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BRANCHING TIME AGENTS' LOGIC, SATISFIABILITY PROBLEM BY RULES IN REDUCED FORM

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Abstract. This paper considers the branching time logic on non–transitive intervals of agents' accessibility relations. The agents' accessibility relations are defined inside transitivity intervals and via neighboring limit points, they may be not complete and lose some states — the lacunas of forgotten time thought they may interfere. This approach is used for modeling computational processes and analysis of incomplete information for individual agents. A logical language for reasoning about models' properties which includes temporal and modal logical operations is suggested. Illustrative examples are provided. Mathematical part of the paper is devoted to the satisfiability and decidability problems for the suggested logic. We use instruments of reduced normal forms for rules and algorithms converting rules to such forms. We find algorithms solving the satisfiability problem. Some open problems are suggested.

Keywords: temporal logic, branching time logic, multi–agent logic, computability, information, satisfiability, decidability.

1. Introduction

Non-classical mathematical logic mostly deals with modal and constructive logic (and its neighbors such as, e.g., Johnson logics and Superintuitionistic logics), many valued logics such as Łukasiewicz logics, etc. Temporal logic is, in a sense, a natural generalization of modal logic when 'possible' is directed to the future and to the past. Historically, multi–valued logics aimed at the representing of the truth relations for the boolean logic, which is the basic logical language. That may be

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dated to Łukasiewicz (1917) and his three–valued and many–valued propositional calculi, as well as to Goedel (1932), who refuted the finite–validness of intuitionistic logic. In their pioneering works, A. Tarski (1951) and S. Kripke (1960th) suggested semantical models for the studies of modal and temporal logics such as topological boolean algebras and relational models (Kripke–Hinttikka models); these models are multi–valued by their nature. Since then non–classical mathematical logic had a long and fruitful history.

We study here a sort of temporal logic. Temporal logic has many strong achieved mathematical results and various applications in Information sciences and CS. In a sense, in a definite form it was introduced by Arthur Prior in the late 1950s. Nowadays, this logic is very popular, highly technical, and fruitful area (cf. e. g. Gabbay and Hodkinson [8, 9, 10]) with various particular areas of applications in CS and in the AI, as semantic web etc. This logic, modal logics, and close multi-agent logics nowadays are used for the verification of correct behavior of computational processes, the verification of correct representation of information and knowledge, etc. (cf. for example Wooldridge et al [28, 29, 30], Lomuscio et al [12, 3], Balbiani and Vakarelov [4], Vakarelov [27]). Concerning the multi–agency, the technique of mathematical logic translated for description logics is useful for the study of otologies, e. g., F. Baader et al [1], for that purpose — the study of otologies — many techniques were applied – from modal–like logics to automatons (cf. eg. [31]).

Our own earlier works also considered some aspects of multi–agency, e.g. the multi–agent logic with distances, the satisfiability problem for it (Rybakov et al [19]), and the models for the conception of Chance Discovery in multi–agent environment (Rybakov [20, 22]). A logic modeling uncertainty via agents' views was also investigated (cf. McLean, Rybakov [14]); the study of the conception of knowledge from the viewpoint of multi–agency based on temporal logic is contained in the works by Rybakov [15, 17, 18]. From the technical point of view, perhaps the very first approach to multi–valued modal logics (when different valuations are taken on algebraic lattices) may be found in the works by M. Fitting [6, 7]; the multi–valued approaches were also used in a such popular area as the model checking (cf. e. g. G. Bruns, P. Godefroid [5]).

Recently we have turned to the case of non–transitive linear temporal logic and its variations (in particular — to multi–agents' versions, versions with multi–valuations and with lacunas in agents' accessibility relations), cf. Rybakov [23, 24, 25, 26].

Earlier, some extensions of the linear time logic LTL with abolishing linearity of the time where investigated, in particular, the branching time (transitive) temporal logic — the full branching time logic (CTL*) (with basic modalities consisting of a path quantifier, either A ("for all paths") of E ("for some path") — was considered in several papers (cf. for the origin, e.g. Emerson et al. [11])

This our paper studies the branching time logic on non–transitive intervals with different agents' accessibility relations. So, in a sense it is an interval logic where the agents' accessibility relations are defined inside transitivity intervals and via neighboring limit points. The innovative points which distinguish this our research from others are that (1) the time is not transitive and branching, (2) the agents' accessibility relations (for distinct agents) and corresponding logical operations are embedded, and (3) the agents accessibility relations may have lacunas, sets of forgotten time. Illustrative examples are provided. We solve the satisfiability and decidability problems for the suggested logic. For this purpose we use instruments of reduced normal forms for rules and algorithms converting rules to such forms. We find algorithms solving the satisfiability problem and thus we prove that the logic is decidable. In the final part of the paper, we formulate some interesting open problems.

2. Syntax and Semantics

Here we start from a formal definition of models. They, in a sense, will extend the models for the logic CTL — branching time logic. CTL models are the so called transition systems whose models are pairs $M := \langle S, \prec \rangle$ together with a labeling function for letters. Here S are states and the binary relation $\prec$ on S is a transition, which is assumed to be serial, i.e. every state has at least one successor (cf. for the origin e.g. [11]).

Actually, we will replace states with finite intervals of states modeling computational runs, sequence of reasoning steps, etc. Besides, we assume our models to be discrete and possessing various accessibility relations for agents (computational agents). For the precise definition we prefer to constructively describe the models starting from a representation of branching time.

Each our branching time model will start from a root — starting state. So, let $S_1 := \{a_{1,1}\}$. Assume S_k to be already constructed and

$$Ad(S_k) := \{a_{k,k_1}, \ldots, a_{k,k_m}\}$$

are all the states added to S_{k-1} at the previous step. Let for any $a_{k,m} \in Ad(S_k)$, $S(a_{k,m}) := \{a, \ldots, b\}$ be new states. Let $a_{k,m} \prec c$ for all $c \in S(a_{k,m})$. So, all the states from $S(a_{k,m})$ are so to say all immediate tomorrow states for $a_{k,m}$. Acting similarly for any $a_{k,m}$ we obtain S_{k+1}. Let $Sl(k+1) := \bigcup_{a_{k,m}} S(a_{k,m}))$ — this is the $k+1$–slay of S_{k+1}. Let

$$T(M) := \bigcup_{k \in N} S_k,$$

it is the specified time flow model. A path within $T(M)$ is a finite sequence of states by $\prec$. This definition reflects very exactly the idea of discrete branching time (within a computation, many paths discussions, knowledge exchange, etc.).

We extend these models in several ways: (1) we consider the case when time is (a) non–transitive (b) not potentially infinite to the future (reflecting always limited resources, (c) not uniformly limited (may have arbitrary though bounded length of possibly paths; (2) we use (a) multi–agent approach assuming that the agents have their own accessibility relations which may differ to each other and differ the general flow of time, and (b) the agents' accessibility relations may have lacunas — intervals of forgotten time.

To reflect these intensions, given an arbitrary model $T(M)$ and all finite paths — the set $Path(T(M))$ in this model — starting from the initial root state $a_{1,1}$, we chop the paths on finite accessibility intervals.

For this, we fix special states $Up(s)$ in each path AP of $T(M)$ and assume that for any state s there is a state $Up(s)$, where $s \prec Up(s)$. Let $BM(T)$ be the set of all such $Up(s)$s.

Definition 1. *For any two $Up(s1), Up(s2) \in BM(T)$ belonging to the same path, $Path(Up(s1), Up(s2)$ is the set of states leading by $\prec$ from $Up(s1)$ to $Up(s2)$.*

We will call these paths $Path(Up(s1), Up(s2))$ b–paths (bounded paths), for any $Path(Up(s1), Up(s2))$, $Up(s1)$ is its lower bound, $Up(s2)$ is its upper bound.

Notice that for any upper bound state $Up(s)$ there is only one path coming to this $Up(s)$; but it could be several paths leaving this $Up(s)$. Thus all the states of the model $T(M)$ now are placed to paths leading from one upper limit $Up(s_1)$ to another one $Up(s_2)$ or they belong to $BM(T)$. Thus

$$|T(M)| = \bigcup_{(Up(s_1),Up(s_2))} Path(Up(s_1), Up(s_2)).$$

For any $Path(Up(s_1), Up(s_2))$, $\preceq_*$ is a linear order within $Path(Up(s_1), Up(s_2))$ by reflexive order and concatenation of all $\prec$.

Definition 2. *A BT–frame is a tuple* $F := \langle T(M), R_j, j \in [1, n]\rangle$, *where any* R_j *is a liner reflexive and transitive relation on any path* $Path(Up(s_1), Up(s_2))$, *where any* R_j *is a subset of* $\preceq_*$ *and any separate* R_j *is the same on all common parts of paths.*

Definition 3. *A model* MBT *is a pair* $\langle F, V\rangle$ *where* F *is a BT-frame and* V *is a valuation of the set* P *of propositional letters in this frame, that is, for any letter* $p \in P$, $V(p) \subseteq |F|$. *Notation:* $(MBT, a) \Vdash_V p$ *iff* $a \in V(p)$.

Now we introduce logical language for our interval temporal branching time logic. It is an extension of linear temporal logic and standard CTL–logic. The language contains the language of Boolean logics, so it has a potentially infinite set of propositional letters P and Boolean logical operations $\wedge, \vee, \to, \neg$. It also has binary temporal operations U_j ("until" for each agent j, where $j \in Ag$ and Ag is a finite set of all agents), and one more unary operations "next": N. Formation rules for formulas are standard. More precisely:

Definition 4. *for any* $p \in P$, p *is a formula; if* φ *and* ψ *are formulas then* $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\neg\varphi$, $N\varphi$, *and* $\varphi U_j \psi$ *for all* $j \in Ag$ *are formulas.*

Thus, everything looks the same as for temporal logics with UNTIL and NEXT, but the difference is that here we consider the temporal logical operations referred to any individual agent and the logic assumes the time to be branching.

Given a model MBT we may extend the valuation V from letters to all formulas as follows:

Definition 5. *For any* $a \in MBT$:

$$MBT \Vdash_V \neg\varphi \quad \Leftrightarrow \quad (MBT, a) \nVdash_{V_j} \varphi;$$

$$(MBT, a) \Vdash_V (\varphi \ \wedge \ \psi) \quad \Leftrightarrow \ (((MBT, a) \Vdash_V \varphi) \wedge (MBT\,(MBT, a) \Vdash_V \psi)\,;$$

$$(MBT, a) \Vdash_V (\varphi \ \vee \ \psi) \quad \Leftrightarrow \ (((MBT, a) \Vdash_V \varphi) \vee (MBT\,(MBT, a) \Vdash_V \psi)\,;$$

$$(MBT, a) \Vdash_V (\varphi \ \to \ \psi) \quad \Leftrightarrow \ (((MBT, a) \nVdash_V \varphi) \vee (MBT\,(MBT, a) \Vdash_V \psi)\,;$$

For all formulas $\varphi U_j \psi$ *we define the truth values as follows:*

$$(MBT, a) \Vdash_V (\varphi \ U_j \ \psi) \quad \Leftrightarrow$$
$$((\exists Path(Up(s_1), Up(s_2)), a \in Path(Up(s_1), Up(s_2)) \ \&$$
$$\exists b \in Path(Up(s_1), Up(s_2)) \ \& \, (aR_jb) \wedge ((MBT, b) \Vdash_V \psi) \ \&$$
$$\forall c(aR_jcR_jb, c \neq b) \Rightarrow \ ((MBT, b) \Vdash_V \varphi)$$
$$(MBT, a) \Vdash_V N\varphi \quad \Leftrightarrow \quad \big[\exists b\,(a \ \prec \ b) \ \& \ (MBT, b) \Vdash_V \varphi\big].$$

As usual, we may definite modal operations via the temporal ones. For example "possibility" may be defined as follows:

$$\Diamond_j p := \top U_j p.$$

The operation "necessary" then has to be expressed as: $\Box_j p := \neg\ U_j \neg p$, and

$$(MBT, a) \Vdash_V \Diamond_j \varphi \;\Leftrightarrow\; \big[\exists b(aR_j b)\ \&\ (MBT, b) \Vdash_V \varphi\big];$$

$$(MBT, a) \Vdash_V \Box_j \varphi \;\Leftrightarrow\; \big[\forall b(aR_j b) \Rightarrow (MBT, b) \Vdash_V \varphi\big],$$

so the defined agents' modal operations work as is expected, in accordance with meaning of modal operations. Now we will give some examples illustrating how the chosen framework may model agents' relations including non–transitivity and possible lacunas in agents' accessibility relations.

EXAMPLES

(1) The formula $Np \wedge \neg\Diamond_1 p \wedge NN\Diamond_1 p$ says that the relation R_1 has lacunas: the next sate is not accessible by R_1 but some next state after the next one is accessible.

(2) Consider the formula $\Diamond_1 p \wedge \neg\Diamond_2 p$ that being true with respect to a valuation V says that the accessibility relation for the agent 2 has a hole (lacuna) which nonetheless has inside states accessible for agent 1.

(3) The formula $\Diamond_1\Diamond_1 p \wedge \neg\Diamond_1 p$ says that the relation R_1 is not transitive for a given state a. More exactly, the truth for p with respect to R_1 is impossible in the transitivity interval (b–path) where a is situated, but in the next b–path p is possible with respect to R_1.

(4) To illustrate the multi–agency, consider a formula $\varphi_{op} := [\Box_1 p \to \Box_2 \neg p] \wedge [\Box_2 p \to \Box_1 \neg p]$. It says that both these agents are totally opposite in their opinion for stable facts at all states accessible for them.

(5) The formula $[(\Box_1 p \to \Box_2 p) \wedge (\Box_2 p \to \Box_1 p)] \wedge \Diamond_1 N\varphi_{op}$ says that the agents may agree at all visible time but after this the agents may be in a complete opposition.

(6) Total recall: $\Diamond_1 p \wedge \Box_1(p \to \Diamond_1[\neg p \wedge \Diamond_1 p]) \wedge \Diamond_1\Diamond_1\Box_1 p$. This formula says that the agent 1 always swapping its opinion about the truth of p from true to false and vise versa or lose p during the whole initial interval of time, but after some time it decides p to be always true.

Definition 6. *The logic BTA is the set of all formulas which are valid in any model MBT for all states and valuations. If there is a model MBT and a state a where a formula φ is true, we say φ is satisfiable in MBT.*

Recall that for any logic L, the satisfiability problem is to determine by any given formula φ if it is satisfiable in L: if there is a model and a state of this model for which this formula is true. If there is an algorithm answering this question for any given formula φ then the satisfiability problem is said to be decidable.

A logic is decidable if there is an algorithm answering questions "$\varphi \in L$?" for any formula. It is clear that if $\varphi \in L$ then $\neg\varphi$ is not satisfiable; vise versa, if $\varphi \notin L$ then $\neg\varphi$ is satisfiable.

3. Finite Clipped Models

To work with the satisfiability, we will need to define some special finite models. A path $Path(Up(s_1), Up(s_2))$ in a model MBT is said to be of depth m if there are exactly m different $Up(s) \in BTTT$ (upper bounds) in the path leading from the root to $Up(s_2)$.

Definition 7. *For any model MBT and any $m \in N$, an m–clipped model $MBT(m)$ is the model based on the frame with the basic set $|MBT|$ in which we have deleted all states of all paths $Path(Up(s_1), Up(s_2))$ of depth strictly bigger than m except the lower bounds of paths $Path(Up(s_1), Up(s_2))$ of depth $m+1$. For the upper bounds $Up(s_2)$ of the remaining paths $Path(Up(s_1), Up(s_2))$ we define the next state to $Up(s_2)$ by $\prec$ to be itself, otherwise we transfer the relations R_j and the valuation from the original model MBT.*

We may transfer the rules for computation of the truth values of formulas for any clipped models without any amendments. For the formulas with bounded temporal degree, these models will give us a useful tool for satisfiability problem.

Definition 8. *For a formula φ, its temporal degree $td(\varphi)$ is defined inductively as follows. If φ is a propositional letter then $td(\varphi) := 0$. If $\varphi = \varphi_1 \circ \varphi_2$ where $\circ$ is a binary Boolean logical operation, then $td(\varphi) := max\{td(\varphi_1), td(\varphi_2)\}$; if $\varphi = \neg\varphi_1$ then $td(\varphi) := td(\varphi_1)$. If $\varphi = N\varphi_1$ then $td(\varphi) := td(\varphi_1) + 1$. If $\varphi = \varphi_1 U_j \varphi_2$ then $td(\varphi) := max\{td(\varphi_1), td(\varphi_2)\} + 1$.*

Lemma 1. *For any path $Path(Up(s_i), Up(s_j))$ of depth k and any $a \in Path(Up(s_i), Up(s_j))$, where $a \neq Up(s_j)$ for any formula α of temporal degree not bigger than m holds*

$$(MBT, a) \Vdash_V \alpha \Leftrightarrow (MBT(k+m, a) \Vdash_V \alpha. \tag{1}$$

Proof. We show this by induction on m. Indeed, the case $m = 0$ is obvious. Assume that the statement of our lemma is proven for all k and all $n \leq m$ and that we have a formula β of temporal degree $m+1$.

Then the formula β is constructed by boolean operations from some formulas β_i with temporal degree at most m and some formulas γ_i with temporal degree $m+1$ where $\gamma_i = N\delta_i$ and $td(\delta_i) = m$, or $\gamma_i = \xi_1 U_j \xi_2$ and $max(td(\xi_1), td(\xi_2) = m$. For all formulas β with temporal degree at most m, for all paths $Path(Up(s_i), Up(s_j))$ of depth k for all k we have for any $a \in Path(Up(s_i), Up(s_j)) \setminus \{s_j\}$

$$(MBT, a) \Vdash_V \beta \Leftrightarrow (MBT(k+m, a) \Vdash_V \beta. \tag{2}$$

by the inductive assumption. So, to prove our lemma we need to consider the formulas γ_i. Let first $\gamma_i = N\delta_i$ and $td(\delta_i) = m$.

Assume that $a \in Path(Up(s_i), Up(s_j)) \setminus \{Up(s_j)\}$ and

$$(MBT, a) \Vdash_V N\delta_i.$$

Then for some b we have $a \prec b$, where

$$(MBT, b) \Vdash_V \delta_i.$$

If $b \in Path(Up(s_i), Up(s)) \setminus \{Up(s_k)\}$ for some s, we have $(MBT, b) \Vdash_V \delta_i$ and using (2) we get $MBT(k+m, b) \Vdash_V \delta_i$ and hence $MBT(k+m, a) \Vdash_V \ N\delta_i$.

If $b = Up(s)$ (for some $Up(s)$) we get $(MBT, Up(s)) \Vdash_V \delta_i$ and using (2) for $k+1$ we obtain $MBT(k+1+m, b) \Vdash_V \delta_i$; and hence $MBT(k+1+m, a) \Vdash_V N\delta_i$

Vise versa, if $MBT(k+m, a) \Vdash_V \ N\delta_i$ then for some b, $a \prec b$ and in $MBT(k+m, a)$ holds $MBT(k+m, b) \Vdash_V \ \delta_i$. If $b \in Path(Up(s_i), Up(s)) \setminus \{Up(s)\}$, for some s, we obtain $MBT(k+m, b) \Vdash_V \delta_i$ and by (2) it follows that $MBT(k+m, a) \Vdash_V \delta_i$ and $(MBT(k+1+m, a) \Vdash_V \delta_i$.

If $b \notin Path(Up(s_i), Up(s_j)) \setminus \{Up(s)\}$ for all s, then $b = Up(s)$ for some s, and by (2) for $k+1$ we have $(MBT(k+1+m, b) \Vdash_V \delta_i$ and hence $(MBT(k+1+m, a)) \Vdash_V N\delta_i$. So the case $\gamma_i = N\delta_i$ is proven.

Let now $\gamma_i = \xi_1 U_j \xi_2$ and $max(td(\xi_1), td(\xi_2) = m$. Assume first that $a \in Path(Up(s_i), Up(s_j)) \setminus \{Up_s\}$ for some $Up(s_j)$ and

$$(MBT, a) \Vdash_V \xi_1 U_j \xi_2.$$

Then for some $b \in Path(Up(s_i), Up(s_j))$ holds $aR_j b$, where $(MBT, b) \Vdash_V \xi_2$ and for all $c \in Path(Up(s_i), Up(s_j)$ where $aR_j c$ and $c \neq b$ we have $(MBT, c) \Vdash_V \xi_1$.

If $b \neq Up(s_j)$ for that (and all possible) $Up(s_j)$ then we may apply (2) to all the states within the path and then $b \in Path(Up(s_i), Up(s_j))$, $aR_j b$, $(MBT(k+m), b) \Vdash_V \xi_2$ and for all $c \in Path(Up(s_i), Up(s_j)$ where $aR_j c$ and $c \neq b$, $MBT(k+m, c) \Vdash_V \xi_1$. Hence, $MBT(k+m, a) \Vdash_V \xi_1 U_j$ and $MBT(k+1+m, a) \Vdash_V \xi_1 U_j$.

Assume now that $b = Up(s_j)$. Then again we can apply (2) to all the states within the path and then $\forall c \in Path(Up(s_i), Up(s_j)$ where $aR_j c$ and $c \neq b$ holds $MBT(k+m, c) \Vdash_V \xi_1$. In addition we have that $(MBT, Up(s_j)) \Vdash_V \xi_2$; and by (2) for $k+1$ then obtain $(MBT(k+1+m), Up(s_j)) \Vdash_V \xi_2$. Summarizing the above we obtain the following. For a being inside the set $Path(Up(s_i), Up(s_j)) \setminus \{Up(s_j)\})$ in the path of depth k we have

$$(MBT(k+1+m), a) \Vdash_V \xi_1 U_j \xi_2.$$

Suppose now that the previous holds. Then for some $b \in Path(Up(s_i), Up(s))$, for some s $aR_j b$ where $(MBT(k+1+m), b) \Vdash_V \xi_2$ and for all $c \in Path(Up(s_i), Up(s_j)$ where $aR_j c$ and $c \neq b$, $(MBT(k+1+m), c) \Vdash_V \xi_1$.

If $b \neq Up(s_j)$ then all the events hold within the same path of depth k and applying (2) for k we get $(MBT, b) \Vdash_V \xi_2$ and for all $c \in Path(Up(s_i), Up(s_j)$ where $aR_j c$ and $c \neq b$, $(MBT, c) \Vdash_V \xi_1$. Hence, $(MBT, a) \Vdash_V \xi_1 U_j$.

If $b = Up(s_j)$ then we will need to apply the inductive hypothesis for states of paths of different (though neighboring) depths. Then $(MBT(k+1+m), Up(s_j)) \Vdash_V \xi_2$ and by (2) for $k+1$ we have $(MBT, Up(s_j)) \Vdash_V \xi_2$. At the same time for all $c \in Path(Up(s_i), Up(s_j)$ where $aR_j c$ and $c \neq b$, $(MBT(k+1+m), c) \Vdash_V \xi_1$ and by (2) for all such c we obtain $(MBT, c) \Vdash_V \xi_1$. So, we get $(MBT, a) \Vdash_V \xi_1 U_j \xi_2$. That completes the proof of our lemma. □.

Using this lemma we immediately infer:

Lemma 2. *Assume that a model MBT based on a frame F is given and a formula α with temporal degree n is satisfied in this model at the root state a from F by a valuation V. Then α is satisfied at the root of the clipped model $MBT(n)$ by the same valuation V.*

Lemma 3. *If a formula α with any temporal degree is satisfied at the root of some clipped model $MBT(k)$ for some k by the a valuation V, then α may be satisfied in the root of the usual not–clipped model MBT obtained from $MBT(k)$ by stretching any final sate to infinite path by $\prec$ of p–paths of length 2.*

Proof. Just do with the clipped model the following: stretch any final sate s to the infinite path by $\prec$ of p–paths of length 2 and define the valuation V on all such states as on s and all the relations R_j on added states as it has been done for s. It is easy to see that this model will satisfy α on the root. □

Using these two last lemmas we are getting closer to the solution of the satisfiability problem. But the problem is that we are not yet able to computably restrict the possible sizes of p–paths in our clipped models. We will resolve it in the next section.

4. Rarefication Technique via Reduced Forms

As we have noticed above, we cannot use clipped models in present form to solve the satisfiability problem. We need to reduce the sizes of b–paths. An immediate work with formulas does not look promising, because, in particular, the non–transitivity of agents' accessibility relations hampers to convert formulas into more suitable and simple forms, to some canonical or similar ones. There we will use the technique of reduction of formulas to rules (which we have already used earlier many times for different purposes (cf. e. g. [2, 18, 21, 23]) and transformation the latter ones in the so–called reduced forms.

This approach efficiently simplifies all the proofs because it allows to consider very simple and uniform formulas without nested temporal operations, so just temporal degree one. We briefly recall this technique.

A *rule* is an expression $\mathbf{r} := \varphi_1(x_1,\ldots,x_n),\ldots,\ \varphi_s(x_1,\ldots,x_n)\ /\ \psi(x_1,\ldots,x_n)$, where all $\varphi_k(x_1,\ldots,x_n)$ and $\psi(x_1,\ldots,x_n)$ are formulas constructed from the letters (variables) $x_1,\ldots,x_n$.

Formulas $\varphi_k(x_1,\ldots,x_n)$ are called *premises* and $\psi(x_1,\ldots,x_n)$ is called the *conclusion.* The rule $\mathbf{r}$ means that $\psi(x_1,\ldots,x_n)$ (conclusion) follows (logically follows) from the assumptions $\varphi_1(x_1,\ldots,x_n)$, $\ldots$, $\varphi_s(x_1,\ldots,x_n)$. The definition of the validness of a rule is the same for any relational model. However we have models with multi–valuations, so we need some modification.

Assume that a clipped model $MBT(m)$ and a rule $\mathbf{r}$ are given.

Definition 9. *The rule* $\mathbf{r} := \varphi_1(x_1,\ldots,x_n),\ldots,\ \varphi_s(x_1,\ldots,x_n)\ /\ \psi(x_1,\ldots,x_n)$, *is valid on the model* $MBT(m)$ *iff*

$$\left[\forall a \left((MBT(m),a) \Vdash_V \textstyle\bigwedge_{1\leq i\leq s}\varphi_i\right)\right] \quad\Rightarrow\quad \left[\forall a\left((MBT(m),a) \Vdash_{V_l} \psi\right)\right].$$

If $\forall a \left((MBT(m),a) \Vdash_V \bigwedge_{1\leq i\leq s}\varphi_i\right)$ *but* $\exists a\left((MBT(m),a) \nVdash_V \psi\right)$, *then we say that* $\mathbf{r}$ *is refuted in* $MBT(m)$ *by* V *and we denote this fact as* $MBT \nVdash_V \mathbf{r}$.

Definition 10. *A rule* $\mathbf{r}$ *is true (or valid) on a frame for* $MBT(m)$ *iff* $\mathbf{r}$ *is true on any model based on* $MBT(m)$.

Lemma 4. *For a formula* φ, φ *is satisfiable iff the rule* $x \to x/\neg\varphi$ *may be refuted in some model* BMT *iff* $x \to x/\neg\varphi$ *may be refuted in a clipped model* $MBT(n)$ *for some* n.

Proof. The first IFF is evident. The second IFF follows from Lemmas 2 and 3. □

Thus we have

Lemma 5. *If there is an algorithm verifying for any given rule r if this rule is valid on all clipped models $MBT(m)$ then there exists an algorithm verifying if any given formula is satisfiable.*

Now we need to have rules in some uniform simple form, in particular — without nested temporal operations.

Definition 11. *A rule* $\mathbf{r}$ *is said to be in* reduced normal form *if* $\mathbf{r} = \varepsilon/x_1$*, where*

$$\varepsilon = \bigvee_{1\leq j\leq m} \Big[\bigwedge_{1\leq i\leq n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\leq i\leq n} (Nx_i)^{t(j,i,1)} \wedge \wedge \bigwedge_{l\in[1,k],1\leq i,k_1\leq n} (x_i\, U_l x_{k_1})^{t(j,i,k_1,l,2)} \Big],$$

$t(j,i,0)$*,* $t(j,i,l,1)$*,* $t(j,i,k_1,l,2) \in \{0,1\}$ *and, for any formula* α *above* $\alpha^0 := \alpha$*,* $\alpha^1 := \neg\alpha$*.*

Definition 12. *For any given rule* $\mathbf{r}$*, a rule* $\mathbf{r_{nf}}$ *in the reduced normal form is said to be a* reduced normal form of $\mathbf{r}$ *iff*

(i) $\mathbf{r_{nf}}$ *contains all variable–letters from* $\mathbf{r}$ *and maybe some extra ones;*
(ii) *For any clipped model* $MBT(m)$*, the rule* $\mathbf{r}$ *may be refuted in* $MBT(m)$ *if and only if the rule* $\mathbf{r_{nf}}$ *may be refuted in this model.*

Theorem 1. *There exists an algorithm running in (single) exponential time which given any rule* $\mathbf{r}$ *constructs some its reduced form* $\mathbf{r_{nf}}$*. The variables of* $\mathbf{r_{nf}}$ *are all variables of* $\mathbf{r}$ *together with the set of new variables denoting all subformulas of* $\mathbf{r}$*.*

Proof. The proofs of the similar statements for various relative relational models and rules was suggested by us quite a while ago (e. g. cf. Lemma 5 in [2], or the proofs of similar statements in [16]). Here, for completeness we give a sketch of the proof. Let a rule $\mathbf{r} = \alpha/\beta$ be given. Let $Sub(\mathbf{r})$ the set of all subformulas of the rule $\mathbf{r}$. We fix a set of variable letters $Z = \{z_\gamma \mid \gamma \in Sub(r)\}$ not occurring in $\mathbf{r}$ and a rule in an intermediate form:

$$\mathbf{r_{if}} = z_\alpha \wedge \bigwedge_{\gamma\in Sub(\mathbf{r})\setminus Var(\mathbf{r})} (z_\gamma \leftrightarrow \gamma^\sharp)/z_\beta,$$

where

$$\gamma^\sharp = \begin{cases} z_\delta * z_\epsilon, & \text{if } \gamma = \delta * \epsilon \text{ for } * \in \{\wedge, \vee, \to, U_j\} \\ *z_\delta, & \text{if } \gamma = *\delta \text{ for } * \in \{\neg, N\}. \end{cases}$$

The rules $\mathbf{r}$ and $\mathbf{r_{if}}$ are true or refuted on the frame of any model simultaneously. If $BTM(m)$ is a model with a valuation V such that $BTM(m) \nVdash_V \mathbf{r}$ then $BTM(m) \Vdash_V \alpha$ and there exists an $a \in |BTM(m)|$ such that $(BTM(m), a) \nVdash_V \beta$. We then choose the valuation $V_1 : Z \to 2^{|BTM(m)|}$ with $V_1(z_\gamma) := V(\gamma)$. Then it is easy to see (computing by induction of the length of formulas) that $BTM(m) \Vdash_{V_1} z_\alpha \wedge \bigwedge\{z_\gamma \leftrightarrow \gamma^\sharp \mid \gamma \in Sub(\mathbf{r}) \setminus Var(\mathbf{r})\}$ and $(BTM(m), w) \nVdash_{V_1} z_\beta$.

From the other hand, let us assume that there is a model $BTM(m)$ with a valuation V_1 such that $V_1 : Z \to 2^{|BTM(m)|}$ and $BTM \Vdash_{V_1} z_\alpha \wedge \bigwedge\{z_\gamma \leftrightarrow \gamma^\sharp \mid \gamma \in Sub(\mathbf{r}) \setminus Var(\mathbf{r})\}$ and $(BTM, w) \nVdash_{V_1} z_\beta$ for some w.

Now define a valuation V as $V : Var(\mathbf{r}) \to 2^{|BTM(m)|}$ and $V(x_i) = V_1(z_{x_i})$. Then (computing by induction of the length of formulas) we obtain $V(\gamma) = V_1(z_\gamma)$, for all $\gamma \in Sub(r)$. Hence, $BMT(m) \Vdash_V \alpha$, $(BMT(m), w) \nVdash_V \beta$ and consequently $BTM(m) \nVdash_V \mathbf{r}$. Next, we transfer the premise of $\mathbf{r_{if}}$ in perfect disjunctive normal form constructed out of formulas of kind x_i, Nx_i and $x_iU_jx_j$. As we know, the latter transformation is single exponential on the number of all formulas of kind x_i, Nx_i and $x_iU_jx_j$, and hence on the number of all subformulas of the original rule, and hence on its length. □

The reduced normal forms of rules constructed by the algorithm shown in the proof of this theorem are defined uniquely.

Thus, if we are interested to investigate the problem of refutation for rules, we may restrict ourselves with considerations of rules in the reduced form only. Recall that now we consider truncated models.

Lemma 6. *If a rule in a reduced normal form* $\mathbf{r_{nf}}$ *is refuted in a clipped model* $MBT(m)$ *then* $\mathbf{r_{nf}}$ *can be refuted in some such model of size computable from the size of the rule.*

Proof. Let $\mathbf{r_{nf}} = \varepsilon/x_1$, where $\varepsilon = \bigvee_{1\leq j\leq m} \theta_j$,

$$\theta_j = \Big[\bigwedge_{1\leq i\leq n} x_i^{t(j,i,0)} \wedge \bigwedge_{1\leq i\leq n} (Nx_i)^{t(j,i,1)} \wedge \\ \wedge \bigwedge_{l\in[1,k],1\leq i,k_1\leq n} (x_i\, U_l x_k)^{t(j,i,k_1,l,2)} \Big],$$

and assume that $\mathbf{r_{nf}}$ is refuted in a given model $MBT(m)$. Then for a valuation V the premise of the rule is true at any state but the conclusion of the rule is refuted by V at the root; we may assume that the conclusion x_1 is refuted at the root $Up(s_0)$ of the frame.

Besides, we may assume that the model $MBT(m)$ has a finite number (though not commutable bounded yet) of paths leaving the root (at least as much as it is necessary to make any required formula $x_iU_jx_k$ or Nx_i from the premise of the rule to be true (if required)).

At the first stage of the proof we will rarefy the frame and achieve that any path $Path(Up(s_i), Up(s_j))$ within the frame will have finite number of states with certain computable upper bound. Let us start from all paths $Path(Up(s_0), Up(s_1))$ leaving from $Up(s_0)$. Consider some of them, $Path(Up(s_0), Up(s_1))$ and the next state a for $Up(s_0)$.

Since the premise of the rule is true at the frame w.r.t. V, at any state b of the frame, there is a unique disjunct θ_j of the premise of the rule which is true at b. Denote it by $\theta(b)$. We have:

$$(MBT(m), b) \Vdash_V \theta(b),$$

so $(MBT(m), Up(s_0)) \Vdash_V \theta(Up(s_0))$ and $(MBT(m), a) \Vdash_V \theta(a)$. Consider the state $Cp(a)$ from the b–path $Path(Up(s_0), Up(s_1))$ closest to the state $Up(s_1)$ such that

$$(MBT(m), Cp(a)) \Vdash_V \theta(a),$$

(if exists). Now we delete all intermediate states between $Up(s_0)$ and $Cp(a)$ together with all paths (and b–paths inside them, completely) erasing from deleted states, and then we define the next state above $Up(s_0)$ to be $Cp(a)$: $Up(s_0) \prec Cp(a)$.

Concerting accessibility relations R_j on $Path(Up(s_0), Up(s_1))$, recall that any of them is a reflexive linear transitive relation on $Path(Up(s_0), Up(s_1))$ which is a subset of $\prec_*$; besides R_j may have some lacunas — states from $Path(Up(s_0), Up(s_1))$ which are not accessible by R_j even from $Up(s_0)$. Now we just transfer any R_j to the remaining part of $Path(Up(s_0), Up(s_1))$. This transformation does not change the previously existed R_j on the states of the remaining part. Denote the obtained model as TM. For any $b \in Path(Up(s_0), Up(s_1)) \cap |TM|$ the following holds:

Lemma 7. *If* $b \in Path(Up(s_0), Up(s_1)) \cap |TM|$ *then for any* θ_j

$$(TM, b) \Vdash_V \theta_j \Leftrightarrow (MBT, b) \Vdash_V \theta_j.$$

Proof. Indeed, the truth of letters x_i is the same, the truth of Nx_i is again the same since $(MBT(m), Cp(a)) \Vdash_V \theta(a)$. The truth of formulas $x_i U_j x_k$ again remains to be the same as it was before, since by our choice of the state $Cp(a)$ it was $(MBT(m), Cp(a)) \Vdash_V \theta(a)$. □

From this point, we consider the state next to $Cp(a)$ in TM within that path and after mowing to $Up(s_1)$ we continue the same transformation; it will preserve the truth values of formulas θ_j. So, this transformation will reduce the size of the path $Path(Up(s_0), Up(s_1)) \cap |\mathrm{TM}|$ to the one whose number of states does not exceed the number of distinct θ_js.

Now we apply the same transformation to all other b–paths coming out of $Up(s_0)$ ($Path(Up(s_0), Up(s_1))$) one by one. The resulting model will again preserve the truth values of formulas θ_j. Next, we keep only different b–paths coming out of $Up(s_0)$ and keep the paths leaving from the internal states of these b–paths and leaving from upper limits $(Up(s_1))$ of such b–paths.

After this we execute similar transformations staring from upper limits of all the b–paths leaving from $Up(0)$ and so on. As a result, we will obtain the finite model preserving truth values of formulas θ_j such that all b–paths $Path(Up(s_i), Up(s_{i+1}))$ will contain at most n_1 states, where n_1 is the number of disjuncts θ_j.

Now we will pull down the b–paths $Path(Up(s_i), Up(s_{i+1}))$ moving from the top of the model to the bottom by replacing all lowermost (belonging to the same path) identical to $Path(Up(s_i), Up(s_{i+1}))$ b–paths and all paths leaving from its states by the upper p–path $Path(Up(s_i), Up(s_{i+1}))$ and paths leaving from it. The resulting model again preserves truth values of formulas θ_j. Now, any complete path of the model consists of different b–paths and the number of states in each path does not exceed the number of distinct θ_js. Hence, the obtained model is finite and its size is computable from the size of the rule. □

Combining Lemmas 5, 6 and Theorem 1 we obtain

Theorem 2. *The satisfiability problem for the logic* BTA *is decidable. The logic* BTA *itself is decidable.*

5. Open problems

Many problems from the framework of this paper are still open; actually, among them is a good set of problems which are actual for any logic, e.g. axiomatization, unifiability problem, decidability with respect to admissible inference rules. We did not yet study the extended versions of our logic for the case with the future and the past. The next open avenue for research is the embedding fuzzy logics in this framework in the case when truth values of formulas at any state are not binary but multi–valued. Here some tools borrowed from Łukasiewicz logic or modern fuzzy–logic with continuous intervals of truth values may be used. In this case it is very interesting to formalize, how different agents interact and, in particular, when, each agent has its own valuation of the basic propositions — propositional letters (but the temporal operations for different agents might be nested in formulas and hence they might interfere). An interesting open problem is the case when the transitivity intervals may have a common overlap, not only chopping boundary states as in this paper.

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