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# UNIQUE DETERMINATION OF CONFORMAL TYPE FOR DOMAINS. II

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Abstract. The article is the second part of a review series entitled "Unique determination of conformal type for domains," initiated by the author's eponymous paper, published in *Sib. Èlektron. Mat. Izv.*, **16**, 692–708 (2019). The main result of the present article is that any convex bounded polyhedral domain in the three-dimensional Euclidean space is uniquely determined by the relative conformal moduli of its boundary condensers.

**Keywords:** $p$-modulus of a family of paths, boundary condenser, quasiconformal mapping, conformal mapping, isometric mapping, unique determination.

## 1. Introduction

In [1], we observed that the main result of [1] is Theorem 1.1, discussed in detail in [2]-[4]. In accordance with this theorem, *if $n \geq 4$ then any bounded convex polyhedral domain $U \subset \mathbb{R}^n$ (i.e., a nonempty bounded intersection of finitely many open $n$-dimensional half-spaces) is uniquely determined by the relative conformal moduli of its boundary condensers in the class $\mathcal{P} = \mathcal{P}(n)$ of all bounded convex polyhedral domains $V \subset \mathbb{R}^n$*. In this article, we keep to the notations and use the notions and assertions of [1]. For this reason, we refer the reader for them to [1].

In the present article, we fully expose an analogous result for the case of bounded convex domains in the three-dimensonal Euclidean space. Moreover, we consider

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the question of unique determination of isometric type for domains in $\mathbb{R}^3$ on the basis of the notion of the $p$-moduli of a family of paths.

## 2. Unique Determination of Three-Dimensional Convex Polyhedral Domains by the Relative Conformal Moduli of Boundary Condensers

In this section, we prove an analog of [1, Theorem 1.1] in the case of $n = 3$ (Theorem 2.1 below), which was announced in [5] and becomes possible due to a criterion obtained by V. V. Aseev (in [6]) for the equality of convex polyhedral angles in the case of $n = 3$ (see Lemma 2.1 below).

Let us list some features of the case $n = 3$. In the proof of [1, Theorem 1.1] (see, for example, [3, Theorem 8.1]), one of the most important points is to prove that if $f : \operatorname{fr} V \to \operatorname{fr} U$ is a homeomorphism preserving the relative conformal moduli of boundary condensers, then on every $(n-1)$-dimensional face $s$ of $\operatorname{fr} V$ there exists a connected subset $\sigma$ open with respect to $s$ such that $f(\sigma)$ is an open subset in some $(n-1)$-dimensional face of $\operatorname{fr} U$; moreover, $f|_\sigma$ is an $(n-1)$-dimensional conformal mapping. Another important point is to prove the identity $\sigma = \operatorname{Int} s$. Here, an essential role is played by the following property of spatial conformal mappings: any conformal mapping $g : \lambda \to \mathbb{R}^m$ of a domain $\lambda$ in $\mathbb{R}^m$, $m \geq 3$, is the restriction $G|_\lambda$ of some Möbius transformation $G : \bar{\mathbb{R}}^n \to \bar{\mathbb{R}}^n$ (see, for example, [7]). However, in the case $m = 2$ ($n = 3$) we need to use plane conformal mappings whose properties are cardinally different from those of spatial conformal mappings.

So, our main aim (in this section) is to prove the following assertion:

**Theorem 2.1.** *Let $n = 3$. Then every bounded convex polyhedral domain $U \subset \mathbb{R}^3$ is uniquely determined by the relative conformal moduli of its boundary condensers in the class $\mathcal{P}$ of all bounded convex polyhedral domains $V \subset \mathbb{R}^3$. Furthermore, $U$ is determined in $\mathcal{P}$ up to an additional similarity transformation $P : \mathbb{R}^3 \to \mathbb{R}^3$.*

The proof of Theorem 2.1 consists of several steps.

The first step is the following assertion:

**Theorem 2.2.** *Let a homeomorphism $f : \operatorname{fr} U \to \operatorname{fr} V$ of the boundaries of bounded convex polyhedral domains $U$ and $V$ in $\mathbb{R}^3$ satisfy the following conditions*:

(1) *$f$ and $f^{-1}$ send edges to edges*;

(2) *the interior dihedral angles between pairs of adjacent faces remain equal under the map $f$*;

(3) *$f$ is a plane conformal mapping on each face of $U$.*

*Then the domains $U$ and $V$ are similar and $f$ extends to a similarity transformation of $\mathbb{R}^3$.*

For proving Theorem 2.2, we need some auxiliary assertions. Let $W$ be a given convex solid polyhedral angle in $\mathbb{R}^3$ with vertex $O$ and edges $w_1, \dots, w_m$ enumerated in the order of circumvention around $O$ along the exterior boundary $\operatorname{fr} W$ in the positive direction (counterclockwise). The face with sides $w_j$, $w_{j+1}$ is denoted by $\sigma_{j,j+1}$ (the operations in the subscripts are understood modulo $m$). Following [8], we construct the polar polyhedral angle $W_*$ with vertex $O$ whose edges are rays directed along the outward normals to the faces $\sigma_{j,j+1}$ ($j = 1, \dots, m$). We note that $W_*$ is also a convex polyhedral angle (see [8]).

**Lemma 2.1.** *Assume that for convex polyhedral angles $W, W' \subset \mathbb{R}^3$ there exists a bijection $f$ between the sets of all edges of $W$ and $W'$ preserving the order of circumvention of edges around the vertex along the exterior boundaries of these angles. Let $\alpha_j$ be the angle between the edges $w_j$ and $w_{j+1}$ ($j = 1, \dots, m$), and let $\beta_j$ be the angle between the edges $f(w_j)$ and $f(w_{j+1})$. If*

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \dots = \frac{\alpha_m}{\beta_m}$$

*and the interior dihedral angles at the edges $w_j$ and $f(w_j)$ are equal, then $W$ and $W'$ are isometric in $\mathbb{R}^3$.*

*Доказательство.* For the polyhedral angles $W$ and $W'$ we construct the polar convex polyhedral angles $W_*$ and $W'_*$. A bijection $f$ between the edges of $W$ and $W_*$ generates a bijection between the edges of $W'$ and $W'_*$. By duality, in $W'$ and $W'_*$, the plane angles in the corresponding faces are equal, and, for the differences of interior dihedral angles $\pi - \alpha_j$ and $\pi - \beta_j$ at the corresponding edges, we have $(\pi - \alpha_j) - (\pi - \beta_j) = (t - 1)\alpha_j$, where $t = \alpha_j/\beta_j$ is independent of $j$. Assume that $t \neq 1$. Then these differences must change the sign at least four times in accordance with [9, Lemma 2a]. However, in our situation, they preserve the sign for all $j$. The so-obtained contradiction shows that $t = 1$, and hence $\alpha_j = \beta_j$ for all $j$. Hence, in the convex polyhedral angles $W$ and $W'$, the corresponding plane and bihedral angles are equal. By the Cauchy lemma (see, for example, [9]), these polyhedral angles are equal. □

**Lemma 2.2.** *Assume that plane bounded polygons $P$ (with vertices $p_1, \dots, p_k$) and $Q$ (with vertices $q_1, \dots, q_k$) are such that their interior angles $\pi\delta_j$ at the corresponding vertices $p_j$ and $q_j$ are equal. If a conformal mapping $f : P \to Q$ takes $p_j$ to $q_j$, $j = 1, \dots, k$, then $f$ is a similarity transformation.*

*Доказательство.* By the Schwarz–Christoffel theorem (see, for example, [10]), there is a collection of finite points $a_1, \dots, a_k$ on the real axis and a conformal mapping $g_1$ from the upper half-plane $\mathbb{C}_+$ to the polygon $P$ such that $g_1(a_j) = p_j$. For $z_0 \in \mathbb{C}_+$ this mapping can be defined by

$$g_1(z) = C_1 \int_{z_0}^{z} (z - a_1)^{\delta_1 - 1} \cdot \dots \cdot (z - a_k)^{\delta_k - 1} dz + C_2,$$

where for every factor $(z - a_j)^{\delta_j - 1} = \exp\{(\delta_j - 1)\operatorname{Log}(z - a_j)\}$ the single branch of the function $\operatorname{Log}(z - a_j)$ in the upper half-plane $\mathbb{C}_+$ is defined by the condition $0 \le \operatorname{Im}(\operatorname{Log}(z - a_j)) \le \pi$. Moreover, the constants $C_1$ and $C_2$ are uniquely determined by the point $z_0$. Applying the Schwarz–Christoffel theorem to an (analogous) conformal mapping $g_2 : \mathbb{C}_+ \to Q$ and taking into account that $f(g_1(a_j)) = q_j$, we get

$$\zeta = g_2(z) = f(g_1(z)) = C_1^* \int_{z_0}^{z} (z - a_1)^{\delta_1 - 1} \cdot \dots \cdot (z - a_k)^{\delta_k - 1} dz + C_2^*.$$

Consequently, for any $w \in P$ we have the equality $(\zeta - C_2^*)/C_1^* = (w - C_2)/C_1$ which means that $f$ is a similarity transformation: $\zeta = C_2^* + (C_1^*/C_1)(w - C_2)$. □

**Lemma 2.3.** *If, under the assumptions of Theorem* 2.2, *$f$ is a similarity transformation of an open subset $u$ of a face $\sigma$ of the domain $U$, then $f$ is extended to a similarity transformation of the whole space $\mathbb{R}^3$.*

*Доказательство.* Let $f|_u$ coincide with the restriction $F|_u$ of some similarity transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$. Then the composition $g = F^{-1} \circ f$ satisfies all the assumptions of Theorem 2.2 for convex polyhedral domains $U$ and $F(V)$ and is identical on the open set $u$. By the interior uniqueness theorem for analytic functions, the mapping $g$ is identical on the whole face $\sigma$. Let the faces $\sigma_1$ and $\sigma$ have a common edge $L$. In the plane, we consider the involute $\Sigma$ of $\operatorname{fr} U$, where the polygons $\sigma$ and $\sigma_1$ are glued together along their common side $L$, and the corresponding involute $\Sigma'$ of $\operatorname{fr} F(V)$, where the polygons $F(\sigma') = g(\sigma)$ and $F(\sigma_1') = g(\sigma_1)$ are glued together along their common side $L' = g(L)$ (see [9, Chapter 1]). The orientation preserving homeomorphism $g$ from the plane domain $\sigma \cup L \cup \sigma_1$ to the plane domain $F(\sigma') \cup L' \cup F(\sigma_1')$ is a conformal mapping on each of the faces $\sigma$ and $\sigma_1$. By the continuous extension principle (see [10, Theorem 1]), $g$ is conformal in the entire domain $\sigma \cup L \cup \sigma_1$ and, since it is identical on $\sigma$, it is identical everywhere in $\sigma \cup L \cup \sigma_1$, in particular, on $\sigma_1$. Thus, passing from face to face, we verify that the homeomorphism $g$ on each face of the domain $U$ is the identity mapping. Hence the mapping $g : \operatorname{fr} U \to \operatorname{fr} F(V)$ is an isometry on each face. By the known result (see [8, Chapter 3]), the mapping $g$ is extended to an isometry $\widetilde{g}$ of the whole space $\mathbb{R}^3$. Therefore, the similarity transformation $\widetilde{f} = F^{-1}(\widetilde{g})$ serves as an extension of the homeomorphism $f$ to the whole space $\mathbb{R}^3$. □

*Proof of Theorem* 2.2. In the polyhedral domain $U$, we consider a vertex $O$ and the adjacent faces $\sigma_1, \dots, \sigma_m$ enumerated in the order of circumvention around $O$ along $\operatorname{fr} U$ in the positive direction. Denote by $\alpha_j$ $(j = 1, \dots, m)$ the interior angle at the vertex $O$ in the polygon $\sigma_j$ and by $L_j = \sigma_j \cap \sigma_{j+1}$ $(j = 1, \dots, m; m+1 \equiv 1)$ the edges outgoing from $O$. We set

$$\Sigma = \bigcup_{j=1}^{m} \sigma_j, \quad \Phi = \bigcup_{j=1}^{m} \alpha_j.$$

In the polyhedral domain $V$, we consider the vertex $O' = f(O)$, the faces $\sigma_j' = f(\sigma)$, the edges $L_j' = f(L_j) = \sigma_j' \cap \sigma_{j+1}'$, and the neighborhood

$$\Sigma' = f(\Sigma) = \bigcup_{j=1}^{m} \sigma_j$$

of the point $O'$ on $\operatorname{fr} V$. We set

$$\Psi = \bigcup_{j=1}^{m} \beta_j,$$

where $\beta_j$ $(j = 1, \dots, m)$ is an interior angle at the vertex $O'$ in the polygon $\sigma_j'$.

We construct an orientation preserving homeomorphism $H : \Sigma \to D \subset \mathbb{C}$ from $\Sigma$ onto the closed neighborhood $D = H(\Sigma)$ of the point 0. The homeomorphism $H$ is uniquely determined by the following conditions:

(1) $H$ takes line segments with origin $O$ and endpoints on $\operatorname{fr} D$ to line segments with origin 0 and endpoints on $\operatorname{fr} D$; moreover, the edge $L_m$ goes to the line segment $H(L_m) \subset \{z \in \mathbb{C} : \operatorname{Im} z = 0, \operatorname{Re} z \geq 0\}$,

(2) $|H(x)| = |x - O|^{2\pi/\Phi}$ for all $x \in \Sigma$,

(3) for any pair of line segments with origin $O$ and endpoints on $\operatorname{fr} \Sigma$ that lie on the same face, the angle $\varphi$ between these segments is connected with the angle $\widetilde{\varphi}$ between their images by the equality $\widetilde{\varphi} = (2\pi/\Phi)\varphi$.

Then, for any face $\sigma_j$, the restriction $H|_{\sigma_j}$ maps $\operatorname{Int}\sigma_j$ conformally onto

$$H(\operatorname{Int}\sigma_j) \subset \mathbb{C}.$$

For the polyhedral domain $V$ with vertex $O'$, we construct an orientation preserving homeomorphism $H' : \Sigma' \ (= f(\Sigma)) \to D' \subset \mathbb{C}$ which is uniquely determined by the following three conditions:

(1′) $H'$ takes line segments with origin $O'$ and endpoints on $\operatorname{fr}\Sigma'$ to line segments with origin 0 and endpoints on $\operatorname{fr} D'$; moreover, the edge $L'_m$ goes to the line segment $H'(L'_m) \subset \{z \in \mathbb{C} : \operatorname{Im} z = 0, \operatorname{Re} z \geq 0\}$;

(2′) $|H'(x)| = |x - O'|^{2\pi/\Psi}$ for all $x \in \Sigma'$,

(3′) for any pair of line segments with origin $O$ and endpoints on $\operatorname{fr}\Sigma'$ that lie on the same face, the angle $\psi$ between these segments is connected with the angle $\widetilde{\psi}$ between their images by the equality $\widetilde{\psi} = (2\pi/\Psi)\psi$.

Then for any face $\sigma'_j$ the restriction $H'|_{\sigma'_j}$ maps $\operatorname{Int}\sigma'_j$ conformally onto

$$H'(\operatorname{Int}\sigma'_j) \subset \mathbb{C}.$$

Let us verify that the homeomorphism $g = H' \circ f \circ H^{-1} : D \to D'$ is a conformal mapping in a neighborhood of any interior point $z_0 \in \operatorname{Int} D$.

If $H^{-1}(z_0)$ is an interior point of some face $\sigma_j$, then $g$ is conformal in a neighborhood of this point because $H^{-1}$ is conformal in a neighborhood of $z_0$, $f|_{\sigma_j}$ is conformal in a neighborhood of $H^{-1}(z_0)$, and $H'$ is conformal in a neighborhood of the point $f(H'(z_0))$ which is an interior point of the face $\sigma'_j$ .

If $H^{-1}(z_0)$ is an interior point of some edge $L_j$, then $z_0$ is an interior point of the line segment $H(L_j)$. Then, in the disk $B(z_0, r)$ of a sufficiently small radius, the function $g$ is analytic in $B(z_0, r) \setminus H(L_j)$ and continuous in $B(z_0, r)$. By the continuous extension principle (see [10, Theorem 1]), $g$ is analytic in the whole disk $B(z_0, r)$. Thus, the function $g$ is analytic in a punctured neighborhood of the point 0. Since $\lim_{z \to z_0} g(z) = g(0) = 0$ exists, 0 is a removable singular point of $g$ and, consequently, $g$ is analytic in a neighborhood of 0 and the homeomorphism $g : \operatorname{Int} D \to \operatorname{Int} D'$ is a conformal mapping. Then $g$ preserves the angles between the directions at the point 0. Therefore, taking into account (3) and (3′), for any pair of edges $L_j$ and $L_{j+1}$ we have $(2\pi/\Phi)\alpha_j = (2\pi/\Psi)\beta_j$ which shows that the ratio $\alpha_j/\beta_j = \Phi/\Psi$ is independent of $j = 1, \dots, m$. Extending the edges $L_j$ and $L'_j$ to rays $w_j$ and $w'_j$ outgoing from the points $O$ and $O'$ respectively, we find polyhedral angles $W$ and $W'$ satisfying all the assumptions of Lemma 2.1. Then for each face $\sigma_j$ $(j = 1, \dots, m)$ we get the equality $\alpha_j = \beta_j$ for interior angles in $\sigma_j$ and $\sigma'_j = f(\sigma_j)$ at the vertices $O$ and $O'$ respectively. Since the vertex $O$ is chosen arbitrarily, in the polyhedral domain $U$, all the corresponding plane angles in the polygons $\sigma_1$ and $\sigma'_1 = f(\sigma_1)$ are equal. By Lemma 2.2, $f|_{\sigma_1}$ is a similarity transformation. Applying Lemma 2.3, we obtain the desired result. □

*Proof of Theorem* 2.1. By definition, for $V \in \mathcal{P}(3)$ there exists a homeomorphism $f : \operatorname{fr} U \to \operatorname{fr} V$ preserving the relative conformal moduli of boundary condensers. Our goal is to prove that the domain $V$ can be conformally mapped onto the domain $U$. For this purpose, we first consider the unions $T_1$ and $T'_1$ of all edges of $\operatorname{fr} U$ and $\operatorname{fr} V$ respectively. The sets $\Sigma = \operatorname{fr} U \setminus \{T_1 \cup f^{-1}(T'_1)\}$ and $\Sigma' = \operatorname{fr} V \setminus \{f(T_1) \cup T'_1\} = f(\Sigma)$ are everywhere dense in $\operatorname{fr} U$ and $\operatorname{fr} V$ (respectively) and are open in the topologies of these boundaries induced by the Euclidean metric in $\mathbb{R}^3$. The restriction $f|_\Sigma$ ($f^{-1}|_{\Sigma'}$) of $f$ ($f^{-1}$) on $\Sigma$ ($\Sigma'$) is a 2-dimensional conformal mapping, which can be proved in the same way as Theorem 8.1 in [3].

Let us prove that $T'_1 = f(T_1)$. Assume that $f(T_1)$ is not a subset of $T'_1$. Since $f$ is continuous, there exists an edge $L$ of $U$ and a point $a \in \operatorname{Int} L$ such that $f(a) \in \operatorname{Int} s'$, where $s'$ is a face of $V$. Since $f(a)$ does not belong to $T'_1$ and $f$ is a homeomorphism, there exists an (open) three-dimensional ball $B_3(a, R)$ with center $a$ and radius $R$ possessing the following properties: $R$ is less than the distance from the point $a$ to the set $f^{-1}(T'_1)$, $f(B_3(a, R) \cap \operatorname{fr} U) \subset s'$, and the intersection of $B_3(a, R)$ with each of the planes $\tau_1$ and $\tau_2$ containing one of the neighboring faces $s_1$ and $s_2$ with the common edge $L$ is a subset of the face $s_1$ (respectively $s_2$). In particular, this means that the intersection of $B_3(a, R)$ with the line $l$ containing the edge $L$ is a subset of $L$. Rotating $h$ around the line $l$, we develop $s_1$ in such a way that the planes of the faces $s_1$ and $s_2$ coincide. Then, without loss of generality, the mapping $\widetilde{f} : B_2(a, R) \to \tau'$ from the disk $B_2(a, R)$ in the plane $\tau_2$ into the plane $\tau'$ containing the face $s'$ of the domain $V$ such that $\widetilde{f}(x) = f(h^{-1}(x))$ for $x \in h(B_2(a, R) \cap s_1)$ and $\widetilde{f}(x) = f(x)$ for $x \in B_2(a, R) \cap s_2$ can be regarded as a mapping from $B_2(a, R) \subset \mathbb{C}$ into $\mathbb{C}$. By construction and the relation $\operatorname{Int}(B_3(a, R) \cap s_j) \subset \Sigma$, $j = 1, 2$, we conclude that $\widetilde{f}$ is a continuous bijection from $B_2(a, R)$ into $f(B_3(a, R) \cap \partial U)$; moreover, $\widetilde{f}$ is a plane conformal mapping in the half-disks $\operatorname{Int} h(B_3(a, R) \cap s_1)$ and $\operatorname{Int}(B_3(a, R) \cap s_2)$. Hence, $L$ is ACL-removable for $\widetilde{f}$. Consequently, $\widetilde{f}$ is conformal in $B_2(a, R)$. Therefore, $\frac{d\widetilde{f}}{dz}(a) \neq 0$, and $f|_{B_3(a,R)\cap L}$ is a smooth mapping.

Let $r_0 > 0$ be such that $r_0 < R$ and $B_3(a, r_0) \cap \widetilde{V_\alpha} \subset U$; moreover, $\widetilde{V_\alpha} = \operatorname{contg}_U a = \operatorname{cl}(P(V_\alpha))$ ($P : \mathbb{R}^3 \to \mathbb{R}^3$ is a similarity transformation) and

$$V_\alpha = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_j \in \mathbb{R}, j = 1, 2, 3, x_2 = \lambda \cos\theta, \\ x_3 = \lambda \sin\theta, 0 < \lambda < \infty, 0 < \theta < \alpha\},$$

where $0 < \alpha < \pi$ (we note that additional conditions on the number $r_0$ will be imposed below). Let $\mu \geq 2$ be a natural number, and let $t > 0$ be a real number such that $2\mu t \leq r_0$. Without loss of generality, we can assume that $l$ is the abscissa axis (in $\mathbb{R}^3$), $\widetilde{V_\alpha} = V_\alpha$, and $a = 0$. We consider the boundary condenser $A_\mu = \{F_1^\mu, F_2^\mu\}$ of both domains $U$ and $V_\alpha$ with components

$$F_1^\mu = \{x \in \mathbb{R}^3 : -t \leq x_1 \leq 0, x_2 = x_3 = 0\},$$
$$F_2^\mu = \{x \in \mathbb{R}^3 : t \leq x_1 \leq \mu t, x_2 = x_3 = 0\}.$$

Since $B_3(0, r_0) \cap V_\alpha \subset U$, we have

$$\Gamma_{F_1^\mu, F_2^\mu, U} \subset \Gamma_{F_1^\mu, F_2^\mu, B_3(0,2\mu t)\cap V_\alpha} \cup \Gamma^\mu \subset \Gamma_{F_1^\mu, F_2^\mu, V_\alpha} \cup \Gamma^\mu,$$

where $\Gamma_{F_1^\mu, F_2^\mu, B_3(0,2\mu t)\cap V_\alpha}$ is a family of paths $\gamma$ in $\Gamma_{F_1^\mu, F_2^\mu, U}$ whose images $\operatorname{Im}\gamma$ are contained in the ball $B_3(0, 2\mu t)$ and $\Gamma^\mu$ is a family of paths $\gamma_0$ in $\Gamma_{F_1^\mu, F_2^\mu, U}$ such that $\{\operatorname{Im}\gamma_0\} \cap \{\mathbb{R}^3 \setminus B_3(0, 2\mu t)\} \neq \varnothing$; moreover, $\Gamma^\mu$ is minorized by the family $\Gamma_{S_1, S_2, \bar{A}_\mu}$ of paths joining the boundary spheres $S_1 = \{x \in \mathbb{R}^3 : |x| = t\}$ and $S_2 = \{x \in \mathbb{R}^3 : |x| = 2\mu t\}$ of the spherical shell $\bar{A}_\mu = \{x \in \mathbb{R}^3 : t < |x| < 2\mu t\}$. By [11, Theorems 6.2, 6.4, and 7.5],

$$M^U(A_\mu) \leq M^{V_\alpha}(A_\mu) + 4\pi\{\log(2\mu)\}^{-2} \leq M^{V_\alpha}(A) + 4\pi(\log\mu)^{-2}. \tag{2.1}$$

We note that estimate (2.1) is independent of the parameter $t$ provided that its values are sufficiently small, for example, $t \leq r_0/(2\mu)$. By the conformal invariance of the modulus (3-modulus) of boundary condensers, we can assume that

$$f(0)=0,\ \ \frac{d\bar f}{dz}(0)=C>0,\ \ \tau'=\{x\in\mathbb{R}^3: x_3=0\},\ \ V\subset\mathbb{R}^3_+=\{x\in\mathbb{R}^3: x_3>0\}.$$

Furthermore, if $r_0$ is sufficiently small, then, because of the smoothness of $f|_{B_3(0,R)\cap L}$, we have $\frac{df}{dt}(r)\neq 0$ for all $r\in[-r_0,r_0]$ and the function $r\mapsto |f(r)|$ is (strictly) decreasing on $[-r_0,0]$ and (strictly) increasing on $[0,r_0]$. We can also assume that $r_0$ satisfies the condition

$$B_3(0,\max_{|z|\le r_0, z\in\mathbb{C}}|\widetilde f(z)|)\subset V.$$

Two cases are possible:

*Case* 1: 0 lies in the closure of the set of $t\in]0,r_0/2\mu[$ such that $|f(-t)|\ge|f(t)|$,

*Case* 2: 0 lies in the closure of the set of $t\in]0,r_0/2\mu[$ such that $|f(-t)|<|f(t)|$.

*Case* 1. Assume that $r_0$ is small enough so that for any $t\in]0,r_0/2\mu[$ the ray $\chi$ outgoing from the point $f(2\mu t)$ in the opposite direction to the direction of the ray that is the contingency of the set $f([0,2\mu t])$ at the point $2\mu t$ and the vector $e_1$ ($e_1$, $e_2$, $e_3$ is the canonical basis for $\mathbb{R}^3$) form an angle less than $\pi/2$. For any such $t$ we extend the curve $f([0,2\mu t])$ by this ray (the obtained curve is denoted by $E_t$) and consider the boundary condensers

$$\widetilde A_\mu=\{\widetilde F_1^\mu,\widetilde F_2^\mu\},\ \ \widetilde F_1^\mu=f(F_1^\mu)=f([-t,0]),\ \ \widetilde F_2^\mu=f(F_2^\mu)$$

of the domain $V$ and, simultaneously, the half-space $\mathbb{R}^3_+$ and

$$\widetilde A^*_\mu=\{\widetilde F_1^\mu,\widetilde F_2^{\mu *}\},\ \ \widetilde A=\{\widetilde F_1,\widetilde F_2\},\ \ \widetilde F_2^{\mu *}=E_t\backslash f([0,\mu t]),\ \ \widetilde F_1=\widetilde F_1^\mu,\ \ \widetilde F_2=E_t\backslash f([0,t[)$$

of the half-space $\mathbb{R}^3_+$. For these condensers

$$M^{V_\pi}(\widetilde A)\le M^V(\widetilde A_\mu)+M_3(\widetilde\Gamma_\mu)+M_3(\widetilde A^*_\mu,\mathbb{R}^3) \tag{2.2}$$

(see [3], formula (8.32)), where $\widetilde\Gamma_\mu$ is the family of paths $\gamma$ joining $\widetilde F_1$ with $\widetilde F_2$ in $\mathbb{R}^3\setminus\{\widetilde F_1\cup\widetilde F_2\}$ and such that $\{\operatorname{Im}\gamma\}\cap\{\mathbb{R}^3\setminus(B_3(0,f(2\mu t)))\}\neq\varnothing$; moreover,

$$M_3(\widetilde A_\mu,\mathbb{R}^3)=M_3(\Gamma_{\widetilde F_1^\mu,\widetilde F_2^{\mu *},\mathbb{R}^3\setminus\{\widetilde F_1^\mu\cup\widetilde F_2^{\mu *}\}}).$$

Indeed,

$$\Gamma_{\widetilde F_1,\widetilde F_2,V_\pi}\subset\Gamma_{\widetilde F_1^\mu,\widetilde F_2^\mu,V_\pi}\cup\Gamma_{\widetilde F_1^\mu,\widetilde F_2^{\mu *},\mathbb{R}^3\setminus\{\widetilde F_1^\mu\cup\widetilde F_2^{\mu *}\}},\ \ \Gamma_{\widetilde F_1^\mu,\widetilde F_2^\mu,V_\pi}\subset\Gamma_{\widetilde F_1^\mu,\widetilde F_2^\mu,V}\cup\widetilde\Gamma_\mu.$$

By [11, Theorem 6.2],

$$M^{V_\pi}(\widetilde A)\le M^{V_\pi}(\widetilde A_\mu)+M_3(\widetilde A^*_\mu,\mathbb{R}^3)\le M^V(\widetilde A_\mu)+M_3(\widetilde\Gamma_\mu)+M_3(\widetilde A^*_\mu,\mathbb{R}^3),$$

which implies (2.2). We assume that $r_0$ also satisfies the inequality $|f(r)-Cre_1|\le Cr/4$ for all $r\in[-r_0,r_0]$. Taking into account that $\Gamma_{\widetilde F_1^\mu,\widetilde F_2^{\mu *},\mathbb{R}^3\setminus\{\widetilde F_1^\mu\cup\widetilde F_2^{\mu *}\}}$ is minorized by the family $\Gamma_{\widetilde S_1,\widetilde S_2',\widetilde A_\mu'}$ of paths joining the boundary spheres $\widetilde S_1=\{x\in\mathbb{R}^3:|x|=|f(-t)|\}$ and $\widetilde S_2'=\{x\in\mathbb{R}^3:|x|=|f(\mu t)|\}$ of the spherical shell $\widetilde A_\mu'=\{x\in\mathbb{R}^3:|f(-t)|<|x|<|f(\mu t)|\}$ and $\widetilde\Gamma_\mu$ is minorized by the family $\Gamma_{\widetilde S_1,\widetilde S_2'',\widetilde A_\mu''}$ of paths joining the boundary spheres $\widetilde S_1$ and $\widetilde S_2''=\{x\in\mathbb{R}^3:|x|=|f(2\mu t)|\}$ of the spherical shell $\widetilde A_\mu''=\{x\in\mathbb{R}^3:|f(-t)|<|x|<|f(2\mu t)|\}$ and using [11, Theorems 6.2, 6.4, and 7.5], we get

$$M_3(\widetilde A^*_\mu,\mathbb{R}^3)\le 4\pi\{\log(|f(\mu t)|/|f(-t)|)\}^{-2}\le 4\pi\{\log(3\mu/5)\}^{-2}, \tag{2.3}$$

$$M_3(\widetilde{\Gamma}_\mu) \le 4\pi\{\log(|f(2\mu t)|/|f(-t)|)\}^{-2} \le 4\pi\{\log(6\mu/5)\}^{-2} \le 4\pi\{\log(3\mu/5)\}^{-2}. \quad (2.4)$$

In turn, inequalities (2.2)-(2.4) imply

$$M^{V_\pi}(\widetilde{A}) \le M^V(\widetilde{A}_\mu) + 8\pi\{\log(3\mu/5)\}^{-2}. \quad (2.5)$$

On the other hand, by [12, Theorem 3.1],

$$M^{V_\pi}(\widetilde{A}) \ge M^{V_\pi}(\widetilde{A}_0), \quad (2.6)$$

where $\widetilde{A}_0$ is the boundary condenser of $V_\pi$ with components

$$F_1^0 = \{x \in \mathbb{R}^3 : -|f(-t)| \le x_1 \le 0, x_2 = x_3 = 0\},$$
$$F_2^0 = \{x \in \mathbb{R}^3 : |f(t)| \le x_1 \le \infty, x_2 = x_3 = 0\}.$$

Further, the conformal invariance of the $n$-moduli of boundary condensers leads to the identity

$$M^{V_\pi}(\widetilde{A}_\mu) = M(\{\{x \in \mathbb{R}^3 : -1 \le x_1 \le 0, x_2 = x_3 = 0\}, \{x \in \mathbb{R}^3 : |f(t)|/|f(-t)| \le x_1 \le \infty, x_2 = x_3 = 0\}\}) = \psi(|f(t)|/|f(-t)|), \quad (2.7)$$

where $\psi$ is the function introduced in [12] as follows. If $u > 0$, then $\psi$ is the (conformal) modulus of the family of arcs joining the segment $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : -1 \le x_1 \le 0, x_2 = x_3 = 0\}$ with the ray $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : u \le x_1 \le \infty, x_2 = x_3 = 0\}$ in the half-space $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -\infty < x_1, x_2 < \infty, x_3 > 0\}$. Taking into account that $f$ leaves the conformal moduli of boundary condensers unchanged and using the relations (2.1) and (2.5)-(2.7), we find

$$\begin{aligned}\psi(|f(t)|/|f(-t)|) &\le M^{V_\pi}(\widetilde{A}) \le M^V(\widetilde{A}_\mu) + 8\pi\{\log(3\mu/5)\}^{-2} \\ &= M^U(A_\mu) + 8\pi\{\log(3\mu/5)\}^{-2} \\ &\le M^{V_\alpha}(A) + 8\pi\{\log(3\mu/5)\}^{-2} + 4\pi(\log\mu)^{-2} \\ &= (\alpha/\pi)\psi(1) + 8\pi\{log(3\mu/5)\}^{-2} + 4\pi(\log\mu)^{-2}, \quad (2.8)\end{aligned}$$

where [12, Lemma 7.1] was used. Taking into account the continuity of $u \mapsto \psi(u)$ and letting $t \to 0$ and then $\mu \to \infty$ in (2.8), we arrive at the contradictory inequality

$$(0 <)\ \psi(1) \le (\alpha/\pi)\psi(1). \quad (2.9)$$

*Case* 2. We argue as in Case 1, but, in this case, the minorizing family for $\Gamma_{\widetilde{F}_1^\mu, \widetilde{F}_2^{\mu*}, \mathbb{R}^3\setminus\{\widetilde{F}_1^\mu \cup \widetilde{F}_2^{\mu*}\}}$ is the family $\Gamma_{\widetilde{S}_1^0, \widetilde{S}_2^0, \widetilde{A}_\mu^0}$ of paths joining the boundary spheres $\widetilde{S}_1^0 = \{x \in \mathbb{R}^3 : |x| = |f(t)|\}$ and $\widetilde{S}_2^0 = \widetilde{S}_2$, of the spherical shell $\widetilde{A}_\mu^0 = \{x \in \mathbb{R}^3 : |f(t)| < |x| < |f(\mu t)|\}$ and the minorizing family for $\widetilde{\Gamma}_\mu$ is the family $\Gamma_{\widetilde{S}_1^{00}, \widetilde{S}_2^{00}, \widetilde{A}_\mu^{00}}$ of paths joining the boundary spheres $\widetilde{S}_1^{00} = \widetilde{S}_1^0$ and $\widetilde{S}_2^{00} = \{x \in \mathbb{R}^3 : |x| = |f(2\mu t)|\}$ of the spherical shell $\widetilde{A}_\mu^{00} = \{x \in \mathbb{R}^3 : |f(t)| < |x| < |f(2\mu t)|\}$; moreover, $\psi(|f(t)|/|f(-t)|)$ should be replaced with $\psi(|f(-t)|/|f(t)|)$ in (2.7) and (2.8). As a result, we arrive at the same contradictory inequality (2.9).

The contradiction obtained (in both cases) shows that $f(T_1) \subset T_1'$. Replacing $f$ with its inverse $f^{-1}$, we get the inclusion $f^{-1}(T_1') \subset T_1$ which is equivalent to the inclusion $T_1' \subset f(T_1)$. Thus, $T_1' = f(T_1)$, which implies that $f$ transforms the edges, vertices, and faces of the polyhedral domain $U$ to the edges, vertices, and faces of the polyhedral domain $V$. But then the connected components of the set

$\Sigma$ are exactly the interiors of the faces of $U$. Consequently, $f$ is a plane conformal mapping in the interior of each face.

The next step of the proof is to show that the dihedral angles between any two faces of $U$ with a common edge are equal to their images. This fact is proved by contradiction in the same way as the equality $T_1' = f(T_1)$ (with slight modifications). As a result, we arrive at the contradictory relation

$$\psi(1) \leq (\alpha/\alpha')\psi(1),$$

where $\alpha$ is the value of the interior dihedral angle between the neighboring faces $s_1$ and $s_2$ in $U$ $(V)$, $\alpha'$ is the value of the dihedral angle between the faces $f(s_1)$ and $f(s_2)$ $(f^{-1}(s_1)$ and $f^{-1}(s_2))$ in $V$ $(U)$ and $\alpha < \alpha'$.

At the final step, we apply Theorem 2.2. □

**Theorem 2.3.** *Suppose that* $U_1 \subset \mathbb{R}^3$ *and* $U_2 \subset \mathbb{R}^3$ *are bounded convex polyhedral domains for which there exist a homeomorphism* $f : \operatorname{fr} U_1 \to \operatorname{fr} U_2$ *between the boundaries of these domains and a number* $p \in \{]1,3[\cup]3,\infty[\}$ *such that* $f$ *preserves both the relative* 3*-moduli and the relative* $p$*-moduli of boundary ring-shaped condensers.*

*Then there exists an isometry* $H : \mathbb{R}^3 \to \mathbb{R}^3$ *satisfying the condition* $H(U_1) = U_2$.

**Remark 2.1.** *In the case where* $n = 3$ *and* $U_1$ *and* $U_2$ *are bounded convex polyhedral domains*, *Theorem* 2.3 *corresponds to* [1, Corollary 3.1]. *It is very likely that*, *for* $n = 3$, *there are complete analogs of* [1, Corollary 3.1] *and* [1, Theorem 2.1]. *Unfortunately*, *at present*, *the author is not able to prove these assertions.*

The proof of Theorem 2.3 is based on Theorems 2.1 and the following

**Lemma 2.4.** *Assume that* $U \subset \mathbb{R}^3$ *is bounded convex polyhedral domain. Then there exists a ring-shaped boundary condenser* $F$ *of* $U$ *such that* $0 < M_p^U(F) < \infty$ *for every* $p \in [1,\infty[$.

The proof of this assertion is similar to that of [1, Corollary 3.1].

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