SOME CALCULATIONS OF ORLICZ COHOMOLOGY
AND POINCARÉ–SOBOLEV–ORLICZ INEQUALITIES

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Abstract. We carry out calculations of Orlicz cohomology for some basic Riemannian manifolds (the real line, the hyperbolic plane, the ball). Relationship between Orlicz cohomology and Poincaré–Sobolev–Orlicz-type inequalities is discussed.

Keywords: differential form, Orlicz cohomology, torsion, Poincaré–Sobolev–Orlicz inequality

Introduction

The article continues the study of Orlicz cohomology of Riemannian manifolds initiated in [7, 8].

Orlicz cohomology is a natural generalization of $L_{qp}$-cohomology (for a detailed discussion of $L_{qp}$-cohomology, the reader is referred, for example, to [4]).

Like Orlicz function spaces, the Orlicz spaces $L^\Phi$ of differential forms are a natural nonlinear generalization of the spaces $L^p$. Orlicz spaces of differential forms on domains in $\mathbb{R}^n$ were first considered by Iwaniec and Martin in [6] and then by Agarwal, Ding, and Nolder in [1]. Orlicz forms on an arbitrary Riemannian manifold were apparently first examined by Kopylov and Panenko in [7].

In [4], Gol’dshtein and Troyanov demonstrated close relationship between $L_{qp}$-cohomology and Sobolev-type inequalities on Riemannian manifolds and, basing on this and some “almost duality” techniques, performed calculations of $L_{qp}$-cohomology for some basic manifolds. It turns out that, with some significant corrections and sometimes under additional constraints on the $N$-functions from which the Orlicz...
cohomology is constructed, these methods prove to be fruitful in computing Orlicz cohomology.

The structure of the article is as follows: In Section 1, we recall the main notions and necessary properties of Orlicz function spaces. In Section 2, we recall some basic information on abstract Banach complexes. Section 3 contains definitions concerning Orlicz spaces of differential forms on a Riemannian manifold, Orlicz cohomology, and its interpretation in terms of Poincaré–Sobolev–Orlicz inequalities (Theorems 3.3 and 3.4). Then we calculate the $L_{\Phi_1, \Phi_2}$-cohomology of $\mathbb{R}$ (Section 4) the hyperbolic plane (Section 5) and the $L_{\Phi}$-cohomology of the ball (“$L^p$-Poincaré inequality”, Section 6).

1. N-Functions and Orlicz Function Spaces

**Definition 1.1.** A nonnegative function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an $N$-function if

(i) $\Phi$ is even and convex;
(ii) $\Phi(x) = 0 \iff x = 0$;
(iii) $\lim_{x \to 0} \frac{\Phi(x)}{x} = 0$; $\lim_{x \to \infty} \frac{\Phi(x)}{x} = \infty$.

An $N$-function $\Phi$ has left and right derivatives (which can differ only on an at most countable set, see, for instance, [10, Theorem 1, p. 7]). The left derivative $\varphi$ of $\Phi$ is left continuous, nondecreasing on $(0, \infty)$, and such that $0 < \varphi(t) < \infty$ for $t > 0$, $\varphi(0) = 0$, $\lim_{t \to \infty} \varphi(t) = \infty$. The function

$$\psi(s) = \inf\{t > 0 : \varphi(t) > s\}, \quad s > 0,$$

is called the left inverse of $\varphi$.

The functions $\Phi, \Psi$ given by

$$\Phi(x) = \int_0^{|x|} \varphi(t)dt, \quad \Psi(x) = \int_0^{|x|} \psi(t)dt$$

are called complementary $N$-functions.

The $N$-function $\Psi$ complementary to an $N$-function $\Phi$ can also be expressed as

$$\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}, \quad y \in \mathbb{R}.$$  

Throughout the article, given an $N$-function $\Phi : \mathbb{R} \to [0, \infty)$, we denote by $\Phi^{-1}$ its “positive” inverse $\Phi^{-1} : [0, \infty) \to [0, \infty)$.

$N$-functions are classified in accordance with their growth rates as follows:

**Definition 1.2.** An $N$-function $\Phi$ is said to satisfy the $\Delta_2$-condition (for all $x$), which is written as $\Phi \in \Delta_2$ if there exists a constant $K > 2$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq 0$; $\Phi$ is said to satisfy the $\nabla_2$-condition (for all $x$), which is denoted symbolically as $\Phi \in \nabla_2$, if there is a constant $c > 1$ such that $\Phi(x) \leq \frac{1}{2c} \Phi(cx)$ for all $x \geq 0$.

It is not hard to see that an $N$-function $\Phi$ satisfies the the $\nabla_2$-condition if and only if its dual $N$-function satisfies the $\Delta_2$-condition.

Henceforth, let $\Phi$ be an $N$-function and let $(\Omega, \Sigma, \mu)$ be a measure space.

**Definition 1.3.** Given a measurable function $f : \Omega \to \mathbb{R}$, we put

$$\rho_\Phi(f) := \int_\Omega \Phi(f)d\mu.$$
Definition 1.4. The linear space
\[ L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{ f : \Omega \to \mathbb{R} \text{ measurable} : \rho_{\Phi}(af) < \infty \text{ for some } a > 0 \} \]
is called an Orlicz space on \((\Omega, \Sigma, \mu)\).

Let \( \Psi \) be the complementary \( N \)-function to \( \Phi \).
Below we as usual identify two functions equal outside a set of measure zero.
If \( f \in L^\Phi \) then the functional \( \| \cdot \|_{\Phi} \) (called the Orlicz norm) defined by
\[ \| f \|_{\Phi} = \| f \|_{L^\Phi(\Omega)} = \sup \left\{ \left\| \int f g \, d\mu \right\| : \rho_{\Psi}(g) \leq 1 \right\} \]
is a seminorm. It becomes a norm if \( \mu \) satisfies the finite subset property (see [10, p. 59]): if \( A \in \Sigma \) and \( \mu(A) > 0 \) then there exists \( B \in \Sigma, B \subset A \), such that \( 0 < \mu(B) < \infty \).

The equivalent gauge (or Luxemburg) norm of a function \( f \in L^\Phi \) is defined by
\[ \| f \|_{(\Phi)} = \| f \|_{L^{(\Phi)}(\Omega)} = \inf \left\{ K > 0 : \rho_{\Phi} \left( \frac{f}{K} \right) \leq 1 \right\} \]
This is a norm without any constraint on the measure \( \mu \) (see [10, p. 54, Theorem 3]).

2. Banach Complexes

Like in the case of \( L_{q,p} \)-cohomology, treated in [4], we apply some abstract facts about Banach complexes to the Orlicz cohomology of Riemannian manifolds.

In this section, we recall some definitions and assertions about abstract Banach complexes given in [4].

Definition 2.1. A Banach complex is a sequence \( F^* = \{ F^k, d^k \}_{k \in \mathbb{N}} \) where \( F^k \) is a Banach space and \( d^k : F^k \to F^{k+1} \) is a bounded operator with \( d^{k+1} \circ d^k = 0 \).

Definition 2.2. Given a Banach complex \( \{ F^k, d^k \} \), introduce the vector spaces:
- \( Z^k := \ker(d : F^k \to F^{k+1}) \) (a closed subspace of \( F^k \));
- \( B^k := \text{Im}(d : F^{k-1} \to F^k) \subset Z^k \);
- \( H^k(F^*) := Z^k/B^k \) is the cohomology of the complex \( F^* = \{ F^k, d^k \} \);
- \( \overline{H}^k(F^*) := Z^k/\overline{B}^k \) is the reduced cohomology of the complex \( F^* \);
- \( T^k(F^*) := \overline{B}^k/B^k = H^k/\overline{H}^k \) is the torsion of the complex \( F^* \).

As was observed in [4], the following easy assertion holds:
(a) \( \overline{H}^k, Z^k \) and \( \overline{B}^k \) are Banach spaces;
(b) The natural (quotient) topology on \( T^k := \overline{B}^k/B^k \) is coarse (any closed set is either empty or \( T^k \));
(c) there is a natural exact sequence
\[ 0 \to T^k \to H^k \to \overline{H}^k \to 0. \]

Lemma 2.3. [4, Lemma 4.4] For any Banach complex \( \{ F^k, d^k \} \), the following are equivalent:
(i) \( T^k = 0 \);
(ii) \( \dim T_k < \infty \);
(iii) \( H^k \) is a Banach space;
(iv) \( B^k \subset F^k \) is closed.
Lemma 2.4. [4, Proposition 4.5] The following are equivalent:

(i) \( H_k = 0 \);
(ii) The operator \( d_{k-1} : F^{k-1}k^{-1} / Z^{k-1} \to Z^k \) admits a bounded inverse \( d_{k-1}^{-1} \);
(iii) There exists a constant \( C_k \) such that if for any \( \theta \in Z^k \) there is an element \( \eta \in F^{k-1}k^{-1} \) with \( d\eta = \theta \) and
\[
\|\eta\|_{F^{k-1}k^{-1}} \leq C_k \|\theta\|_{F^k}.
\]

Lemma 2.5. [4, Propositions 4.6 and 4.7] The following conditions (i) and (ii) are equivalent:

(i) \( T_k = 0 \).
(ii) The operator \( d_{k-1} : F^{k-1}k^{-1} / Z^{k-1} \to B^k \) admits a bounded inverse \( d_{k-1}^{-1} \).

Any of these conditions implies
(iii) There exists a constant \( C'_k \) such that for any \( \xi \in F^{k-1}k^{-1} \) there is an element \( \zeta \in Z^{k-1} \) such that
\[
\|\xi - \zeta\|_{F^{k-1}k^{-1}} \leq C'_k \|d\xi\|_{F^k}.
\]

Moreover, if \( F^{k-1}k^{-1} \) is a reflexive Banach space then conditions (i)-(iii) are equivalent.

3. Orlicz Spaces of Differential Forms and Orlicz Cohomology

Let \( X \) be a Riemannian manifold of dimension \( n \). Given \( x \in X \), denote by \( (\omega(x), \theta(x)) \) the scalar product of exterior \( k \)-forms \( \omega(x) \) and \( \theta(x) \) on \( T_x X \). This gives a function \( x \mapsto (\omega(x), \theta(x)) \) on \( X \).

Let \( \Phi : \mathbb{R} \to \mathbb{R} \) and \( \Psi : \mathbb{R} \to \mathbb{R} \) be two complementary \( N \)-functions. Given a measurable \( k \)-form \( \omega \), we put
\[
\rho_{\Phi}(\omega) := \int_X \Phi(|\omega(x)|)d\mu_X.
\]
Here \( d\mu_X \) stands for the volume element of the Riemannian manifold \( X \). We will identify \( k \)-forms differing on a set of measure zero.

Given a (not necessarily orientable) Riemannian manifold \( X \), introduce the space \( L^\Phi(X, \Lambda^k) \) as the class of all measurable \( k \)-forms \( \omega \) satisfying the condition
\[
\rho_{\Phi}(\alpha \omega) < \infty \text{ for some } \alpha > 0.
\]

As in the case of Orlicz function spaces, the space \( L^\Phi(X, \Lambda^k) \) is endowed with two equivalent norms: the gauge norm
\[
\|\omega\|_{\Phi} = \inf \left\{ K > 0 : \rho_{\Phi}\left(\frac{\omega}{K}\right) \leq 1 \right\},
\]
and the Orlicz norm (\( \Psi \) is the complementary \( N \)-function to \( \Phi \)):
\[
\|\omega\|_{\Psi} = \sup \left\{ \int_X (\omega(x), \theta(x)) d\mu_X : \rho_{\Psi}(\theta) \leq 1 \right\}
\]
As in the case of function spaces, it can be proved that \( L^\Phi(X, \Lambda^k) \) endowed with one of these norms is a Banach space.

Obviously, the gauge norm of a \( k \)-form \( \omega \) is nothing but the gauge norm of its modulus function \( |\omega| \). The same holds for the Orlicz norm ([7, Lemma 2.1]).
Unless otherwise specified, we endow the $L^k$ spaces with the gauge norms; the quotient (semi)norm on each of the cohomology spaces to be defined below depends on the choice of the norms on $L^q_j$ and $L^{q,j}$ but the resulting topology does not.

**Definition 3.1.** A form $\theta \in L^{q,j+1}_1(X)$ is called the (weak) differential $d\omega$ of $\omega \in L^q_{1,\text{loc}}(X)$ if
$$\int_U \omega \wedge du = (-1)^j \int_U \theta \wedge u$$
for every orientable domain $U \subset \text{Int} X$ and every form $u \in D^{n-j-1}(X)$ having support in $U$.

Let $\Phi_I$ and $\Phi_{II}$ be $N$-functions. For $0 \leq k \leq n$, put
$$\Omega_{\Phi_I, \Phi_{II}}^k(X) = \{ \omega \in L^{q,I}(X, \Lambda^k) : d\omega \in L^{q,II}(X, \Lambda^{k+1}) \}.$$  
This is a Banach space with the norm
$$\| \omega \|_{(\Phi_I), (\Phi_{II})} = \| \omega \|_{(\Phi_I)} + \| d\omega \|_{(\Phi_{II})}.$$
Consider also the spaces
$$Z^k_{\Phi_{II}}(X) = \{ \omega \in L^{q,II}(X, \Lambda^k) : d\omega = 0 \};$$
$$B^k_{\Phi_I, \Phi_{II}}(X) = \{ \omega \in L^{q,II}(X, \Lambda^k) : \omega = d\beta \text{ for some } \beta \in L^{q,I}(X, \Lambda^{k-1}) \}.$$
Denote by $\overline{B}^k_{\Phi_I, \Phi_{II}}(X)$ the closure of $B^k_{\Phi_I, \Phi_{II}}(X)$ in $L^{q,II}(X, \Lambda^k)$.

**Definition 3.2.** The quotient spaces
$$H^k_{\Phi_I, \Phi_{II}}(X) := Z^k_{\Phi_{II}}(X)/B^k_{\Phi_I, \Phi_{II}}(X)$$
and
$$\overline{H}^k_{\Phi_I, \Phi_{II}}(X) := Z^k_{\Phi_{II}}(X)/\overline{B}^k_{\Phi_I, \Phi_{II}}(X)$$
are called the $k$th $L_{\Phi_I, \Phi_{II}}$-cohomology and the $k$th reduced $L_{\Phi_I, \Phi_{II}}$-cohomology of the Riemannian manifold $X$, the latter cohomology being a Banach space. Define the $L_{\Phi_I, \Phi_{II}}$-torsion as
$$T^k_{\Phi_I, \Phi_{II}}(X) := \overline{H}^k_{\Phi_I, \Phi_{II}}(X)/B^k_{\Phi_I, \Phi_{II}}(X).$$

The torsion $T^k_{\Phi_I, \Phi_{II}}(X)$ can be either $\{0\}$ or infinite-dimensional. In fact, if $\dim T^k_{\Phi_I, \Phi_{II}}(X) < \infty$ then $B^k_{\Phi_I, \Phi_{II}}(X)$ is closed, hence $T^k_{\Phi_I, \Phi_{II}}(X) = \{0\}$. In particular, if $\dim T^k_{\Phi_I, \Phi_{II}}(X) \neq 0$ then $\dim H^k_{\Phi_I, \Phi_{II}}(X) = \infty$.

If $\Phi_I = \Phi_{II} = \Phi$ then we use the notations $\Omega^k_{\Phi}(X)$, $H^k_{\Phi}(X)$, and $\overline{H}^k_{\Phi}(X)$ instead of $\Omega^k_{\Phi_I, \Phi_{II}}(X)$, $H^k_{\Phi_I, \Phi_{II}}(X)$, and $\overline{H}^k_{\Phi_I, \Phi_{II}}(X)$ respectively. Thus, the $L_{\Phi,\Phi}$-cohomology $H^k_{\Phi}(X)$ (respectively, the reduced $L_{\Phi,\Phi}$-cohomology $\overline{H}^k_{\Phi}(X)$) is the $k$th cohomology (respectively, the $k$th reduced cohomology) of the cochain complex $\{ \Omega^k_{\Phi}(X), d \}$.

In [4], Gol’dshtein and Troyanov realized the $k$th $L_{q,p}$-cohomology as the $k$th cohomology of some Banach complex. Here we apply this approach to $L_{\Phi_I, \Phi_{II}}$-cohomology.

Fix an $(n + 1)$-tuple of $N$-functions $F = \{ \Phi_0, \Phi_1, \ldots, \Phi_n \}$ and put
$$\Omega^k_F(X) = \Omega^k_{\Phi_0, \Phi_{n+1}}(X);$$
Since the weak exterior differential is a bounded operator $d : \Omega^k_\mathcal{F}(X) \to \Omega^{k+1}_\mathcal{F}(X)$, we obtain a Banach complex

$$0 \to \Omega^0_\mathcal{F}(X) \to \Omega^1_\mathcal{F}(X) \to \cdots \to \Omega^k_\mathcal{F}(X) \to \cdots \to \Omega^{\infty}_\mathcal{F}(X) \to 0.$$ 

The $L_\mathcal{F}$-cohomology $H^k_\mathcal{F}(X)$ (respectively, the reduced $L_\mathcal{F}$-cohomology $\overline{H}^k_\mathcal{F}(X)$) of $X$ is the $k$th cohomology (respectively, the $k$th reduced cohomology) of the Banach complex $(\Omega^k_\mathcal{F}, d)$.

The above-defined cohomology spaces $H^k_\mathcal{F}(X)$ and $\overline{H}^k_\mathcal{F}(X)$ in fact depend only on $\Phi_{k-1}$ and $\Phi_k$:

$$H^k_\mathcal{F}(X) = H^k_{\Phi_{k-1}, \Phi_k}(X) = Z^k_{\Phi_k}(X) / B^k_{\Phi_{k-1}, \Phi_k} ;$$

$$\overline{H}^k_\mathcal{F}(X) = \overline{H}^k_{\Phi_{k-1}, \Phi_k}(X) = Z^k_{\Phi_k}(X) / B^k_{\Phi_{k-1}, \Phi_k} .$$

The results on abstract Banach complexes by Gol’dstein and Troyanov enable us to interpret Orlicz cohomology in terms of a Poincaré–Sobolev–Orlicz type inequality for differential forms on a Riemannian manifold $X$:

**Theorem 3.3.** $H^k_{\Phi_1, \Phi_1}(X) = 0$ if and only if there exists a constant $C < \infty$ such that for any closed differential form $\omega \in L^{\frac{1}{\Phi_1}}(X, \Lambda^k)$ there exists a differential form $\theta \in L^{\frac{1}{\Phi_1}}(X, \Lambda^{k-1})$ such that $d\theta = \omega$ and

$$\|\theta\|_{L^{\frac{1}{\Phi_1}}} \leq C\||\omega\|_{L^{\frac{1}{\Phi_1}}} .$$

This result is an immediate consequence of Lemma 2.4.

**Theorem 3.4.** (A) If $T^k_{\Phi_1}(X) = 0$ then there exists a constant $C'$ such that for any differential form $\theta \in \Omega^{k-1}_{\Phi_1}(X)$ there exists a closed form $\zeta \in Z^k_{\Phi_1}(X)$ such that

$$\|\theta - \zeta\|_{L^{\frac{1}{\Phi_1}}} \leq C'\|d\theta\|_{L^{\frac{1}{\Phi_1}}} .$$

(B) Conversely, if $\Phi_1 \in \Delta_2 \cap \nabla_2$ and there exists a constant $C'$ such that for any form $\theta \in \Omega^{k-1}_{\Phi_1}(X)$ there exists $\zeta \in Z^k_{\Phi_1}(X)$ such that (3.1) holds then $T^k_{\Phi_1}(X) = 0$.

**Proof.** Considering the Banach complex $\Omega^\ast_\mathcal{F}$ with $\mathcal{F} = \{\Phi_1, \ldots, \Phi_{k-1}, \Phi_k, \ldots, \Phi_1\}$, where $\Phi_1$ changes to $\Phi_1$ at the $k$th position, we get

$$H^k_\mathcal{F}(X) = H^k_{\Phi_1, \Phi_1}(X); \quad \overline{H}^k_\mathcal{F}(X) = \overline{H}^k_{\Phi_1, \Phi_1}(X).$$

Since $\Phi_1 \in \Delta_2 \cap \nabla_2$, the Banach space $\Omega^{k-1}_{\Phi_1, \Phi_1}(X)$ is reflexive. Theorem 3.4 now stems from Lemma 2.5.

4. The $L_{\Phi_1, \Phi_2}$-COHOMOLOGY OF $\mathbb{R}$

Let $\Phi_1$ and $\Phi_2$ be $N$-functions.

**Proposition 4.1.** $T^1_{\Phi_1, \Phi_2}(\mathbb{R}) \neq 0$.

**Proof.** Suppose on the contrary that $T^1_{\Phi_1, \Phi_2}(\mathbb{R}) = 0$. In accordance with Theorem 3.4, then there is a Sobolev inequality for functions on $\mathbb{R}$

$$\inf_{z \in \mathbb{R}} \|f - z\|_{(\Phi_1)} \leq C\|f\|_{(\Phi_2)} .$$
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for some real positive constant $C$.

Consider the function

$$
\theta(x) = \omega_{1/2}(x) = \begin{cases} 
Ce^{-\frac{1}{2-x^2}} & \text{if } |x| \leq 1/2, \\
0 & \text{if } |x| > 1/2.
\end{cases}
$$

Here the constant $C$ is chosen so that

$$
\int_{-\infty}^{\infty} \theta(x) \, dx = \frac{c}{2} \int_{-1}^{1} e^{-\frac{1}{1-x^2}} \, dt = 1.
$$

Now, consider the family of smooth functions with compact support

$\{f_a : \mathbb{R} \to \mathbb{R} : a > 1\}$, where

$$
f_a(x) = \int_{-\infty}^{x} \left( \theta \left( x + \frac{3}{2} \right) + \theta \left( -x + a + \frac{1}{2} \right) \right) \, dx
$$

(we owe this construction to [2, pp. 8–9]). Then $f_a(x) = 1$ if $x \in [1, a]$, $f_a(x) = 0$ if $x \not\in [0, a+1]$, and $\|f'_a\|_{L^\infty} = L < \infty$. Clearly, $\|f_a - z\|_{(\Phi_1)}$ is finite only for $z = 0$.

Estimate the Orlicz norms involved in (4.1). We have

$$
\rho_{\Phi_1} \left( \frac{f_a}{K} \right) = \int_{-\infty}^{\infty} \Phi_1 \left( \frac{f_a(x)}{K} \right) \, dx \geq \int_{1}^{a} \Phi_1 \left( \frac{1}{K} \right) \, dx = (a-1)\Phi_1 \left( \frac{1}{K} \right).
$$

If $\rho_{\Phi_1} \left( \frac{f_a}{K} \right) \leq 1$ then $(a-1)\Phi_1 \left( \frac{1}{K} \right) \leq 1$, which is equivalent to

$$
K \geq \frac{1}{\Phi_1^{-1} \left( \frac{1}{a-1} \right)}.
$$

Hence,

$$
\|f_a\|_{(\Phi)} = \inf \left\{ K : \rho_{\Phi_1} \left( \frac{f_a}{K} \right) \leq 1 \right\} \geq \frac{1}{\Phi_1^{-1} \left( \frac{1}{a-1} \right)}.
$$

On the other hand,

$$
\rho_{\Phi_2} \left( \frac{f'_a}{K} \right) = \int_{-\infty}^{\infty} \Phi_2 \left( \frac{f'_a(x)}{K} \right) \, dx
$$

$$
= \int_{0}^{1} \Phi_2 \left( \frac{f'_a(x)}{K} \right) \, dx + \int_{a}^{a+1} \Phi_2 \left( \frac{f'_a(x)}{K} \right) \, dx \leq 2\Phi_2 \left( \frac{L}{K} \right).
$$

We have

$$
2\Phi_2 \left( \frac{L}{K} \right) \leq 1 \iff K \geq \frac{L}{\Phi_2^{-1} \left( \frac{1}{2} \right)}.
$$

Put $\mathcal{M}_{f'_a} = \left\{ K : \rho_{\Phi_2} \left( \frac{f'_a}{K} \right) \leq 1 \right\}$. We have shown that if $K \geq \frac{L}{\Phi_2^{-1} \left( \frac{1}{2} \right)}$ then $K \in \mathcal{M}_{f'_a}$. Therefore,

$$
\|f'_a\|_{(\Phi_2)} = \inf \mathcal{M}_{f'_a} \leq \frac{L}{\Phi_2^{-1} \left( \frac{1}{2} \right)}.
$$

Thus,

$$
C \geq \frac{\Phi_2^{-1} \left( \frac{1}{2} \right)}{L\Phi_1^{-1} \left( \frac{1}{a-1} \right)} \to \infty \quad \text{as } a \to \infty.
$$

The obtained contradiction proves the proposition. \qed
Corollary 4.2. If \( \Phi_1 \) and \( \Phi_2 \) are \( N \)-functions then the space \( H_{\Phi_1,\Phi_2}^1(\mathbb{R}) \) is not separated; in particular, \( H_{\Phi_1,\Phi_2}^1(\mathbb{R}) \neq 0 \).

Proposition 4.3. If \( \Phi_1 \) and \( \Phi_2 \) are \( N \)-functions and \( \Phi_2 \in \Delta_2 \) then \( \overline{H}_{\Phi_1,\Phi_2}^1(\mathbb{R}) = 0 \).

Proof. Let \( \omega = a(x)dx \in \mathcal{L}_{\Phi_2}(\mathbb{R}) \). For each \( n \), put

\[
C_n = \int_{-n}^{n} a(x) \, dx.
\]

If \( C_n = 0 \) then put \( \lambda_n(x) \equiv 0 \) for all \( x \in \mathbb{R} \). If \( C_n \neq 0 \) then put

\[
\lambda_n(x) = \text{sign} \, C_n \varepsilon_n \chi \left( -\frac{|C_n|}{2\varepsilon_n}, \frac{|C_n|}{2\varepsilon_n} \right),
\]

where \( \varepsilon_n = t_n/m \) and \( t_n \) is the only root of the equation

\[
\frac{\Phi_2(t_n)}{t_n} = \frac{1}{m|C_n|}.
\]

(The function \( t \mapsto \Phi_2(t)/t \) is strictly increasing; see, for example, [9]). We obviously have

\[
\int_{\mathbb{R}} \lambda_n(x) \, dx = C_n = \int_{-n}^{n} a(x) \, dx.
\]

Compute the norm \( \|\lambda_n\|_{(\Phi_2)} \). We have

\[
\rho_{\Phi_2} \left( \frac{\lambda_n}{K} \right) = \int_{|C_n|/2\varepsilon_n}^{C_n/2\varepsilon_n} \Phi_2 \left( \frac{\varepsilon_n}{K} \right) \, dx = \frac{|C_n|}{\varepsilon_n} \Phi_2 \left( \frac{\varepsilon_n}{K} \right).
\]

Thus,

\[
\rho_{\Phi_2} \left( \frac{\lambda_n}{K} \right) \leq 1 \iff \frac{|C_n|}{\varepsilon_n} \Phi_2 \left( \frac{\varepsilon_n}{K} \right) \leq 1 \iff \Phi_2 \left( \frac{\varepsilon_n}{|C_n|} \right) \leq \frac{\varepsilon_n}{|C_n|} \\
\iff \frac{\varepsilon_n}{K} \leq \Phi_2^{-1} \left( \frac{\varepsilon_n}{|C_n|} \right) \iff K \geq \frac{\varepsilon_n}{\Phi_2^{-1} \left( \frac{\varepsilon_n}{|C_n|} \right)}.
\]

Here \( \Phi_2^{-1} \) stands for the positive inverse function to \( \Phi_2 : [0, \infty) \rightarrow [0, \infty) \). Hence, \( \|\lambda_n\|_{(\Phi_2)} = \frac{\varepsilon_n}{\Phi_2^{-1}(\varepsilon_n/|C_n|)} \). By the choice of \( \varepsilon_n \),

\[
\frac{\Phi_2(m\varepsilon_n)}{m\varepsilon_n} = \frac{1}{m|C_n|},
\]

and so \( \|\lambda_n\|_{(\Phi_2)} = \frac{1}{m} \).

Let \( b_n(x) := \int_{-\infty}^{x} (\chi_{[-m,m]}(t)a(t) - \lambda_n(t)) \, dt \). Since \( b_n \) has compact support, \( b_n \in \mathcal{L}_{\Phi_1}^1(\mathbb{R}) \) for each \( m \). Furthermore, \( \|db_n - \omega\|_{(\Phi_2)} \leq \|a\|_{(\mathcal{L}^{\Phi_2}(\mathbb{R}[-m,m]))} + \|\lambda_n\|_{(\mathcal{L}^{\Phi_2}(\mathbb{R}))} \rightarrow 0 \) as \( m \rightarrow \infty \) since for \( \Phi_2 \in \Delta_2 \) all functions in \( \mathcal{L}^{\Phi_2} \) have absolutely continuous norm ([9, Theorem 10.3]). Thus, \( \overline{H}_{\Phi_1,\Phi_2}^1(\mathbb{R}) = 0 \). \( \square \)

All the results of this section are also valid for the half-line \( \mathbb{R}_+ \) (with similar proofs).
5. The $L_{\Phi_1, \Phi_2}$-Cohomology of the Hyperbolic Plane

We will need the following Orlicz versions of Propositions 8.3 and 8.4 in [4], which are proved in absolutely the same manner:

**Proposition 5.1.** Let $M$ be a complete manifold of dimension $n$ and let $(\Phi_1, \Psi_1)$ and $(\Phi_2, \Psi_2)$ be two pairs of complementary Orlicz functions. Suppose that $\alpha \in Z^k_{\Phi_2}(X)$ and there exists a smooth closed $(n-k)$-form $\gamma$ such that $\gamma \in Z^{n-k}_{\Psi_1}(X)$, $\gamma \wedge \alpha \in L^1(X, \Lambda^n)$, and
\[
\int_M \gamma \wedge \alpha \neq 0,
\]
then $\alpha \notin B^k_{\Phi_1; \Phi_2}(X)$. In particular, $H^k_{\Phi_1; \Phi_2}(X) \neq 0$.

**Proposition 5.2.** Let $M$ be a complete manifold of dimension $n$ and let $(\Phi_1, \Psi_1)$ and $(\Phi_2, \Psi_2)$ be two pairs of complementary Orlicz functions. Suppose that $\alpha \in Z^k_{\Phi_2}(X)$ and there exists a smooth closed $(n-k)$-form $\gamma \in Z^{n-k}_{\Psi_1}(X) \cap Z^{n-k}_{\Psi_2}(X)$ such that
\[
\int_M \gamma \wedge \alpha \neq 0,
\]
then $\alpha \notin B^k_{\Phi_1; \Phi_2}(X)$. In particular, $\Pi^k_{\Phi_1; \Phi_2}(X) \neq 0$.

The hyperbolic plane $\mathbb{H}^2$ is the Riemannian manifold that can be modelled as the space $\mathbb{R}^2$ endowed with the Riemannian metric
\[
ds^2 = e^{2z}dy^2 + dz^2.
\]

For an $N$-function $\Phi$, introduce the condition
\[
\int_0^1 \Phi(v) \frac{dv}{v^2} < \infty. \tag{A}
\]
(The upper integration limit 1 can be replaced by any positive number.)

**Theorem 5.3.** If $\Phi_1$ and $\Phi_2$ are $N$-functions such that their complementary $N$-functions $\Psi_1$ and $\Psi_2$ and the function $\Phi_2$ satisfy condition (A) then
\[
\dim(\bar{H}^1_{\Phi_1, \Phi_2}(\mathbb{H}^2)) = \infty.
\]

We will need the following lemma, which is in fact Lemma 10.2 in [4]:

**Lemma 5.4.** There exist two smooth functions $f$ and $g$ on $\mathbb{H}^2$ such that
(1) $f$ and $g$ are nonnegative;
(2) $f(y, z) = g(y, z) = 0$ if $z \leq 0$ or $|y| \geq 1$;
(3) $df$ and $dg \in L^r(\mathbb{H}^2, \Lambda^1)$ for any $1 < r \leq \infty$;
(4) the support of $df \wedge dg$ is contained in $\{(y, z) : |y| \leq 1, 0 \leq z \leq 1\}$;
(5) $df \wedge dg \geq 0$;
(6) $\int_{\mathbb{H}^2} df \wedge dg = 1$;
(7) $\frac{df}{dy}$ and $\frac{dg}{dy} \in L^\infty(\mathbb{H}^2)$;
(8) $\frac{df}{dz}$ and $\frac{dg}{dz}$ have compact support.

We will also need the following generalization of item (3) above:

**Lemma 5.5.** If $\Phi$ is an $N$-function satisfying condition (A) then $df$, $dg \in L^\Phi(\mathbb{H}^2, \Lambda^1)$. 

Proof. Recall the construction of the functions $f$ and $g$ of [4, Lemma 10.2].

Choose smooth functions $h_1$, $h_2$, and $k : \mathbb{R} \to \mathbb{R}$ with the following properties:

1. $h_1$, $h_2$, and $k$ are nonnegative;
2. $h_1(y) = 0$ if $|y| \geq 1$;
3. $h_1(y)h_2(y) \geq 0$ and $h_1(y)h_2'(y) \leq 0$ for all $y$;
4. the support of the function $h_1'(y)h_2(y) - h_1(y)h_2'(y)$ is not empty;
5. $k'(z) \geq 0$ for all $z$;
6. $k(z) = 1$ if $z \geq 1$ and $k(z) = 0$ if $z \leq 0$.

Then $f$ and $g$ are defined as $f(y, z) := h_1(y)k(z)$ and $g(y, z) := h_2(y)k(z)$ respectively.

We will now prove that $df \in L^p$ by modifying the argument of the proof of [4, Lemma 10.2].

Indeed,

$$df = h_1(y)k'(z)dz + k(z)h_1'(y)dy.$$  

The first summand $h_1(y)k'(z)dz$ has compact support, and the second summand $k(z)h_1'(y)dy$ is zero outside the infinite rectangle $Q = \{ |y| \leq 1; z \geq 0 \}$.

Choose $D < \infty$ such that $|k(z)h_1'(y)| \leq D$ on $Q$. We have

$$|k(z)h_1'(y)dy| \leq D |dy| = D e^{-z}.$$  

Since the area element of $\mathbb{H}^2$ is $dA = e^z dydz$, for any $a > 0$ we infer

$$\int_{\mathbb{H}^2} \Phi(a|k(z)h_1'(y)dy|)dA \leq \int_Q \Phi(aDe^{-z})e^z dydz = 2aD \int_0^\infty \frac{\Phi(aDe^{-z})}{aDe^{-z}} dz.$$  

Putting $aDe^{-z} = v$ in the last integral, we get

$$2aD \int_0^\infty \frac{\Phi(aDe^{-z})}{aDe^{-z}} dz = 2aD \int_0^{aD} \frac{\Phi(v)}{v^2} dv < \infty.$$  

Thus, $\rho_4(ak(z)h_1'(y)dy) < \infty$ for any $a$. Consequently, $k(z)h_1'v \in L^p(\mathbb{H}^2, \Lambda^1)$. Thus, $df = h_1(y)k'(z)dz + k(z)h_1'(y)dy$ also lies in $L^p(\mathbb{H}^2, \Lambda^1)$.

The lemma is proved.

Proof of Theorem 5.3. Take the functions $f$ and $g$ on $\mathbb{H}^2$ defined in Lemma 5.4 and consider the 1-forms $\alpha = df$ and $\gamma = dg$ on $\mathbb{H}^2$. Obviously, $d\alpha = d\gamma = 0$.

By Lemmas 5.4 and 5.5, $\alpha \in L^p$ for any N-function $\Phi$ such that $\int_0^1 \Phi(v)/v^2 dv < \infty$ and $\gamma$ is smooth and $\gamma \in L^p(\mathbb{H}^2, \Lambda^1)$ if $\int_0^1 \Phi_{\alpha}(v)/v^2 dv < \infty$ and $\int_0^1 \Phi_{\gamma}(v)/v^2 dv < \infty$.

Since $\int_{\mathbb{H}^2} \alpha \wedge \gamma \neq 0$, Proposition 5.1 shows that $\alpha \notin \overline{\mathcal{T}}_{\Phi_1, \Phi_2}(\mathbb{H}^2)$.

Now, using the isometry group of $\mathbb{H}^2$, we obtain an infinite family of linearly independent classes in $\overline{\mathcal{T}}_{\Phi_1, \Phi_2}(\mathbb{H}^2)$.

6. THE $L^p$-COHOMOLOGY OF THE BALL

In this section, we prove the "$L^p$-Poincaré lemma", i.e., the vanishing of the $L^p$-cohomology of the unit ball $B^n \subset \mathbb{R}^n$.

Since $B^n$ has finite volume, $H^0_{\Phi_1, \Phi_2}(\mathbb{R}^n) = \overline{\mathcal{T}}^0_{\Phi_1, \Phi_2}(\mathbb{R}^n) = \mathbb{R}$ for any N-functions $\Phi_1$ and $\Phi_2$.

For the case of $L^p$ spaces, Gol’dshein, Kuz’minov, and Shvedov proved the vanishing of the $L^p$-cohomology of the ball in [3, Lemma 3.2]; for $p \neq q$, Gol’dshein and Troyanov found necessary and sufficient conditions on $p$ and $q$ for the vanishing
of the $L^{q,p}$-cohomology of $\mathbb{B}^n$. Their proof is based on the following fact, established by Iwaniec and Lutoborski in [5]:

**Proposition 6.1.** For any bounded convex domain $U \subset \mathbb{R}^n$ and any $k = 1, 2, \ldots, n$, there exists an operator

$$T = T_U : L^1_{\text{loc}}(U, \Lambda^k) \to L^1_{\text{loc}}(U, \Lambda^{k-1})$$

with the following properties:

(a) $T(d\theta) + dT\theta = \theta$ (in the sense of currents);

(b) $|T\theta(x)| \leq C \int_U |\theta(y)| |y - x|^{n-1} dy$.

We prove

**Corollary 6.2.** If $\Phi$ is an $N$-function then the operator $T$ maps $L^\Phi(U, \Lambda^k)$ continuously into $L^\Phi(U, \Lambda^{k-1})$.

**Proof.** The following Orlicz space version of Young’s inequality for convolution holds (see the proof of Corollary 7 in [10, pp. 230–231]: If $f \in L^\Phi$ and $g \in L^1$ then $f \ast g \in L^\Phi$ and

$$\|f \ast g\| \leq \|f\| \|g\|_1.$$

Applying this inequality to $f = |\theta|$ and $g(x) = |x|^{1-n}$, we obtain the corollary from Proposition 6.1. In the Orlicz norms, the norm of the operator $T$ is bounded by $\|g\|_1$.

**Corollary 6.3.** The operator $T : \Omega^\Phi(U, \Lambda^k) \to \Omega^\Phi(U, \Lambda^{k-1})$ is bounded and $Td\omega + dT\omega = \omega$ for any $\omega \in \Omega^\Phi(U)$.

Corollary 6.3 gives the following

**Theorem 6.4.** If $\Phi$ is an $N$-function then $H^k_\Phi(\mathbb{B}^n) = 0$ for all $k = 1, \ldots, n$.

**Proof.** Let $\omega \in Z^k_\Phi(\mathbb{B}^n)$. By Corollary 6.3, $T\omega \in L^\Phi(\mathbb{B}^n, \Lambda^{k+1})$. Since $\omega = dT\omega + Td\omega = d(T\omega)$, we conclude that $[\omega] = [dT\omega] = 0 \in H^k_\Phi(\mathbb{B}^n)$ and so $H^k_\Phi(\mathbb{B}^n) = 0$.

**Remark 6.5.** The Sobolev space analog of Theorem 6.4, Theorem 11.5 in [4] (see also [4, Proposition 11.4]) gives the following criterion: for $1 < p, q \leq \infty$, the $L^{q,p}$-cohomology spaces $H^{k,q,p}(\mathbb{B}^n)$, $k = 1, 2, \ldots, n$ are trivial if and only if $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. It would be interesting to obtain a criterion like [4, Theorem 11.5] for the Orlicz cohomology $H^k_{\Phi_1, \Phi_2}(\mathbb{B}^n)$ for different functions $\Phi_1$ and $\Phi_2$.

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**References**


\[1\] Though it is required in Corollary 7 in [10, pp. 230–231] that $\Phi \in \Delta_2$, the proof of Young’s inequality works for general $N$-functions $\Phi$. 


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