S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 16, стр. 1079–1090 (2019) УДК 517, 515.168
DOI 10.33048/semi.2019.16.075 MSC 58A12, 46E30, 22E25

SOME CALCULATIONS OF ORLICZ COHOMOLOGY AND POINCARÉ–SOBOLEV–ORLICZ INEQUALITIES

V.GOL′DSHTEIN, YA.A.KOPYLOV

Abstract. We carry out calculations of Orlicz cohomology for some basic Riemannian manifolds (the real line, the hyperbolic plane, the ball). Relationship between Orlicz cohomology and Poincaré–Sobolev–Orlicz-type inequalities is discussed.
Keywords: differential form, Orlicz cohomology, torsion, Poincaré–Sobolev–Orlicz inequality

Introduction

The article continues the study of Orlicz cohomology of Riemannian manifolds initiated in [7, 8].

Orlicz cohomology is a natural generalization of L_{qp}-cohomology (for a detailed discussion of L_{qp}-cohomology, the reader is referred, for example, to [4]).

Like Orlicz function spaces, the Orlicz spaces L^{Φ} of differential forms are a natural nonlinear generalization of the spaces L^p. Orlicz spaces of differential forms on domains in $\mathbb{R}^n$ were first considered by Iwaniec and Martin in [6] and then by Agarwal, Ding, and Nolder in [1]. Orlicz forms on an arbitrary Riemannian manifold were apparently first examined by Kopylov and Panenko in [7].

In [4], Gol′dshtein and Troyanov demonstrated close relationship between L_{qp}-cohomology and Sobolev-type inequalities on Riemannian manifolds and, basing on this and some "almost duality" techniques, performed calculations of L_{qp}-cohomology for some basic manifolds. It turns out that, with some significant corrections and sometimes under additional constraints on the N-functions from which the Orlicz

Gol′dshtein V., Kopylov Ya.A., Some calculations of Orlicz cohomology and Poincaré–Sobolev–Orlicz inequalities.

The second author was supported by the Program of Basic Scientific Research of the Siberian Branch of the Russian Academy of Sciences (Grant No. 1.1.2, Project 0314–2016–0007).

Received February, 18, 2019, published August, 13, 2019.

cohomology is constructed, these methods prove to be fruitful in computing Orlicz cohomology.

The structure of the article is as follows: In Section 1, we recall the main notions and necessary properties of Orlicz function spaces. In Section 2, we recall some basic information on abstract Banach complexes. Section 3 contains defininitions concerning Orlicz spaces of differential forms on a Riemannian manifold, Orlicz cohomology, and its interpretation in terms of Poincaré–Sobolev–Orlicz inequalities (Theorems 3.3 and 3.4). Then we calculate the L_{Φ_1,Φ_2}-cohomology of $\mathbb{R}$ (Section 4) the hyperbolic plane (Section 5) and the L_Φ-cohomology of the ball ("L^Φ-Poincaré inequality", Section 6).

1. N-Functions and Orlicz Function Spaces

Definition 1.1. A nonnegative function $\Phi : \mathbb{R} \to \mathbb{R}$ is called an *N-function* if

(i) Φ is even and convex;
(ii) $\Phi(x) = 0 \Longleftrightarrow x = 0$;
(iii) $\lim\limits_{x\to 0} \frac{\Phi(x)}{x} = 0$; $\lim\limits_{x\to\infty} \frac{\Phi(x)}{x} = \infty$.

An N-function Φ has left and right derivatives (which can differ only on an at most countable set, see, for instance, [10, Theorem 1, p. 7]). The left derivative φ of Φ is left continuous, nondecreasing on $(0,\infty)$, and such that $0 < \varphi(t) < \infty$ for $t > 0$, $\varphi(0) = 0$, $\lim\limits_{t\to\infty} \varphi(t) = \infty$. The function

$$\psi(s) = \inf\{t > 0 \,:\, \varphi(t) > s\}, \quad s > 0,$$

is called the *left inverse* of φ.

The functions Φ, Ψ given by

$$\Phi(x) = \int_0^{|x|} \varphi(t)dt, \quad \Psi(x) = \int_0^{|x|} \psi(t)dt$$

are called *complementary N-functions.*

The N-function Ψ complementary to an N-function Φ can also be expressed as

$$\Psi(y) = \sup\{x|y| - \Phi(x) \,:\, x \geq 0\}, \quad y \in \mathbb{R}.$$

Throughout the article, given an N-function $\Phi : \mathbb{R} \to [0,\infty)$, we denote by Φ^{-1} its "positive" inverse $\Phi^{-1} : [0,\infty) \to [0,\infty)$.

N-functions are classified in accordance with their growth rates as follows:

Definition 1.2. An N-function Φ is said to satisfy the Δ_2-condition (for all x), which is written as $\Phi \in \Delta_2$ if there exists a constant $K > 2$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq 0$; Φ is said to satisfy the ∇_2-condition (for all x), which is denoted symbolically as $\Phi \in \nabla_2$, if there is a constant $c > 1$ such that $\Phi(x) \leq \frac{1}{2c}\Phi(cx)$ for all $x \geq 0$.

It is not hard to see that an N-function Φ satisfies the the ∇_2-condition if and only if its dual N-function satisfies the Δ_2-condition.

Henceforth, let Φ be an N-function and let (Ω, Σ, μ) be a measure space.

Definition 1.3. Given a measurable function $f : \Omega \to \mathbb{R}$, we put

$$\rho_\Phi(f) := \int_\Omega \Phi(f)d\mu.$$

Definition 1.4. The linear space

$$L^\Phi = L^\Phi(\Omega) = L^\Phi(\Omega, \Sigma, \mu) = \{f : \Omega \to \mathbb{R} \text{ measurable } : \rho_\Phi(af) < \infty \text{ for } \textit{some } a > 0\}$$

is called an *Orlicz space* on (Ω, Σ, μ).

Let Ψ be the complementary N-function to Φ.

Below we as usual identify two functions equal outside a set of measure zero.

If $f \in L^\Phi$ then the functional $\|\cdot\|_\Phi$ (called *the Orlicz norm*) defined by

$$\|f\|_\Phi = \|f\|_{L^\Phi(\Omega)} = \sup\left\{\left|\int_\Omega fg\,d\mu\right| : \rho_\Psi(g) \le 1\right\}$$

is a seminorm. It becomes a norm if μ satisfies the *finite subset property* (see [10, p. 59]): if $A \in \Sigma$ and $\mu(A) > 0$ then there exists $B \in \Sigma$, $B \subset A$, such that $0 < \mu(B) < \infty$.

The equivalent *gauge* (or *Luxemburg*) *norm* of a function $f \in L^\Phi$ is defined by the formula

$$\|f\|_{(\Phi)} = \|f\|_{L^{(\Phi)}(\Omega)} = \inf\left\{K > 0 \,:\, \rho_\Phi\left(\frac{f}{K}\right) \le 1\right\}.$$

This is a norm without any constraint on the measure μ (see [10, p. 54, Theorem 3]).

2. Banach Complexes

Like in the case of $L_{q,p}$-cohomology, treated in [4], we apply some abstract facts about Banach complexes to the Orlicz cohomology of Riemannian manifolds.

In this section, we recall some definitions and assertions about abstract Banach complexes given in [4].

Definition 2.1. A *Banach complex* is a sequence $F^* = \{F^k, d_k\}_{k\in\mathbb{N}}$ where F^k is a Banach space and $d = d^k : F^k \to F^{k+1}$ is a bounded operator with $d^{k+1} \circ d^k = 0$.

Definition 2.2. Given a Banach complex $\{F^k, d\}$, introduce the vector spaces:

- $Z^k := \ker(d : F^k \to F^{k+1})$ (a closed subspace of F^k);
- $B^k :=\mathrm{Im}(d : F^{k-1} \to F^k) \subset Z^k$;
- $H^k(F^*) := Z^k/B^k$ is the *cohomology* of the complex $F^* = \{F^k, d\}$;
- $\overline{H}^k(F^*) := Z^k/\overline{B}^k$ is *the reduced cohomology* of the complex F^*;
- $T^k(F^*) := \overline{B}^k/B^k = H^k/\overline{H}^k$ is the *torsion* of the complex F^*.

As was observed in [4], the following easy assertion holds:

(a) $\overline{H}^k, Z^k$ and $\overline{B}^k$ are Banach spaces;
(b) The natural (quotient) topology on $T^k := \overline{B}^k/B^k$ is coarse (any closed set is either empty or T^k);
(c) there is a natural exact sequence

$$0 \to T^k \to H^k \to \overline{H}^k \to 0.$$

Lemma 2.3. [4, Lemma 4.4] *For any Banach complex $\{F^k, d\}$, the following are equivalent:*

(i) $T^k = 0$*;*
(ii) $\dim T_k < \infty$*;*
(iii) H^k *is a Banach space;*
(iv) $B^k \subset F^k$ *is closed.*

Lemma 2.4. [4, Proposition 4.5] *The following are equivalent:*

(i) $H^k = 0$;

(ii) *The operator* $d_{k-1} : F^{k-1}/Z^{k-1} \to Z^k$ *admits a bounded inverse* d_{k-1}^{-1};

(iii) *There exists a constant* C_k *such that if for any* $\theta \in Z^k$ *there is an element* $\eta \in F^{k-1}$ *with* $d\eta = \theta$ *and*

$$\|\eta\|_{F^{k-1}} \le C_k \|\theta\|_{F^k}.$$

Lemma 2.5. [4, Propositions 4.6 and 4.7] *The following conditions (i) and (ii) are equivalent:*

(i) $T^k = 0$.

(ii) *The operator* $d_{k-1} : F^{k-1}/Z^{k-1} \to B^k$ *admits a bounded inverse* d_{k-1}^{-1}.

Any of these conditions implies

(iii) *There exists a constant* C_k' *such that for any* $\xi \in F^{k-1}$ *there is an element* $\zeta \in Z^{k-1}$ *such that*

$$\|\xi - \zeta\|_{F^{k-1}} \le C_k' \|d\xi\|_{F^k}. \tag{2.1}$$

Moreover, if F^{k-1} *is a reflexive Banach space then conditions (i)-(iii) are equivalent.*

3. Orlicz Spaces of Differential Forms and Orlicz Cohomology

Let X be a Riemannian manifold of dimension n. Given $x \in X$, denote by $(\omega(x), \theta(x))$ the scalar product of exterior k-forms $\omega(x)$ and $\theta(x)$ on T_xX. This gives a function $x \mapsto (\omega(x), \theta(x))$ on X.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \mathbb{R} \to \mathbb{R}$ be two complementary N-functions. Given a measurable k-form ω, we put

$$\rho_\Phi(\omega) := \int_X \Phi(|\omega(x)|) d\mu_X.$$

Here $d\mu_X$ stands for the volume element of the Riemannian manifold X. We will identify k-forms differing on a set of measure zero.

Given a (not necessarily orientable) Riemannian manifold X, introduce the space $L^\Phi(X, \Lambda^k)$ as the class of all measurable k-forms ω satisfying the condition

$$\rho_\Phi(\alpha\omega) < \infty \text{ for some } \alpha > 0.$$

As in the case of Orlicz function spaces, the space $L^\Phi(X, \Lambda^k)$ is endowed with two equivalent norms: the *gauge norm*

$$\|\omega\|_{(\Phi)} = \inf\left\{K > 0 \,:\, \rho_\Phi\left(\frac{\omega}{K}\right) \le 1\right\}.$$

and the *Orlicz norm* (Ψ is the complementary N-function to Φ):

$$\|\omega\|_\Phi = \sup\left\{\left|\int_X (\omega(x), \theta(x))\, d\mu_X\right| : \rho_\Psi(\theta) \le 1\right\}$$

As in the case of function spaces, it can be proved that $L^\Phi(X, \Lambda^k)$ endowed with one of these norms is a Banach space.

Obviously, the gauge norm of a k-form ω is nothing but the gauge norm of its modulus function $|\omega|$. The same holds for the Orlicz norm ([7, Lemma 2.1]).

Unless otherwise specified, we endow the L^{Φ} spaces with the gauge norms; the quotient (semi)norm on each of the cohomology spaces to be defined below depends on the choice of the norms on L^{Φ_I} and $L^{\Phi_{II}}$ but the resulting topology does not.

Definition 3.1. A form $\theta \in L^{j+1}_{1,\mathrm{loc}}(X)$ is called the (*weak*) *differential* $d\omega$ of $\omega \in L^{j}_{1,\mathrm{loc}}(X)$ if

$$\int\limits_U \omega \wedge du = (-1)^{j+1} \int\limits_U \theta \wedge u$$

for every orientable domain $U \subset \mathit{Int}\, X$ and every form $u \in D^{n-j-1}(X)$ having support in U.

Let Φ_I and Φ_{II} be N-functions. For $0 \le k \le n$, put

$$\Omega^k_{\Phi_I,\Phi_{II}}(X) = \left\{\omega \in L^{\Phi_I}(X, \Lambda^k) \,:\, d\omega \in L^{\Phi_{II}}(X, \Lambda^{k+1})\right\}.$$

This is a Banach space with the norm

$$\|\omega\|_{(\Phi_I),(\Phi_{II})} = \|\omega\|_{(\Phi_I)} + \|d\omega\|_{(\Phi_{II})}.$$

Consider also the spaces

$$Z^k_{\Phi_{II}}(X) = \{\omega \in L^{\Phi_{II}}(X, \Lambda^k) \,:\, d\omega = 0\};$$

$$B^k_{\Phi_I,\Phi_{II}}(X) = \{\omega \in L^{\Phi_{II}}(X, \Lambda^k) \,:\, \omega = d\beta \text{ for some } \beta \in L^{\Phi_I}(X, \Lambda^{k-1})\}.$$

Denote by $\overline{B}^k_{\Phi_I,\Phi_{II}}(X)$ the closure of $B^k_{\Phi_I,\Phi_{II}}(X)$ in $L^{\Phi_{II}}(X, \Lambda^k)$.

Definition 3.2. The quotient spaces

$$H^k_{\Phi_I,\Phi_{II}}(X) := Z^k_{\Phi_{II}}(X)/B^k_{\Phi_I,\Phi_{II}}(X)$$

and

$$\overline{H}^k_{\Phi_I,\Phi_{II}}(X) := Z^k_{\Phi_{II}}(X)/\overline{B}^k_{\Phi_I,\Phi_{II}}(X)$$

are called the kth $L_{\Phi_I,\Phi_{II}}$-*cohomology* and the kth *reduced* $L_{\Phi_I,\Phi_{II}}$-*cohomology* of the Riemannian manifold X, the latter cohomology being a Banach space. Define the $L_{\Phi_I,\Phi_{II}}$-*torsion* as

$$T^k_{\Phi_I,\Phi_{II}}(X) := \overline{B}^k_{\Phi_I,\Phi_{II}}(X)\,/B^k_{\Phi_I,\Phi_{II}}(X).$$

The torsion $T^k_{\Phi_I,\Phi_{II}}(X)$ can be either $\{0\}$ or infinite-dimensional. In fact, if $\dim T^k_{\Phi_I,\Phi_{II}}(X) < \infty$ then $B^k_{\Phi_I,\Phi_{II}}(X)$ is closed, hence $T^k_{\Phi_I,\Phi_{II}}(X) = \{0\}$. In particular, if $\dim T^k_{\Phi_I,\Phi_{II}}(X) \neq 0$ then $\dim H^k_{\Phi_I,\Phi_{II}}(X) = \infty$.

If $\Phi_I = \Phi_{II} = \Phi$ then we use the notations $\Omega^k_{\Phi}(X)$, $H^k_{\Phi}(X)$, and $\overline{H}^k_{\Phi}(X)$ instead of $\Omega^k_{\Phi,\Phi}(X)$, $H^k_{\Phi,\Phi}(X)$, and $\overline{H}^k_{\Phi,\Phi}(X)$ respectively. Thus, the L_{Φ}-*cohomology* $H^k_{\Phi}(X)$ (respectively, the *reduced* L_{Φ}-*cohomology* $\overline{H}^k_{\Phi}(X)$) is the kth cohomology (respectively, the kth reduced cohomology) of the cochain complex $\{\Omega^*_{\Phi}(X), d\}$.

In [4], Gol′dshtein and Troyanov realized the kth $L_{q,p}$-cohomology as the kth cohomology of some Banach complex. Here we apply this approach to $L_{\Phi_I,\Phi_{II}}$-cohomology.

Fix an $(n+1)$-tuple of N-functions $\mathcal{F} = \{\Phi_0, \Phi_1, \dots, \Phi_n\}$ and put

$$\Omega^k_{\mathcal{F}}(X) = \Omega^k_{\Phi_k,\Phi_{k+1}}(X);$$

Since the weak exterior differential is a bounded operator $d : \Omega^k_{\mathcal{F}}(X) \to \Omega^{k+1}_{\mathcal{F}}(X)$, we obtain a Banach complex

$$0 \to \Omega^0_{\mathcal{F}}(X) \to \Omega^1_{\mathcal{F}}(X) \to \cdots \to \Omega^k_{\mathcal{F}}(X) \to \cdots \to \Omega^n_{\mathcal{F}}(X) \to 0.$$

The $L_{\mathcal{F}}$*-cohomology* $H^k_{\mathcal{F}}(X)$ (respectively, the *reduced* $L_{\mathcal{F}}$*-cohomology* $\overline{H}^k_{\mathcal{F}}(X)$) of X is the kth cohomology (respectively, the kth reduced cohomology) of the Banach complex $(\Omega^*_{\mathcal{F}}, d)$.

The above-defined cohomology spaces $H^k_{\mathcal{F}}(X)$ and $\overline{H}^k_{\mathcal{F}}(X)$ in fact depend only on Φ_{k-1} and Φ_k:

$$H^k_{\mathcal{F}}(X) = H^k_{\Phi_{k-1},\Phi_k}(X) = Z^k_{\Phi_k}(X) \Big/ B^k_{\Phi_{k-1},\Phi_k} ;$$
$$\overline{H}^k_{\mathcal{F}}(X) = \overline{H}^k_{\Phi_{k-1},\Phi_k}(X) = Z^k_{\Phi_k}(X) \Big/ \overline{B}^k_{\Phi_{k-1},\Phi_k} .$$

The results on abstract Banach complexes by Gol′dhstein and Troyanov enable us to interprete Orlicz cohomology in terms of a Poincaré–Sobolev–Orlicz type inequality for differential forms on a Riemannian manifold X:

Theorem 3.3. $H^k_{\Phi_I,\Phi_{II}}(X) = 0$ *if and only if there exists a constant* $C < \infty$ *such that for any closed differential form* $\omega \in L^{\Phi_{II}}(X, \Lambda^k)$ *there exists a differential form* $\theta \in L^{\Phi_I}(X, \Lambda^{k-1})$ *such that* $d\theta = \omega$ *and*

$$\|\theta\|_{L^{(\Phi_I)}} \le C\|\omega\|_{L^{(\Phi_{II})}}.$$

This result is an immediate consequence of Lemma 2.4.

Theorem 3.4. *(A) If* $T^k_{\Phi_{II}}(X) = 0$ *then there exists a constant* C' *such that for any differential form* $\theta \in \Omega^{k-1}_{\Phi_I,\Phi_{II}}(X)$ *there exists a closed form* $\zeta \in Z^{k-1}_{\Phi_I}(X)$ *such that*

$$\|\theta - \zeta\|_{L^{(\Phi_I)}} \le C'\|d\theta\|_{L^{(\Phi_{II})}}. \tag{3.1}$$

(B) Conversely, if $\Phi_{II} \in \Delta_2 \cap \nabla_2$ *and there exists a constant* C' *such that for any form* $\theta \in \Omega^{k-1}_{\Phi_I,\Phi_{II}}(X)$ *there exists* $\zeta \in Z^{k-1}_{\Phi_I}(X)$ *such that (3.1) holds then* $T_{\Phi_I,\Phi_{II}}(X) = 0$.

Proof. Considering the Banach complex $\Omega^*_{\mathcal{F}}$ with $\mathcal{F} = \{\Phi_{II}, \dots, \Phi_{II}, \Phi_I, \dots, \Phi_I\}$, where Φ_{II} changes to Φ_I at the kth position, we get

$$H^k_{\mathcal{F}}(X) = H^k_{\Phi_I,\Phi_{II}}(X); \quad \overline{H}^k_{\mathcal{F}}(X) = \overline{H}^k_{\Phi_I,\Phi_{II}}(X).$$

Since $\Phi_I \in \Delta_2 \cap \nabla_2$, the Banach space $\Omega^{k-1}_{\Phi_{II},\Phi_{II}}(X)$ is reflexive. Theorem 3.4 now stems from Lemma 2.5. □

4. The L_{Φ_1,Φ_2}-Cohomology of $\mathbb{R}$

Let Φ_1 and Φ_2 be N-functions.

Proposition 4.1. $T^1_{\Phi_1,\Phi_2}(\mathbb{R}) \ne 0$.

Proof. Suppose on the contrary that $T^1_{\Phi_1,\Phi_2}(\mathbb{R}) = 0$. In accordance with Theorem 3.4, then there is a Sobolev inequality for functions on $\mathbb{R}$

$$\inf_{z\in\mathbb{R}} \|f - z\|_{(\Phi_1)} \le C\|f'\|_{(\Phi_2)} \tag{4.1}$$

for some real positive constant C.

Consider the function

$$\theta(x) = \omega_{1/2}(x) = \begin{cases} Ce^{-\frac{1}{1-4x^2}} & \text{if } |x| \le 1/2, \\ 0 & \text{if } |x| > 1/2. \end{cases}$$

Here the constant C is chosen so that

$$\int_{-\infty}^{\infty} \theta(x)\, dx = \frac{c}{2} \int_{-1}^{1} e^{-\frac{1}{1-t^2}}\, dt = 1.$$

Now, consider the family of smooth functions with compact support $\{f_a : \mathbb{R} \to \mathbb{R} : a > 1\}$, where

$$f_a(x) = \int_{-\infty}^{x} \left(\theta\left(x + \frac{3}{2}\right) + \theta\left(-x + a + \frac{1}{2}\right)\right) dx$$

(we owe this construction to [2, pp. 8–9]). Then $f_a(x) = 1$ if $x \in [1, a]$, $f_a(x) = 0$ if $x \notin [0, a+1]$, and $\|f_a'\|_{L^\infty} =: L < \infty$. Clearly, $\|f_a - z\|_{(\Phi_1)}$ is finite only for $z = 0$. Estimate the Orlicz norms involved in (4.1). We have

$$\rho_{\Phi_1}\left(\frac{f_a}{K}\right) = \int_{-\infty}^{\infty} \Phi_1\left(\frac{f_a(x)}{K}\right) dx \ge \int_1^a \Phi_1\left(\frac{1}{K}\right) dx = (a-1)\Phi_1\left(\frac{1}{K}\right).$$

If $\rho_{\Phi_1}\left(\frac{f_a}{K}\right) \le 1$ then $(a-1)\Phi_1\left(\frac{1}{K}\right) \le 1$, which is equivalent to

$$K \ge \frac{1}{\Phi_1^{-1}\left(\frac{1}{a-1}\right)}.$$

Hence,

$$\|f_a\|_{(\Phi)} = \inf\left\{K : \rho_{\Phi_1}\left(\frac{f_a}{K}\right) \le 1\right\} \ge \frac{1}{\Phi_1^{-1}\left(\frac{1}{a-1}\right)}.$$

On the other hand,

$$\begin{aligned} \rho_{\Phi_2}\left(\frac{f_a'}{K}\right) &= \int_{-\infty}^{\infty} \Phi_2\left(\frac{f_a'(x)}{K}\right) dx \\ &= \int_0^1 \Phi_2\left(\frac{f_a'(x)}{K}\right) dx + \int_a^{a+1} \Phi_2\left(\frac{f_a'(x)}{K}\right) dx \le 2\Phi_2\left(\frac{L}{K}\right). \end{aligned}$$

We have

$$2\Phi_2\left(\frac{L}{K}\right) \le 1 \iff K \ge \frac{L}{\Phi_2^{-1}\left(\frac{1}{2}\right)}.$$

Put $\mathcal{M}_{f',a} = \left\{K : \rho_{\Phi_2}\left(\frac{f_a'}{K}\right) \le 1\right\}$. We have shown that if $K \ge \frac{L}{\Phi_2^{-1}\left(\frac{1}{2}\right)}$ then $K \in \mathcal{M}_{f',a}$. Therefore,

$$\|f_a'\|_{(\Phi_2)} = \inf \mathcal{M}_{f',a} \le \frac{L}{\Phi_2^{-1}\left(\frac{1}{2}\right)}.$$

Thus,

$$C \ge \frac{\Phi_2^{-1}\left(\frac{1}{2}\right)}{L\Phi_1^{-1}\left(\frac{1}{a-1}\right)} \to \infty \quad \text{as } a \to \infty.$$

The obtained contradiction proves the proposition. $\square$

Corollary 4.2. *If Φ_1 and Φ_2 are N-functions then the space $H^1_{\Phi_1,\Phi_2}(\mathbb{R})$ is not separated; in particular, $H^1_{\Phi_1,\Phi_2}(\mathbb{R}) \neq 0$.*

Proposition 4.3. *If Φ_1 and Φ_2 are N-functions and $\Phi_2 \in \Delta_2$ then $\overline{H}^1_{\Phi_1,\Phi_2}(\mathbb{R}) = 0$.*

Proof. Let $\omega = a(x)dx \in L^{\Phi_2}(\mathbb{R})$. For each n, put

$$C_m = \int_{-m}^{m} a(x)\, dx.$$

If $C_m = 0$ then put $\lambda_m(x) \equiv 0$ for all $x \in \mathbb{R}$. If $C_m \neq 0$ then put

$$\lambda_m(x) = \operatorname{sign} C_m \varepsilon_m\, \chi\left[-\frac{|C_m|}{2\varepsilon_m}, \frac{|C_m|}{2\varepsilon_m}\right],$$

where $\varepsilon_m = t_m/m$ and t_m is the only root of the equation

$$\frac{\Phi_2(t_m)}{t_m} = \frac{1}{m|C_m|}.$$

(The function $t \mapsto \Phi_2(t)/t$ is strictly increasing; see, for example, [9]). We obviously have

$$\int_{\mathbb{R}} \lambda_m(x)\, dx = C_m = \int_{-m}^{m} a(x)\, dx.$$

Compute the norm $\|\lambda_m\|_{(\Phi_2)}$. We have

$$\rho_{\Phi_2}\left(\frac{\lambda_m}{K}\right) = \int_{-|C_m|/2\varepsilon_m}^{|C_m|/2\varepsilon_m} \Phi_2\left(\frac{\varepsilon_m}{K}\right) dx = \frac{|C_m|}{\varepsilon_m}\Phi_2\left(\frac{\varepsilon_m}{K}\right).$$

Thus,

$$\rho_{\Phi_2}\left(\frac{\lambda_m}{K}\right) \le 1 \Longleftrightarrow \frac{|C_m|}{\varepsilon_m}\Phi_2\left(\frac{\varepsilon_m}{K}\right) \le 1 \Longleftrightarrow \Phi_2\left(\frac{\varepsilon_m}{K}\right) \le \frac{\varepsilon_m}{|C_m|}$$
$$\Longleftrightarrow \frac{\varepsilon_m}{K} \le \Phi_2^{-1}\left(\frac{\varepsilon_m}{|C_m|}\right) \Longleftrightarrow K \ge \frac{\varepsilon_m}{\Phi_2^{-1}\left(\frac{\varepsilon_m}{|C_m|}\right)}.$$

Here Φ_2^{-1} stands for the positive inverse function to $\Phi_2 : [0,\infty) \to [0,\infty)$. Hence, $\|\lambda_m\|_{(\Phi_2)} = \frac{\varepsilon_m}{\Phi_2^{-1}(\varepsilon_m/|C_m|)}$. By the choice of ε_m,

$$\frac{\Phi_2(m\varepsilon_m)}{m\varepsilon_m} = \frac{1}{m|C_m|},$$

and so $\|\lambda_m\|_{(\Phi_2)} = \frac{1}{m}$.

Let $b_m(x) := \int_{-\infty}^{x} \left(\chi_{[-m,m]}(t)a(t) - \lambda_m(t)\right) dt$. Since b_m has compact support, $b_m \in L^{\Phi_1}(\mathbb{R})$ for each m. Furthermore, $\|db_m - \omega\|_{(\Phi_2)} \le \|a\|_{L^{(\Phi_2)}(\mathbb{R}\setminus[-m,m])} + \|\lambda_m\|_{L^{(\Phi_2)}(\mathbb{R})} \to 0$ as $m \to \infty$ since for $\Phi_2 \in \Delta_2$ all functions in L^{Φ_2} have absolutely continuous norm ([9, Theorem 10.3]). Thus, $\overline{H}^1_{\Phi_1,\Phi_2}(\mathbb{R}) = 0$. □

All the results of this section are also valid for the half-line $\mathbb{R}_+$ (with similar proofs).

5. The L_{Φ_1,Φ_2}-Cohomology of the Hyperbolic Plane

We will need the following Orlicz versions of Propositions 8.3 and 8.4 in [4], which are proved in absolutely the same manner:

Proposition 5.1. *Let M be a complete manifold of dimension n and let (Φ_1, Ψ_1) and (Φ_2, Ψ_2) be two pairs of complementary Orlicz functions. Suppose that $\alpha \in Z^k_{\Phi_2}(X)$ and there exists a smooth closed $(n-k)$-form γ such that $\gamma \in Z^{n-k}_{\Psi_1}(X)$, $\gamma \wedge \alpha \in L^1(X, \Lambda^n)$, and*

$$\int_M \gamma \wedge \alpha \neq 0,$$

then $\alpha \notin B^k_{\Phi_1,\Phi_2}(X)$. In particular, $H^k_{\Phi_1,\Phi_2}(X) \neq 0$.

Proposition 5.2. *Let M be a complete manifold of dimension n and let (Φ_1, Ψ_1) and (Φ_2, Ψ_2) be two pairs of complementary Orlicz functions. Suppose that $\alpha \in Z^k_{\Phi_2}(X)$ and there exists a smooth closed $(n-k)$-form $\gamma \in Z^{n-k}_{\Psi_1}(X) \cap Z^{n-k}_{\Psi_2}(X)$ such that*

$$\int_M \gamma \wedge \alpha \neq 0,$$

then $\alpha \notin \overline{B}^k_{\Phi_1,\Phi_2}(X)$. In particular, $\overline{H}^k_{\Phi_1,\Phi_2}(X) \neq 0$.

The hyperbolic plane $\mathbb{H}^2$ is the Riemannian manifold that can be modelled as the space $\mathbb{R}^2$ endowed with the Riemannian metric

$$ds^2 = e^{2z} dy^2 + dz^2.$$

For an N-function Φ, introduce the condition

$$\int_0^1 \frac{\Phi(v)}{v^2}\, dv < \infty. \tag{A}$$

(The upper integration limit 1 can be replaced by any positive number.)

Theorem 5.3. *If Φ_1 and Φ_2 are N-functions such that their complementary N-functions Ψ_1 and Ψ_2 and the function Φ_2 satisfy condition (A) then*

$$\dim(\bar{H}^1_{\Phi_1,\Phi_2}(\mathbb{H}^2)) = \infty\,.$$

We will need the following lemma, which is in fact Lemma 10.2 in [4]:

Lemma 5.4. *There exist two smooth functions f and g on $\mathbb{H}^2$ such that*

(1) *f and g are nonnegative;*
(2) *$f(y,z) = g(y,z) = 0$ if $z \leq 0$ or $|y| \geq 1$;*
(3) *df and $dg \in L^r(\mathbb{H}^2, \Lambda^1)$ for any $1 < r \leq \infty$;*
(4) *the support of $df \wedge dg$ is contained in $\{(y,z) : |y| \leq 1\,,\, 0 \leq z \leq 1\}$;*
(5) *$df \wedge dg \geq 0$;*
(6) *$\int_{\mathbb{H}^2} df \wedge dg = 1$;*
(7) *$\frac{\partial f}{\partial y}$ and $\frac{\partial g}{\partial y} \in L^\infty(\mathbb{H}^2)$;*
(8) *$\frac{\partial f}{\partial z}$ and $\frac{\partial g}{\partial z}$ have compact support.*

We will also need the following generalization of item (3) above:

Lemma 5.5. *If Φ is an N-function satisfying condition (A) then df, $dg \in L^\Phi(\mathbb{H}^2, \Lambda^1)$.*

Proof. Recall the construction of the functions f and g of [4, Lemma 10.2].

Choose smooth functions h_1, h_2, and $k : \mathbb{R} \to \mathbb{R}$ with the following properties:

(1) h_1, h_2, and k are nonnegative;

(2) $h_i(y) = 0$ if $|y| \geq 1$;

(3) $h_1'(y)h_2(y) \geq 0$ and $h_1(y)h_2'(y) \leq 0$ for all y;

(4) the support of the function $h_1'(y)h_2(y) - h_1(y)h_2'(y)$ is not empty;

(5) $k'(z) \geq 0$ for all z;

(6) $k(z) = 1$ if $z \geq 1$ and $k(z) = 0$ if $z \leq 0$.

Then f and g are defined as $f(y,z) := h_1(y)k(z)$ and $g(y,z) := h_2(y)k(z)$ respectively.

We will now prove that $df \in L^\Phi$ by modifying the argument of the proof of [4, Lemma 10.2].

Indeed,

$$df = h_1(y)k'(z)dz + k(z)h_1'(y)dy.$$

The first summand $h_1(y)k'(z)dz$ has compact support, and the second summand $k(z)h_1'(y)dy$ is zero outside the infinite rectangle $Q = \{|y| \leq 1\,; z \geq 0\}$.

Choose $D < \infty$ such that $|k(z)h_1'(y)| \leq D$ on Q. We have

$$|k(z)h_1'(y)dy| \leq D\,|dy| = D\,e^{-z}.$$

Since the area element of $\mathbb{H}^2$ is $dA = e^z dydz$, for any $a > 0$ we infer

$$\int_{\mathbb{H}^2} \Phi(a|k(z)h_1'(y)dy|)dA \leq \int_Q \Phi(aDe^{-z})e^z\,dy\,dz = 2aD\int_0^\infty \frac{\Phi(aDe^{-z})}{aDe^{-z}}\,dz.$$

Putting $aDe^{-z} = v$ in the last integral, we get

$$2aD\int_0^\infty \frac{\Phi(aDe^{-z})}{aDe^{-z}}\,dz = 2aD\int_0^{aD} \frac{\Phi(v)}{v^2}\,dv < \infty.$$

Thus, $\rho_\Phi(ak(z)h_1'(y)\,dy) < \infty$ for any a. Consequently, $k(z)h_1'v \in L^\Phi(\mathbb{H}^2, \Lambda^1)$. Thus, $df = h_1(y)k'(z)dz + k(z)h_1'(y)dy$ also lies in $L^\Phi(\mathbb{H}^2, \Lambda^k)$.

The lemma is proved. □

Proof of Theorem 5.3. Take the functions f and g on $\mathbb{H}^2$ defined in Lemma 5.4 and consider the 1-forms $\alpha = df$ and $\gamma = dg$ on $\mathbb{H}^2$. Obviously, $d\alpha = d\gamma = 0$. By Lemmas 5.4 and 5.5, $\alpha \in L^\Phi$ for any N-function Φ such that $\int_0^1 \Phi(v)/v^2\,dv < \infty$ and γ is smooth and $\gamma \in L^{\Psi_1} \cap L^{\Psi_2}$ if $\int_0^1 \Psi_1(v)/v^2\,dv < \infty$ and $\int_0^1 \Psi_2(v)/v^2\,dv < \infty$.

Since $\int_{\mathbb{H}^2} \alpha \wedge \gamma \neq 0$, Proposition 5.1 shows that $\alpha \notin \overline{B}^1_{\Phi_1,\Phi_2}(\mathbb{H}^2)$.

Now, using the isometry group of $\mathbb{H}^2$, we obtain an infinite family of linearly independent classes in $\overline{H}^1_{\Phi_1,\Phi_2}(\mathbb{H}^2)$. □

6. The L_Φ-Cohomology of the Ball

In this section, we prove the "L^Φ-Poincaré lemma", i.e., the vanishing of the L^Φ-cohomology of the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$.

Since $\mathbb{B}^n$ has finite volume, $H^0_{\Phi_1,\Phi_2}(\mathbb{B}^n) = \overline{H}^0_{\Phi_1,\Phi_2}(\mathbb{B}^n) = \mathbb{R}$ for any N-functions Φ_1 and Φ_2.

For the case of L^p spaces, Gol′dshtein, Kuz′minov, and Shvedov proved the vanishing of the L^p-cohomology of the ball in [3, Lemma 3.2]; for $p \neq q$, Gol′dshtein and Troyanov found necessary and sufficient conditions on p and q for the vanishing of the $L^{q,p}$-cohomology of $\mathbb{B}^n$. Their proof is based on the following fact, established by Iwaniec and Lutoborski in [5]:

Proposition 6.1. *For any bounded convex domain $U \subset \mathbb{R}^n$ and any $k = 1, 2, \dots, n$, there exists an operator*

$$T = T_U : L^1_{loc}(U, \Lambda^k) \to L^1_{loc}(U, \Lambda^{k-1})$$

with the following properties:

(a) $T(d\theta) + dT\theta = \theta$ (in the sense of currents);

(b) $|T\theta(x)| \le C \int_U \frac{|\theta(y)|}{|y-x|^{n-1}} dy.$

□

We prove

Corollary 6.2. *If Φ is an N-function then the operator T maps $L^\Phi(U, \Lambda^k)$ continuously into $L^\Phi(U, \Lambda^{k-1})$.*

Proof. The following Orlicz space version of Young's inequality for convolution holds (see the proof of Corollary 7 in [10, pp. 230–231][1]: If $f \in L^\Phi$ and $g \in L^1$ then $f * g \in L^\Phi$ and

$$\|f * g\|_\Phi \le \|f\|_\Phi \|g\|_1.$$

Applying this inequality to $f = |\theta|$ and $g(x) = |x|^{1-n}$, we obtain the corollary from Proposition 6.1. In the Orlicz norms, the norm of the operator T is bounded by $\|g\|_1$. □

Corollary 6.3. *The operator $T : \Omega_\Phi(U, \Lambda^k) \to \Omega_\Phi(U, \Lambda^{k-1})$ is bounded and $Td\omega + dT\omega = \omega$ for any $\omega \in \Omega^k_\Phi(U)$.*

Corollary 6.3 gives the following

Theorem 6.4. *If Φ is an N-function then $H^k_\Phi(\mathbb{B}^n) = 0$ for all $k = 1, ..., n$.*

Proof. Let $\omega \in Z^k_\Phi(\mathbb{B}^n)$. By Corollary 6.3, $T\omega \in L^\Phi(\mathbb{B}^n, \Lambda^{k+1})$. Since $\omega = dT\omega + Td\omega = d(T\omega)$, we conclude that $[\omega] = [dT\omega] = 0 \in H^k_\Phi(\mathbb{B}^n)$ and so $H^k_\Phi(\mathbb{B}^n) = 0$. □

Remark 6.5. The Sobolev space analog of Theorem 6.4, Theorem 11.5 in [4] (see also [4, Proposition 11.4]) gives the following criterion: *for $1 < p, q \le \infty$, the $L_{q,p}$-cohomology spaces $H^k_{q,p}(\mathbb{B}^n)$, $k = 1, 2, \dots, n$ are trivial if and only if $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$.* It would be interesting to obtain a criterion like [4, Theorem 11.5] for the Orlicz cohomology $H^k_{\Phi_1,\Phi_2}(\mathbb{B}^n)$ for different functions Φ_1 and Φ_2.

Acknowledgments. The authors are grateful to the referee for useful remarks and comments.

References

[1] R. P. Agarwal, S. Ding, and C. A. Nolder, *Inequalities for Differential Forms*, Berlin: Springer, 2009. Zbl 1184.53001

[2] P. G. Dolya, *Mathematical Methods of Computer Tomography. Supplement I. Introduction to the Theory of Distributions*, Kharkiv National University, Kharkiv, 2012 (http://geometry.karazin.ua/resources/documents/20140424194043_9a57d424.pdf).

[1]Though it is required in Corollary 7 in [10, pp. 230–231] that $\Phi \in \Delta_2$, the proof of Young's inequality works for general N-functions Φ.

[3] V. M. Gol′dshtein, V. I. Kuz′minov, and I. A. Shvedov, *Differential forms on Lipschitz manifolds*, Siberian Math. J., **23**:2 (1982), 151–161. Zbl 0522.58001

[4] V. Gol′dshtein V. and M. Troyanov, *Sobolev inequalities for differential forms and $L_{q,p}$-cohomology*, J. Geom. Anal., **16**:4 (2006), 597–632. Zbl 1105.58008

[5] T. Iwaniec and A. Lutoborski, *Integral estimates for null Lagrangians*, Arch. Rational Mech. Anal., **125**:1 (1993), 25–79. Zbl 0793.58002

[6] T. Iwaniec and G. Martin, *Geometric Function Theory and Nonlinear Analysis*, Oxford: Oxford University Press, 2001. Zbl 1045.30011

[7] Ya. A. Kopylov and R. A. Panenko, *De Rham regularization operators in Orlicz spaces of differential forms on Riemannian manifolds*, Sib. Élektron. Mat. Izv., **12** (2015), 361–371. Zbl 1408.58003

[8] Ya. A. Kopylov, *Orlicz spaces of differential forms on Riemannian manifolds: duality and cohomology*, Probl. Anal. Issues Anal., **6 (24)**:2 (2017), 57–80. Zbl 1394.58002

[9] M. A. Krasnosel′skiĭ and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Groningen: P. Noordhoff Ltd, 1961. Zbl 0095.09103

[10] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces. Pure and Applied Mathematics,* 146, New York etc.: Marcel Dekker, 1991. Zbl 0724.46032

Vladimir Gol′dshtein
Department of Mathematics,
Ben Gurion University of the Negev,
Beer Sheva, P.O.Box 653, Israel
E-mail address: vladimir@bgumail.bgu.ac.il

Yaroslav Anatol′evich Kopylov
Sobolev Institute of Mathematics,
4, Koptyuga ave.,
Novosibirsk, 630090, Russia
Novosibirsk State University,
1, Pirogova str.,
Novosibirsk, 630090, Russia
E-mail address: yakop@math.nsc.ru