.6) \quad \underline{\lim}_{n \rightarrow \infty} J_0^{V_n}\left(g_n^{\mathbf{t}_K^{(n)}}\right) \geq J_0^U(f^{\mathbf{t}_K}).$$

Now (5.5), (5.6), (5.3) imply

$$(5.7) \quad \underline{\lim}_{n \rightarrow \infty} J_0^V(f_n) \geq \min\{J_0^U(f) - \delta, N\}.$$

Thus (5.1) is established.

II. For a fixed $U \geq 1$ consider the set

$$B_U := \{g \in \mathbb{V} : g(U) = f(U)\}.$$

$J_0^U(g)$ achieves its minimum over $g \in B_U$ on the function $g_0(t) = f(U)\frac{t}{U}$ for $0 \leq t \leq U$, by equation (10.3), Appendix, Lemma 10.1. Therefore

$$N \geq J(f) \geq J_0^U(f) \geq J_0^U(g_0) = U\Lambda\left(\frac{f(U)}{U}\right).$$

In view of $[\mathbf{C}_0]$ and $\mathbf{E}\xi = 0$, for some $c > 0$

$$\Lambda(\alpha) \geq c \min\{\alpha^2, |\alpha|\},$$

hence

$$N \geq c \min\left\{\frac{f^2(U)}{U}, |f(U)|\right\}.$$

It now follows that if $\frac{|f(U)|^2}{U} \leq |f(U)|$, then

$$c\frac{|f(U)|^2}{U} \leq N, \quad |f(U)| \leq \frac{1}{\sqrt{c}}\sqrt{UN};$$

if $\frac{|f(U)|^2}{U} > |f(U)|$, then

$$c|f(U)| \leq N, \quad |f(U)| \leq \frac{1}{c}N.$$

Hence for $U \geq 1$ we have

$$|f(U)| \leq \max\left\{\frac{1}{\sqrt{c}}\sqrt{UN}, \frac{1}{c}N\right\} \leq C\sqrt{U}(N + \sqrt{N}),$$

where $C := \frac{1}{\sqrt{c}} + \frac{1}{c}$. Statement *II* is proved.

III. If $J(f) = \infty$, then since any $f \in \mathbb{V}$ is a limit of absolutely continuous $f_n \in \mathbb{V}$, $\rho(f_n, f) \rightarrow 0$ as $n \rightarrow \infty$, and using lower semi-continuity established in *I*,

$$\lim_{n \rightarrow \infty} J(f_n) = \infty = J(f).$$

Take $J(f) < \infty$. Using the already proven results *I* and *II* we can show (following the proof of Theorem 4.2.1 in [5]) that for any $\delta > 0$ there is a sequence of piecewise linear $f_n \in \mathbb{V}$ going through the points

$$(0, 0), (t_1^{(n)}, f(t_1^{(n)})), \dots, (t_{K_n}^{(n)}, f(t_{K_n}^{(n)})), \quad f_n(t) = f(t_{K_n}^{(n)}) \quad \text{for } t \geq t_{K_n}^{(n)},$$

such that

$$\rho(f_n, f) \leq \frac{1}{n}, \quad J(f_n) = J_0^{t_{K_n}^{(n)}}(f_n) \leq J(f) + \frac{1}{n}.$$

Therefore for this sequence

$$\overline{\lim}_{n \rightarrow \infty} J(f_n) \leq J(f),$$

which together with (3.1) gives

$$\lim_{n \rightarrow \infty} J(f_n) = J(f).$$

Lemma 3.1 is proved. □

6. Auxillary statements.

To proceed we need the following results. Denote for $U \geq 2$, $V \geq 2$

$$A(U, \varepsilon) := \left\{ g \in \mathbb{V} : \sup_{t \geq U} |\hat{g}(t)| \leq \varepsilon \right\},$$

$$B(2U, V) := \left\{ g \in \mathbb{V} : \sup_{t \leq 2U} |\hat{g}(t)| \leq V \right\}.$$

Lemma 6.1.

(1) For any $N < \infty$ and $\varepsilon > 0$ there is $U = U_{N, \varepsilon} < \infty$ such that

$$(6.1) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \notin A(U, \varepsilon)) \leq -N.$$

(2) For any $N < \infty$ and $U < \infty$ there is $V = V_{N, U} < \infty$ such that

$$(6.2) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \notin B(2U, V)) \leq -N.$$

Proof. of Lemma 6.1.

(1) Since

$$\sup_{t \geq U} \frac{1}{1+t} |\xi_T(t)| \leq \sup_{tT \geq UT} \frac{1}{tT} |\xi(tT)| = \sup_{v \geq UT} \frac{1}{v} |\xi(v)|,$$

we have

$$(6.3) \quad \{\xi_T \notin A_T(U, \varepsilon)\} = \left\{ \sup_{t \geq U} \frac{1}{1+t} |\xi_T(t)| > \varepsilon \right\} \subset \cup_{k \geq [UT]} B_k(\varepsilon),$$

where

$$B_k(\varepsilon) := \left\{ \sup_{k \leq v \leq k+1} \frac{1}{v} |\xi(v)| \geq \varepsilon \right\}.$$

Since

$$B_k(\varepsilon) = B_k^+(\varepsilon) \cup B_k^-(\varepsilon),$$

where

$$B_k^+(\varepsilon) := \left\{ \sup_{k \leq v \leq k+1} \frac{1}{v} \xi(v) \geq \varepsilon \right\}, \quad B_k^-(\varepsilon) := \left\{ \inf_{k \leq v \leq k+1} \frac{1}{v} \xi(v) \leq -\varepsilon \right\},$$

we have

$$(6.4) \quad \mathbf{P}(\xi_T \notin A_T(U, \varepsilon)) \leq \sum_{k \geq UT} \mathbf{P}(B_k^+(\varepsilon)) + \sum_{k \geq UT} \mathbf{P}(B_k^-(\varepsilon)),$$

and it is enough to bound probabilities

$$\mathbf{P}(B_k^+(\varepsilon)), \quad \mathbf{P}(B_k^-(\varepsilon)).$$

We bound $\mathbf{P}(B_k^+(\varepsilon))$. Since

$$B_k^+(\varepsilon) \subset \left\{ \frac{1}{k} \xi(k) \geq \frac{\varepsilon}{2} \right\} \cup \left\{ \frac{1}{k} \sup_{0 \leq u \leq 1} (\xi(k+u) - \xi(k)) \geq \frac{\varepsilon}{2} \right\},$$

we have

$$(6.5) \quad \mathbf{P}(B_k^+(\varepsilon)) \leq \mathbf{P}\left(\frac{1}{k} \xi(k) \geq \frac{\varepsilon}{2}\right) + \mathbf{P}\left(\frac{1}{k} \bar{\xi} \geq \frac{\varepsilon}{2}\right),$$

where $\bar{\xi} := \sup_{0 \leq u \leq 1} \xi(u)$.

We start with the first term in (6.5). For the rate function of ξ

$$\Lambda(\alpha) := \sup_{\lambda} \{\lambda\alpha - \ln \mathbf{E}e^{\lambda\xi}\}, \quad \alpha \in \mathbb{R},$$

using exponential Chebyshev's (Chernof's) inequality

$$(6.6) \quad \mathbf{P}\left(\frac{1}{k}\xi(k) \geq \frac{\varepsilon}{2}\right) \leq e^{-k\Lambda(\frac{\varepsilon}{2})},$$

noting that since $\mathbf{E}\xi = 0$, $\Lambda(\frac{\varepsilon}{2}) > 0$ for any $\varepsilon > 0$.

For the next term $\mathbf{P}(\frac{1}{k}\bar{\xi} \geq \frac{\varepsilon}{2})$ in (6.5), first verify that $\bar{\xi}$ satisfies Cramer's condition $[\mathbf{C}_0]$. To see this, consider a compound Poisson process $\xi^+(t)$, which jumps at the same times as $\xi(t)$ but the size of jumps is absolute value of the original jumps. Then for all $t \geq 0$

$$\sup_{0 \leq v \leq t} \xi(v) \leq \xi^+(t),$$

and

$$(6.7) \quad 0 \leq \bar{\xi} \leq \xi^+(1).$$

It is clear that $\xi^+(1)$ satisfies $[\mathbf{C}_0]$, and by (6.7) so does $\bar{\xi}$.

Hence just as above for all k such that $\frac{\varepsilon}{2}k \geq \mathbf{E}\bar{\xi}$

$$(6.8) \quad \mathbf{P}\left(\frac{1}{k}\bar{\xi} \geq \frac{\varepsilon}{2}\right) \leq e^{-\Lambda_{\bar{\xi}}(\frac{\varepsilon}{2}k)}.$$

Due to $[\mathbf{C}_0]$ for $\bar{\xi}$, the function $\Lambda_{\bar{\xi}}(\frac{\varepsilon}{2}k)$ grows as $k \rightarrow \infty$ faster than some linear function, and we obtain from (6.5), (6.6), (6.8) that for some $c > 0$, $C < \infty$ and all $k \geq 1$

$$(6.9) \quad \mathbf{P}(B_k^+(\varepsilon)) \leq Ce^{-kc}.$$

Obviously, a similar bound holds for $\mathbf{P}(B_k^-(\varepsilon))$ and (6.4) yields

$$\mathbf{P}(\xi_T \notin A_T(U, \varepsilon)) \leq 2 \sum_{k \geq [UT]} Ce^{-kc} \leq 2 \frac{C}{1 - e^{-c}} e^{-[UT]c}.$$

(6.1) now follows, and statement (1) of Lemma 6.1 is proved. Statement (2) has a similar proof. \square

In what follows we fix $N < \infty$ and $\varepsilon \in (0, \frac{1}{10})$, constants $U \geq 2$ and $V \geq 2$ so that (6.1), (6.2) hold.

For the next auxiliary result we need further notations. Denote for a function $g \in \mathbb{V}$

$$(6.10) \quad \bar{g} = \bar{g}(t) := \begin{cases} g(t), & t \leq 2U; \\ g(2U+), & t > 2U, \end{cases}$$

and for a set $B \subset \mathbb{V}$

$$\bar{B} := \{\bar{g} : g \in B\} = \cup_{g \in B} \{\bar{g}\}.$$

Then $\bar{\mathbb{V}}$ consists of functions $g \in \mathbb{V}$, which are constant on the half-line $(2U, \infty)$.

Lemma 6.2.

- (1) If $f \in A(U, \varepsilon) \cap B(2U_1, V)$ and $2U_1 \geq U$, then $f \in B(\infty, V + \varepsilon)$.
- (2) If $g \in (f)_\varepsilon$, $f \in A(U, \delta)$, then $g \in A(U + \varepsilon, \varepsilon + \delta)$.
- (3) If $g \in (f)_\varepsilon$, $f \in B(2U, V)$, then $g \in B(2U - \varepsilon, V + \varepsilon)$.

(4) Let $g \in (f)_{10\varepsilon}$, $f \in A(U + \varepsilon, 2\varepsilon) \subset A(U + \varepsilon, 10\varepsilon)$. Then by (2) we have $g \in A(U + 11\varepsilon, 20\varepsilon)$.

(5) If $f \in A(U + \varepsilon, 2\varepsilon)$, then

$$(6.11) \quad \rho(\bar{f}, f) \leq 4\varepsilon < 5\varepsilon.$$

(6) Let $h \in (\bar{g})_{\mathbb{B}, \delta}$, $\bar{g} \in B(\infty, V_1)$, then $\rho(h, \bar{g}) \leq \delta(1 + V_1)$.

(7) If $h \in B(\infty, V)$ and $\rho(h, g) < \varepsilon$, then $g \in B(\infty, V + \varepsilon)$.

Proof. Lemma 6.2. (1) is obvious.

(2) By definition of ρ , for any t there is $(u, \hat{\beta}) \in \Gamma_{\hat{f}}$, such that

$$(6.12) \quad |t - u| < \varepsilon, \quad |g(t) - \hat{\beta}| < \varepsilon.$$

Therefore, for $t \geq U + \varepsilon$ we have $u \geq U$, $|\hat{\beta}| \leq \delta$,

$$|\hat{g}(t)| \leq |\hat{g}(t) - \hat{\beta}| + |\hat{\beta}| \leq \varepsilon + \delta.$$

In other words, $g \in A(U + \varepsilon, \varepsilon + \delta)$.

(3) By (6.12) if $t \leq 2U - \varepsilon$ then $u \leq 2U$, $|\hat{\beta}| \leq V$,

$$|\hat{g}(t)| \leq |\hat{g}(t) - \hat{\beta}| + |\hat{\beta}| \leq \varepsilon + V.$$

so that $g \in B(2U - \varepsilon, V + \varepsilon)$.

(4) is straight forward.

(5) For $t \leq 2U$ we have $|\hat{f}(t) - \hat{f}(t)| = 0$, and for $t > 2U$ we have

$$|\hat{f}(t) - \hat{f}(t)| \leq |\hat{f}(2U+)| + |\hat{f}(t)| \leq 2\varepsilon + 2\varepsilon = 4\varepsilon.$$

therefore

$$\sup_t |\hat{f}(t) - \hat{f}(t)| \leq 4\varepsilon,$$

and (6.11) follows.

(6) For any $(t, \alpha) \in \Gamma_h$ there is $(u, \beta) \in \Gamma_{\bar{g}}$ such that

$$|t - u| < \delta, \quad |\alpha - \beta| < \delta.$$

Then

$$\begin{aligned} |\hat{\alpha} - \hat{\beta}| &= \left| \frac{\alpha}{1+t} - \frac{\beta}{1+u} \right| \leq \left| \frac{\alpha}{1+t} - \frac{\beta}{1+t} \right| + \frac{|\beta|}{1+u} \left| \frac{1}{1+t} - \frac{1}{1+u} \right| (1+u) \leq \\ &\frac{\delta}{1+t} + V_1 \frac{|t-u|}{1+t} < \delta + \delta V_1 = \delta(1 + V_1). \end{aligned}$$

We have shown that for any $(t, \hat{\alpha}) \in \Gamma_{\hat{h}}$ there is $(u, \hat{\beta}) \in \Gamma_{\hat{g}}$ such that

$$(6.13) \quad |t - u| < \delta(1 + V_1), \quad |\hat{\alpha} - \hat{\beta}| < \delta(1 + V_1).$$

In the same way, for any $(u, \hat{\beta}) \in \Gamma_{\hat{g}}$ there is $(t, \hat{\alpha}) \in \Gamma_{\hat{h}}$ such that (6.13) holds.

(7) is obvious. □

7. Proof of Lemma 4.1.

Proof. If $J(f) = \infty$, then (4.1) holds. Let $J(f) < \infty$. Consider first an absolutely continuous f , for which

$$J(f) = \int_0^\infty \Lambda(f'(t)) dt < \infty.$$

By property *II* of Lemma 3.1 there is $U_0 < \infty$ such that

$$\sup_{t \geq U_0} |\hat{f}(t)| < \frac{\varepsilon}{4}.$$

Denote for $U \geq U_0$

$$C(U) := \left\{ g \in \mathbb{V} : \sup_{t \leq U} |\hat{g}(t) - \hat{f}(t)| < \frac{\varepsilon}{4} \right\},$$

$$D(U) := \left\{ g \in \mathbb{V} : \sup_{t \geq U} |\hat{g}(t)| < \frac{\varepsilon}{4} \right\},$$

and use

$$(7.1) \quad (f)_\varepsilon \supset C(U) \cap D(U),$$

which follows from

$$(f)_\varepsilon \supset (f)_{\hat{\rho}, \varepsilon} = \{g \in \mathbb{V} : \hat{\rho}(f, g) < \varepsilon\},$$

which in turn follows from (2.3) and an obvious

$$(f)_{\hat{\rho}, \varepsilon} \supset C(U) \cap D(U).$$

We obtain from (7.1)

$$(7.2) \quad \mathbf{P}(\xi_T \in (f)_\varepsilon) \geq \mathbf{P}(\xi_T \in C(U)) - \mathbf{P}(\xi_T \notin D(U)).$$

By (1) of Lemma 6.1 there is $U_1 \geq U_0$ such that

$$(7.3) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \notin D(U_1)) \leq -2J(f).$$

On the other hand, by the local LDP on compacts which holds in the subclass of continuous functions, eg. Theorem 4.9.3 of [5] we have

$$(7.4) \quad \underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log \mathbf{P}(\xi_T \in C(U_1)) \geq - \int_0^{U_1} \Lambda(f'(t)) dt \geq -J(f).$$

It follows from (7.3) and (7.4) that the second term in (7.2) is o -little of the first. The desired lower bound (4.1) now follows.

If $f \in \mathbb{V}$ is not absolutely continuous, then by property *III* of Lemma 3.1 for an arbitrary $\delta > 0$ take an absolutely continuous g such that

$$\rho(g, f) < \frac{\varepsilon}{2}, \quad |J(f) - J(g)| < \delta.$$

Then

$$\mathbf{P}(\xi_T \in (f)_\varepsilon) \geq \mathbf{P}(\xi_T \in (g)_{\frac{\varepsilon}{2}}),$$

and applying the lower bound proved above, we obtain

$$(7.5) \quad \underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathbf{P}(\xi_T \in (f)_\varepsilon) \geq \underline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathbf{P}(\xi_T \in (g)_{\frac{\varepsilon}{2}}) \geq -J(g) \geq -J(f) - \delta.$$

(4.1) now follows and Lemma 4.1 is proved. \square

8. Proof of Lemma 4.2.

Proof. For any measurable $B \subset \mathbb{V}$ the following clearly holds (recall that operations \bar{f} , \bar{B} depend on U , see definition (6.10))

$$\{\xi_T \in B\} \subset \{\bar{\xi}_T \in \bar{B}\}, \quad \mathbf{P}(\xi_T \in B) \leq \mathbf{P}(\bar{\xi}_T \in \bar{B}).$$

Taking $B = (f)_\varepsilon$, we obtain

$$(8.1) \quad \{\xi_T \in (f)_\varepsilon\} \subset \{\bar{\xi}_T \in \overline{(f)_\varepsilon}\}, \quad \mathbf{P}(\xi_T \in (f)_\varepsilon) \leq \mathbf{P}(\bar{\xi}_T \in \overline{(f)_\varepsilon}).$$

By the ELDP for processes on a compact proven in [11], it follows that for any measurable $\bar{B} \subset \bar{\mathbb{V}}$ and any $\delta > 0$

$$(8.2) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathbf{P}(\bar{\xi}_T \in \bar{B}) \leq -J(\overline{(\bar{B})_{\mathbb{B}, \delta}}).$$

It is easy to see (with notations from Section 6) that

$$\begin{aligned} \mathbf{P}(\xi_T \in (f)_\varepsilon) &\leq \mathbf{P}(\xi_T \notin A(U, \varepsilon)) + \mathbf{P}(\xi_T \notin B(2U, V)) + \\ &\mathbf{P}(\xi_T \in (f)_\varepsilon, \quad \xi_T \in A(U, \varepsilon), \quad \xi_T \in B(2U, V)) =: P_1 + P_2 + P_3, \end{aligned}$$

so that

$$(8.3) \quad \mathbf{P}(\xi_T \in (f)_\varepsilon) \leq P_1 + P_2 + P_3.$$

By (1) and (2) of Lemma 6.1 P_1, P_2 admit the exponential bound for suitable $U = U_{\varepsilon, N}$ and $V = V_{\varepsilon, N}$

$$(8.4) \quad P_1 + P_2 \leq O(e^{-T(N+o(1))}) \quad \text{as } T \rightarrow \infty.$$

We bound P_3 . When it is not equal to zero by (2), (3) of Lemma 6.2 we have

$$(8.5) \quad f \in A(U + \varepsilon, 2\varepsilon) \cap B(2U - \varepsilon, V + \varepsilon),$$

and

$$P_3 \leq P(\xi_T \in (f)_\varepsilon).$$

Using (8.1) we obtain

$$P_3 \leq \mathbf{P}(\xi_T \in (f)_\varepsilon) \leq \mathbf{P}(\bar{\xi}_T \in \overline{(f)_\varepsilon}).$$

Therefore by (8.2) for any $\delta > 0$

$$(8.6) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln P_3 \leq -\bar{J},$$

where

$$\bar{J} := J(\overline{((f)_\varepsilon)_{\mathbb{B}, \delta}}).$$

We bound \bar{J} from below, taking into account (8.5).

By (5) of Lemma 6.2 we have

$$\rho(f, \bar{f}) < 5\varepsilon, \quad (f)_\varepsilon \subset (\bar{f})_{6\varepsilon}, \quad \overline{((f)_\varepsilon)_{\mathbb{B}, \delta}} \subset \overline{((\bar{f})_{6\varepsilon})_{\mathbb{B}, \delta}},$$

therefore

$$\bar{J} = J(\overline{((f)_\varepsilon)_{\mathbb{B}, \delta}}) \geq J(\overline{((\bar{f})_{6\varepsilon})_{\mathbb{B}, \delta}}),$$

and we need a lower bound for

$$\bar{J}_1 := J(\overline{((\bar{f})_{6\varepsilon})_{\mathbb{B}, \delta}}).$$

By (8.5) and (6.11), taking into account (2) and (3) of Lemma 6.2, we obtain

$$\bar{f} \in A(U + 6\varepsilon, 7\varepsilon) \cap B(2U - 6\varepsilon, V + 6\varepsilon).$$

Note that this implies by (1) of Lemma 6.2 that

$$(8.7) \quad \overline{f} \in B(\infty, V + 13\varepsilon).$$

Further, let $\overline{g} \in \overline{(\overline{f})_{6\varepsilon}}$, and (8.7) to hold. Then by (7) of Lemma 6.2, we have $\overline{g} \in B(\infty, V + 19\varepsilon)$. By (6) of Lemma 6.2, for $h \in (\overline{g})_{\mathbb{B}, \delta}$ it holds that $\rho(h, \overline{g}) < \delta(1 + V + 19\varepsilon)$. This means that

$$\overline{((\overline{f})_{6\varepsilon})_{\mathbb{B}, \delta}} \subset \overline{((\overline{f})_{6\varepsilon})_{\delta(1+V+19\varepsilon)}}, \quad \overline{((\overline{f})_{6\varepsilon})_{\mathbb{B}, \delta}} \subset \overline{((\overline{f})_{6\varepsilon})_{\delta(1+V+19\varepsilon)}}.$$

Thus, taking $\delta = \frac{\varepsilon}{(1+V+19\varepsilon)}$, we obtain

$$\overline{J} \geq \overline{J}_1 = J(\overline{((\overline{f})_{6\varepsilon})_{\mathbb{B}, \delta}}) \geq J(\overline{((\overline{f})_{6\varepsilon})_{\varepsilon}}).$$

Now we need to bound below

$$\overline{J}_2 := J(\overline{((\overline{f})_{6\varepsilon})_{\varepsilon}}).$$

We use the following result, which will be shown later: *for any $\gamma > 0$, $\nu > 0$ it holds that*

$$(8.8) \quad \overline{((\overline{f})_{\gamma})_{\nu}} \subset \overline{(\overline{f})_{\gamma+\nu}}.$$

By (8.8)

$$\overline{((\overline{f})_{6\varepsilon})_{\varepsilon}} \subset \overline{(\overline{f})_{7\varepsilon}}, \quad \overline{J} \geq \overline{J}_2 \geq J(\overline{((\overline{f})_{6\varepsilon})_{\varepsilon}}) \geq J(\overline{(\overline{f})_{7\varepsilon}}),$$

and we need to bound

$$\overline{J}_3 := J(\overline{(\overline{f})_{7\varepsilon}}).$$

Taking into account that $\rho(\overline{f}, f) \leq 4\varepsilon$ (see V Lemma 6.2), we obtain

$$\overline{(\overline{f})_{7\varepsilon}} \subset \overline{(f)_{11\varepsilon}}.$$

Therefore

$$\overline{J} \geq \overline{J}_3 \geq J(\overline{(f)_{11\varepsilon}}),$$

and we need to bound

$$\overline{J}_4 := J(\overline{(f)_{11\varepsilon}}).$$

Let $g \in (f)_{11\varepsilon}$, $f \in A(U + \varepsilon, 2\varepsilon) \subset A(U + \varepsilon, 10\varepsilon)$. then by II Lemma 6.2, $g \in A(U + 12\varepsilon, 21\varepsilon)$. Further, by V Lemma 6.2, if $g \in A(U + 12\varepsilon, 21\varepsilon) \subset A(U + 12\varepsilon, 24\varepsilon)$ then $\rho(\overline{g}, g) \leq 48\varepsilon$. Since also $\rho(g, f) < 11\varepsilon$, we obtain $\rho(\overline{g}, f) < 59\varepsilon$. We proved that

$$\overline{(f)_{11\varepsilon}} \subset (f)_{59\varepsilon}, \quad \overline{J}_4 = J(\overline{(f)_{11\varepsilon}}) \geq J((f)_{59\varepsilon}),$$

so that

$$(8.9) \quad \overline{J} \geq J((f)_{59\varepsilon}).$$

Taking into account (8.3), (8.4), (8.6), (8.9), we obtain for any $N < \infty$

$$\overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathbf{P}(\xi_T \in (f)_{\varepsilon}) \leq -\min\{N, J((f)_{59\varepsilon})\}.$$

Since N is arbitrary, the Lemma is proved.

It remains to show (8.8). To this end, note that in the space $\overline{\mathbb{V}} \subset \mathbb{V}$ there is the triangular inequality, for any $\overline{f}, \overline{g}, \overline{h} \in \overline{\mathbb{V}}$

$$\rho(\overline{f}, \overline{g}) \leq \rho(\overline{f}, \overline{h}) + \rho(\overline{h}, \overline{g}).$$

(8.8) follows by the triangular inequality, and the proof of Lemma 4.2 is complete. \square

9. Proof of Lemma 4.3.

We continue to use notations and results of the previous Sections. Fix an $\varepsilon \in (0, 0.1)$ and $N < \infty$. By Lemma 6.1 there are $U = U_{\varepsilon, N} < \infty$ and $V = V_{\varepsilon, N} < \infty$ so that (6.1) and (6.2) hold.

Proof. We use a result from [11] that there are finitely many $\{\bar{f}_1, \dots, \bar{f}_M\}$ functions $\bar{f}_i \in \bar{\mathbb{V}}$ such that

$$(9.1) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \ln \mathbf{P}(\xi_T \notin E) \leq -N,$$

where

$$E := \cup_{i=1}^M E_i, \quad E_i := \left\{ g \in \mathbb{V} : \rho_{\mathbb{B}}(\bar{g}, \bar{f}_i) < \frac{\varepsilon}{V} \right\}, \quad i \in \{1, \dots, M\}.$$

Define $g_i := \bar{f}_i$ for $i = 1, \dots, M$;

$$F := \cup_{i=1}^M F_i, \quad F_i := \{g \in \mathbb{V} : \rho(g, g_i) < 5\varepsilon\} = (g_i)_{5\varepsilon}.$$

Then

$$\begin{aligned} \mathbf{P}(\xi_T \notin F) &\leq \mathbf{P}(\xi_T \notin A(U, \varepsilon)) + \mathbf{P}(\xi_T \in A(U, \varepsilon), \xi_T \notin B(2U, V)) + \\ &\quad \mathbf{P}(\xi_T \in A(U, \varepsilon), \xi_T \in B(2U, V), \xi_T \notin E) + \\ &\quad \mathbf{P}(\xi_T \in A(U, \varepsilon), \xi_T \in B(2U, V), \xi_T \in E, \xi_T \notin F) \\ &\leq \mathbf{P}(\xi_T \notin A(U, \varepsilon)) + \mathbf{P}(\xi_T \notin B(2U, V)) + \mathbf{P}(\xi_T \notin E) + \\ &\quad \mathbf{P}(\xi_T \in A(U, \varepsilon) \cap B(2U, V) \cap E, \xi_T \notin F) =: P_1 + P_1 + P_3 + P_4. \end{aligned}$$

Using inequalities (6.1), (6.2), (9.1) we have for $j = 1, 2, 3$

$$(9.2) \quad \overline{\lim}_{T \rightarrow \infty} \frac{1}{T} \log P_j \leq -N.$$

We bound P_4 , and show that it is nil. To this end, use the following result, which will be proven later.

For any $i \in \{1, \dots, M\}$

$$(9.3) \quad A(U, \varepsilon) \cap B(2U, V) \cap E_i \subset F_i.$$

Then

$$\{\xi_T \in A(U, \varepsilon) \cap B(2U, V) \cap E\} \subset \{\xi_T \in F\};$$

and therefore

$$\{\xi_T \in A(U, \varepsilon) \cap B(2U, V) \cap E\} \cap \{\xi_T \notin F\} \subset \{\xi_T \in F\} \cap \{\xi_T \notin F\} = \emptyset.$$

Thus we established that (9.3) implies

$$P_4 = \mathbf{P}(\xi_T \in A(U, \varepsilon) \cap B(2U, V) \cap E, \xi_T \notin F) = 0.$$

Therefore bounds in (9.2) imply the Lemma. It remains to show (9.3).

Let $f \in A(U, \varepsilon) \cap B(2U, V) \cap E_i$, $0 \leq \varepsilon < 0.1$, $U \geq 2$. We show first that for all $t \geq 2U - 2$, $u \geq 2U - 1$ we have

$$(9.4) \quad \frac{|f(t)|}{1+t} \leq \varepsilon, \quad \frac{|\bar{f}(t)|}{1+t} \leq \varepsilon, \quad \frac{|\bar{f}_i(u)|}{1+u} \leq 3\varepsilon.$$

The first inequality in (9.4) follows from $f \in A(U, \varepsilon)$, the second follows from the first and the definition of \bar{f} . We show the third inequality in (9.4).

Due to $f \in E_i$ we have $\rho_{\mathbb{B}}(\bar{f}, \bar{f}_i) < \frac{\varepsilon}{V}$, therefore for any $(u, \bar{f}_i(u))$ there is $(t, \alpha) \in \Gamma_{\bar{f}}$ such that

$$|u - t| < \frac{\varepsilon}{V}, \quad |\bar{f}_i(u) - \alpha| < \frac{\varepsilon}{V}.$$

Therefore for $u \geq 2U - 1$ we have

$$\begin{aligned} \frac{|\bar{f}_i(u)|}{1+u} &\leq \frac{|\bar{f}_i(u) - \alpha|}{1+u} + \frac{|\alpha|}{1+u} \leq \\ &\frac{|\bar{f}_i(u) - \alpha|}{1+u} + \frac{|\alpha|}{1+t} + \frac{|\alpha|}{1+t} \left| \frac{1}{1+t} - \frac{1}{1+u} \right| (1+t) = \\ &\frac{|\bar{f}_i(u) - \alpha|}{1+u} + \frac{|\alpha|}{1+t} + \frac{|\alpha|}{1+t} \frac{|t-u|}{(1+t)(1+u)} (1+t) \leq \\ &\frac{\varepsilon}{V(1+u)} + \varepsilon + \varepsilon \frac{\varepsilon}{V(1+u)} \leq 3\varepsilon. \end{aligned}$$

(9.4) is proved.

Two results follow from (9.4).

(1)₊. For any $(t, \hat{\alpha}) \in \Gamma_{\hat{f}}$ for $t \geq 2U - 1$ there is $(u, \hat{\beta}) = (t, \hat{\beta}) \in \Gamma_{\hat{f}_i}$ such that

$$(9.5) \quad |t - u| < 5\varepsilon, \quad |\hat{\alpha} - \hat{\beta}| < 5\varepsilon.$$

(2)₊. For any $(u, \hat{\beta}) \in \Gamma_{\hat{f}_i}$ for $u \geq 2U - 1$ there is $(t, \hat{\alpha}) = (u, \hat{\alpha}) \in \Gamma_{\hat{f}}$ such that (9.5) holds.

Further, since $f \in B(2U, V) \cap E_i$, for any $(t, \alpha) \in \Gamma_f$ for $t \leq 2U - 1$, evidently $(t, \alpha) \in \Gamma_{\bar{f}}$, therefore for any $(t, \alpha) \in \Gamma_f$ for $t \leq 2U - 1$ there is $(u, \beta) \in \Gamma_{\bar{f}_i}$ such that

$$|t - u| < \frac{\varepsilon}{V}, \quad |\alpha - \beta| < \frac{\varepsilon}{V}.$$

Hence

$$\begin{aligned} |\hat{\alpha} - \hat{\beta}| &= \left| \frac{\alpha}{1+t} - \frac{\beta}{1+u} \right| \leq \left| \frac{\alpha}{1+u} - \frac{\beta}{1+u} \right| + \left| \frac{\alpha}{1+u} - \frac{\alpha}{1+t} \right| \leq \\ &\frac{\varepsilon}{V(1+u)} + \frac{|\alpha|}{1+t} \left| \frac{t-u}{(1+t)(1+u)} \right| (1+t) \leq \varepsilon + V \frac{\varepsilon}{V(1+u)} < 2\varepsilon < 5\varepsilon. \end{aligned}$$

Thus we showed the following.

(1)₋. For any $(t, \hat{\alpha}) \in \Gamma_{\hat{f}}$ for $t \leq 2U - 1$ there is $(u, \hat{\beta}) \in \Gamma_{\hat{f}_i}$ such that (9.5) holds.

Similarly, since $f \in B(2U, V) \cap E_i$ for any $(u, \beta) \in \Gamma_{\bar{f}_i}$ for $u \leq 2U - 1$ there is $(t, \alpha) \in \Gamma_{\bar{f}}$ such that

$$|t - u| < \frac{\varepsilon}{V}, \quad |\alpha - \beta| < \frac{\varepsilon}{V}.$$

Evidently $(t, \alpha) \in \Gamma_f$. Therefore for any $(u, \beta) \in \Gamma_{\bar{f}_i}$ for $u \leq 2U - 1$ there is $(t, \alpha) \in \Gamma_f$ such that

$$\begin{aligned} |\hat{\alpha} - \hat{\beta}| &= \left| \frac{\alpha}{1+t} - \frac{\beta}{1+u} \right| \leq \left| \frac{\alpha}{1+u} - \frac{\beta}{1+u} \right| + \left| \frac{\alpha}{1+u} - \frac{\alpha}{1+t} \right| \leq \\ &\frac{\varepsilon}{V(1+u)} + \frac{|\alpha|}{1+t} \left| \frac{t-u}{(1+t)(1+u)} \right| (1+t) \leq \varepsilon + V \frac{\varepsilon}{V(1+u)} < 2\varepsilon < 5\varepsilon. \end{aligned}$$

Hence we established

(2)₋. For any $(u, \hat{\beta}) \in \Gamma_{\hat{f}_i}$ for $u \leq 2U - 1$ there is $(t, \hat{\alpha}) \in \Gamma_{\hat{f}}$ such that (9.5) holds.

It follows from (1)₊, (1)₋; (2)₊, (2)₋ that $\rho(f, \bar{f}_i) = \rho(f, g_i) < 5\varepsilon$, so that $f \in (g_i)_{5\varepsilon}$. Consequently (9.3) and Lemma 4.3 are proved. \square

10. Appendix.

Lemma 10.1. Let $\mathbf{E}\xi(1) = a$, $B_1 := \{f \in \mathbb{V} : \sup_{0 \leq t \leq 1} f(t) \geq 1\}$, and g_v be continuous piece-wise linear with $g_v(t) = \frac{t}{v}$ for $t \in [0, v]$, and $g'_v(t) = a$ for $t \in [v, 1]$. Then

$$(10.1) \quad J(B_1) = \inf_{0 < v \leq 1} I(g_v) = \inf_{0 < v \leq 1} v\Lambda\left(\frac{1}{v}\right) =: v_0\Lambda\left(\frac{1}{v_0}\right).$$

Proof. Take an arbitrary $g \in B_1$ and let $v = \eta_g := \inf\{t \in [0, 1] : g(t) \geq 1\}$ be the first time of hitting or jumping over level 1 and $b = \chi_g := g(v) - 1$ be the overshoot. Consider the function

$$\bar{g}(t) := g(t) - b\mathbf{1}_{(v,1]}(t),$$

Then $\bar{g} \in B_1$ with the first hitting time of 1 being v , but with zero overshoot. It is clear from the definition of the rate function (3.1) that $J(g) \geq J(\bar{g})$. Therefore we conclude that

$$(10.2) \quad J(B_1) = \inf_{g \in B_1: \chi_g = 0} J(g).$$

Consider next $g \in B_1$, with $\eta_g = v$, $\chi_g = 0$, and also consider $g_v = g_v(t)$, defined above in Lemma 10.1. Write the rate function as

$$J(g) = J_0^v(g) + J_v^1(g), \quad \text{where } J_v^1(g) := J_0^1(g) - J_0^v(g),$$

It follows from the definition of the rate function $J_0^v(g)$ (see (5.2), Theorem 4.2.1 of [5]) that there exists a sequence of piece-wise linear $g_{(n)}$ on $[0, v]$ such that $g_{(n)}(v) = g(v)$ and

$$J_0^v(g) = \lim_{n \rightarrow \infty} I_0^v(g_{(n)}),$$

where

$$I_0^v(g_{(n)}) := \int_0^v \Lambda(g'_{(n)}(t)) dt.$$

By convexity of $\Lambda(\alpha)$ and definition of the function g_v for any n

$$I_0^v(g_{(n)}) \geq I_0^v(g_v).$$

Taking limit as $n \rightarrow \infty$ we obtain

$$J_0^v(g) \geq I_0^v(g_v).$$

Notice next that due to $\Lambda(a) = 0$,

$$I_v^1(g_v) := I_0^1(g_v) - I_0^v(g_v) = \int_v^1 \Lambda(g'_v(t)) dt = (1 - v)\Lambda(a) = 0.$$

Thus it follows

$$(10.3) \quad J_0^v(g) \geq J_0^v(g_v) = I_0^v(g_v), \quad J_v^1(g) \geq J_v^1(g_v) = I_v^1(g_v).$$

Therefore for any $g \in B_1$ with $\chi_g = 0$,

$$(10.4) \quad J(g) \geq J(g_v).$$

The result now follows by (10.2) and (10.4). The second equality in (10.1) is due to

$$J(g_v) = I(g_v) = v\Lambda\left(\frac{1}{v}\right),$$

so that

$$J(B_1) = \inf_{0 < v \leq 1} v\Lambda\left(\frac{1}{v}\right).$$

Convexity of $v\Lambda\left(\frac{1}{v}\right)$ follows from convexity of Λ . Hence there is $v_0 \in [0, 1]$ in (10.1), and the proof is complete. \square

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